Axiomatic Set Theory

Sheet 1 — TT21

Section A

1. Most of these do not have unique or 'right'answers.

   For each of the concepts below give a formula in LST which is a 'natural' definition. Where possible for you, give a Δ_0-formula which is equivalent under **ZF**.

   (a) Concepts:

   (i) $z = \{x_0, \ldots, x_n\}$

   (ii) $z = \langle x_0, \ldots, x_n \rangle$

   (iii) z is an n-tuple

   (iv) z is an n-tuple and $\pi_i(z) = x$

   (v) $z = x \cup y$;

   (vi) $z = x \cap y$;

   (vii) $z = \bigcup x$;

   (viii) $z = \bigcap x$;

   (ix) $z = x \setminus y$;

   (x) z is an n-ary relation on y;

   (xi) z is a function;

   (xii) $z = x \times y$;

   (xiii) z is a function and $dom(z) = x$;

   (xiv) z is a function and $ran(z) = x$;

   (xv) z is transitive;

   (xvi) z is a successor ordinal;

   (xvii) z is a limit ordinal;

   (xviii) $z = \omega$;

2. Deduce Pairing from the other axioms of $\mathbf{ZF}^-$.

3. Suppose ϕ is a formula of LST. Write an LST formula for '$z = \{t : \phi(t)\}$', relativize it to a class A and then write the abbreviation in terms of z and $\{. : .\}$.

   If A is a transitive class, ϕ, ψ are formulae and $x, d, z \in A$, what are the relativizations of $z = \{t \in x : \phi(t)\}$ and $z = \{y : \exists x \in d \; \phi(x, y)\}$ to A?

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4. Show that the transitive set $(M, \in)$ constructed in question 6 below satisfies **Extensionality**, **Emptyset** and **Pairing**.

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Section B

5. Interpret the strict total order $(\mathbb{Q}, <)$ as a model of the LST, i.e. interpret the binary predicate $\in$ as $<$.

   Which axioms of **ZF** hold? Give brief proofs or counterexamples.

6. Work in **ZF**$^-$.

   Show that there exists a transitive set M such that

$$\begin{aligned} \emptyset &\in M; \\ \forall x, y \in M \;\; \{x, y\} &\in M; \\ \forall x \in M \;\; |x| &\leq 2. \end{aligned}$$

   Carefully show that neither **Union** nor **Powerset** is satisfied in $(M, \in)$.

7. Work in **ZF**.

   Show that if a is a non-empty transitive set then $\emptyset \in a$.

   Explain why the following sketch proof is not correct: Suppose $\emptyset \notin a$. By recursion on $n \in \omega$ find $x_n \in a$ such that $x_{n+1} \in x_n$ (in the inductive step we use that $x_n \in a$ means that $x_n \neq \emptyset$ and then transitivity of a to get $x_{n+1} \in a$ as well). Then $\{x_n : n \in \omega\}$ is a subset of a with no minimal element, contradicting **Foundation**.

8. Explain how you would formally express the statement of the following informal meta-theorem (we will take it as a fact and use it freely):

   If $A \subseteq B$ are non-empty transitive classes satisfying (enough of) **ZF**, F is a class function that is absolute for A, B and $a \in A$ then the class function G given by recursion on On is absolute for A, B.

9. Work in **ZF**.

   We define 'x is an ordinal' to mean 'x is a transitive set well-ordered by $\in$'.

   (a) Show that 'x is an ordinal' is equivalent to 'x is a transitive set totally ordered by $\in$'.

   (b) Deduce that 'x is an ordinal' is absolute for non-empty transitive classes $A \subseteq B$ satisfying (enough of) **ZF**. Does this imply that $\mathrm{On}^A = \mathrm{On}^B$ (no full proof or counterexample expected)?

   (c) Assuming that it is consistent with **ZF**$^-$ that there is x with $x = \{x\}$, show that the equivalence in part (a) requires **Foundation**.

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Section C

10. In your answers to question 1:

    Is the Δ_0-formula you gave still equivalent without assuming **ZF**? Which 'bit' of **ZF** is sufficient to give an equivalent Δ_0-formula.

    Is the concept (downward resp. upward) absolute for any classes $A \subseteq B$? For any transitive classes $A \subseteq B$? For any transitive classes $A \subseteq B$ satisfying enough of **ZF**?

11. For an arbitrary strict partial order $(P, <)$, find the order theoretic interpretations of $\emptyset$, $\{x, y\}$, $\bigcup x$, $\mathcal{P}(x)$ and conditions for their existence.

    Given a substructure $(Q, <)$ of $(P, <)$, can you find general conditions on Q (and P) are these absolute?

12. Show that in the theory of weak partial orders $(P, \leq)$ the concepts of minimum, maximum, greatest element and least element are not absolute.

    Note that in the theory of lattices $(L, \leq, \wedge, \vee)$ where $\wedge$ and $\vee$ are binary functions the concepts of minimum and maximum are absolute (by definition of what a substructure is).

13. In your favourite area of mathematics, think about different ways to axiomatize the theory and how these may give rise to different notions of 'substructure' and 'absoluteness' of important concepts.

14. In $(M, \in)$ as constructed in question 6, which of the other axioms of **ZFC** hold? Does it depend on the precise formulation of the axiom?

15. Prove the theorem in question 8.

16. Suppose $V \models$ **ZF**.

    Suppose $F : V \to V$ is a bijective class function.

    Define the relation E by xEy if and only if $x \in F^{-1}(y)$ and consider the structure (V, E) and we write ϕ^E for the formula ϕ where $\in$ is replaced by E (so we interpret ϕ in the structure (V, E)).

    (a) Show that $(V, E) \models$ **Extensionality**.

    (b) Compute $\emptyset^E$, $\{x, y\}^E$, $(\bigcup x)^E$, $\mathcal{P}(x)^E$, ω^E.

    (c) Show that (V, E) satisfies **ZF**$^-$.

    (d) Find a concrete F so that there is x such that $(V, E) \models x = \{x\}$.

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