Axiomatic Set Theory

Sheet 1 — TT21

Section A

1. Most of these do not have unique or 'right'answers.

   For each of the concepts below give a formula in LST which is a 'natural' definition. Where possible for you, give a Δ_0-formula which is equivalent under **ZF**.

   (a) Concepts:

   (i) $z = \{x_0, \ldots, x_n\}$

   (ii) $z = \langle x_0, \ldots, x_n \rangle$

   (iii) z is an n-tuple

   (iv) z is an n-tuple and $\pi_i(z) = x$

   (v) $z = x \cup y$;

   (vi) $z = x \cap y$;

   (vii) $z = \bigcup x$;

   (viii) $z = \bigcap x$;

   (ix) $z = x \setminus y$;

   (x) z is an n-ary relation on y;

   (xi) z is a function;

   (xii) $z = x \times y$;

   (xiii) z is a function and $dom(z) = x$;

   (xiv) z is a function and $ran(z) = x$;

   (xv) z is transitive;

   (xvi) z is a successor ordinal;

   (xvii) z is a limit ordinal;

   (xviii) $z = \omega$;

   Solution: See lecture notes.

2. Deduce Pairing from the other axioms of $\mathbf{ZF}^-$.

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Solution: We write $0 = \emptyset$ and $1 = \{\emptyset\} = \mathcal{P}(\emptyset)$, $2 = \{0, 1\} = \mathcal{P}(1)$ so $0, 1, 2$ exist by **Emptyset** and **Powerset** (and these are well-defined by **Extensionality**).

Let

$$\phi(a, b, x, y) \equiv (x = 0 \to y = a) \wedge (x = 1 \to y = b).$$

Fix a, b.

Apply **Replacement** with ϕ and parameters a, b to 2 to obtain z such that $z = \{y : \exists x \in 2\ \phi(a, b, x, y)\}$. We should of course check that ϕ codes a function on 2: if $x \in 2$ then $x = 0$ or $x = 1$ and hence a (resp b) witness $\exists y\ \phi(a, b, x, y)$ and if $\phi(a, b, 0, y)$ then $y = a$ (and similar for b).

Thus z exists and we can verify that $z = \{a, b\}$: with $x = 0 \in 2$ (resp $x = 1$) we get $a \in z$ (resp $b \in z$) and if $t \in z$ then $\phi(a, b, 0, t)$ or $\phi(a, b, 1, t)$ so that $t = a$ or $t = b$.

3. Suppose ϕ is a formula of LST. Write an LST formula for '$z = \{t : \phi(t)\}$', relativize it to a class A and then write the abbreviation in terms of z and $\{. : .\}$.

   If A is a transitive class, ϕ, ψ are formulae and $x, d, z \in A$, what are the relativizations of $z = \{t \in x : \phi(t)\}$ and $z = \{y : \exists x \in d\ \phi(x, y)\}$ to A?

Solution:

$$z = \{t : \phi(t)\} \equiv \forall t\ (t \in z \leftrightarrow \phi(t)).$$

Thus

$$\begin{aligned}[z = \{t : \phi(t)\}]^A &\equiv \forall t \in A\ (t \in z \leftrightarrow \phi^A(t)) \\ &\leftrightarrow \forall t\ t \in z \cap A \leftrightarrow t \in A \wedge \phi^A(t) \\ &\equiv z \cap A = \left\{t \in A : \phi^A(t)\right\}\end{aligned}$$

If A is transitive and $b \in A$ then $b \cap A = b$ so

$$[z = \{t \in x : \phi(t)\}]^A \leftrightarrow z = \left\{t \in x : \phi^A(t)\right\}$$

and

$$\left[z = \left\{y : \exists x \in d\ \psi^A(x, y)\right\}\right]^A \leftrightarrow z = \left\{y : \exists x \in d\ y \in A \wedge \psi^A(x, y)\right\}$$

4. Show that the transitive set $(M, \in)$ constructed in question 6 below satisfies **Extensionality**, **Emptyset** and **Pairing**.

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Solution:

Extensionlity: Since M is transitive, **Extensionality** holds.

Emptyset: We note that $z = \emptyset^U \in M$ and M is transitive so that the formula $z = \emptyset \equiv \forall t \in z\ t \neq t$ is absolute for M, U. Hence z witnesses **Emptyset**^{M}.

Pairing: Let $x, y \in M$ and set $z = \{x, y\}^U \in M$ by the properties of M. Transitivity of M and thus absoluteness of $z = \{x, y\}$ gives that z witnesses **Pairing** (for x, y).

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Section B

5. Interpret the strict total order $(\mathbb{Q}, <)$ as a model of the LST, i.e. interpret the binary predicate $\in$ as $<$.

   Which axioms of **ZF** hold? Give brief proofs or counterexamples.

   Solution:

   (a) **Extensionality**: True. We decode the axiom: $\forall t \in \mathbb{Q}\,[(t < p \leftrightarrow t < q) \rightarrow p = q]$ and note that this is true for any $p, q \in \mathbb{Q}$ e.g. by totality and anti-reflexivity of $<$: if $p \neq q$ then one of $p < q$ or $q < p$ must hold and this t contradicts the premise.

   Emptyset: False. Take any $x \in \mathbb{Q}$ and let $y = x - 1 \in \mathbb{Q}$. Then $y < x$ so that $(\exists y\ (y \in x))^{(\mathbb{Q},<)}$.

   Pairing: False. We think about the meaning of $z = \{x, y\}$ in $(\mathbb{Q}, <)$. This says $x < z \wedge y < z \wedge \forall t < z\ \ [t = x \vee t = y]$. So let $x = y = 0$ and $z \in \mathbb{Q}$. We must have $z > 0$ by the first two conjuncts, but then $0 \neq z/2 < z$ contradicting the last conjunct.

   Union: True. Take $x \in \mathbb{Q}$ and let $z = x$. If $t < y < x$ then $t < x = z$ and if $t < z$ then $t < x$, so $(z = \bigcup x)^{(\mathbb{Q},<)}$.

   Powerset: False. First note that $(a \subseteq b)^{(\mathbb{Q},<)} \equiv a \leq b \equiv a < b \vee a = b$: Suppose $(a \subseteq b)^{(\mathbb{Q},b)}$, i.e. $\forall t\ t < a \rightarrow t < b$. If $a > b$ then then consider $t = b$ for a contradiction. Thus by totality we must have $a < b \vee a = b$. Conversely suppose $a \leq b$. Then by transitivity $t < a \rightarrow t < b$.

   Now take $x = 0 \in \mathbb{Q}$ and assume $(z = \mathcal{P}\,(x))^{(\mathbb{Q},<)}$: then $x \leq x$, i.e. $(x \subseteq x)^{(\mathbb{Q},<)}$ so $(x \in z)^{(\mathbb{Q},<)} \equiv x < z$ by assumption. But then let $y = \frac{x+z}{2} < z$ and $t = x \not\leq y$ gives $(y \in z)^{(\mathbb{Q},z)}$ but $(\neg y \subseteq x)^{(\mathbb{Q},<)}$.

   Separation: False. Applying **Separation** to $\phi(t) \equiv t \neq t$ and any $z \in \mathbb{Q}$ would produce $\emptyset^{(\mathbb{Q},<)}$ which does not exist.

   Replacement: False. Applying **Replacement** with $\phi(a, x, y) \equiv y = a$, $a = 0$ and $z = 0$ would yield $z = \{0\}^{(\mathbb{Q},<)}$ so that $0 < z$. But then $-1 < z$ gives a contradiction.

   Infinity: False. Since there is no emptyset, the condition $\emptyset inz$ (formally $\exists x \in z\ x = \emptyset$ cannot hold.

   Foundation: False. Let $x = 0$. We clearly have $-1 < x$ so $x \neq \emptyset^{(\mathbb{Q},<)}$. But for any $m < x$ we consider $t = m - 1$ to get $t < m \wedge t < x$ contradicting $\in$-minimality of m.

6. Work in $\mathbf{ZF}^-$.

   Show that there exists a transitive set M such that

$$\begin{aligned} \emptyset &\in M; \\ \forall x, y \in M \ \ \{x, y\} &\in M; \\ \forall x \in M \ \ |x| &\leq 2. \end{aligned}$$

   Carefully show that neither **Union** nor **Powerset** is satisfied in $(M, \in)$.

   Solution: First note that

$$F(x) = x \cup \{y \in \mathcal{P}(x) : |x| \leq 2\}$$

   (where $|x| \leq 2$ could be expressed by $x = \emptyset \vee \exists a, b \ \forall t \in y \ t = a \vee t = b$) is a class function by **Powerset**, **Separation** and the existence of finite unions (following from **Pairing** and **Union**).

   Other choices are possible (using **Replacement** instead of **Powerset** could be interesting).

   By recursion on $\omega + 1$ we define $G : \omega + 1 \to U$ such that

$$\begin{aligned} G_0 &= \emptyset \\ G_{n+1} &= F(G_n) \\ G_\omega &= \bigcup_{n \in \omega} G_n \end{aligned}$$

   and let $M = G_\omega$.

   Note that by construction $G_n \subseteq G_{n+1}$ and hence by induction $n < m \leq \omega \to G_n \subseteq G_m$. We will use that silently below.

   Transitivity: By induction on n we claim that each G_n is transitive (i.e. $y \in G_n \to y \subseteq G_n$): this is vacuously true for G_0. For the inductive step, let $y \in G_{n+1}$: if $y \in G_n$ then by IH $g \subseteq G_n \subseteq G_{n+1}$. Otherwise $y \in \mathcal{P}(G_n)$ so $y \subseteq G_n$.

   $\emptyset \in M$: Thus $M = G_\omega$ is transitive as the union of transitive sets (if $y \in G_\omega$ then $y \in G_n$ for some n then $y \subseteq G_n \subseteq G_\omega$).

   $x, y \in M \to \{x, y\} \in M$: Next $\emptyset \in G_1 \subseteq G_\omega = M$ since $\emptyset \subseteq \emptyset = G_0$ and $|\emptyset| \leq 2$ (e.g. take $a = b = \emptyset$).

   Now fix $x, y \in M = G_\omega = \bigcup_{n \in \omega} G_n$: find $n_x, n_y \in \omega$ such that $x \in G_{n_x}$ and $y \in G_{n_y}$ and let $n = \max\{n_x, n_y\}$ to get $x, y \in G_n$. Then $\{x, y\} \in G_{n+1}$ by construction.

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$x \in M \to |x| \leq 2$: Finally suppose $x \in M = G_\omega$. Let $n \in \omega$ be minimal such that $x \in G_n$. Note that $n \neq 0$ as $x \notin \emptyset$. Thus $n = m + 1$ and by minimality $x \notin G_m$. Therefore $x \subseteq G_m$ such that $|x| \leq 2$ by construction of $G_n = G_{m+1}$.

Union False. Note that $a = 0, b = 1, c = \{1\}, d = \{1\} \in M$ are four elements of M and they are distinct. Thus $u = \{a, b\}, v = \{c, d\}, x = \{u, v\} \in M$. Now suppose $z \in M$ with $(z = \bigcup x)^M$. Since M is transitive the formula $z = \bigcup x$ is absolute for M, U so that $z = \bigcup^U x = \{a, b, c, d\}$ and clearly $|z| > 2$ contradicting $z \in M$.

Powerset False. Note that $x = 2 = \{0, 1\} \in M$ (since as above $0, 1 \in M$). Suppose $z \in M$ with $(z = \mathcal{P}(x))^M$, i.e.

$$\forall t \in M \ \left[t \in z \leftrightarrow (t \subseteq 2)^M\right].$$

Since M is transitive the formula $t \subseteq x$ is absolute for M, U so that

$$\forall t \in M \ \left[t \in z \leftrightarrow (t \subseteq 2)^U\right].$$

But $0, 1, 2 \in M$ and $(0 \subseteq 2)^U, (1 \subseteq 2)^U, (2 \subseteq 2)^U$ giving that $|z| \geq 3 > 2$ a contradiction.

Separation True. Observe that if $z \subseteq x \in M$ then $z \in M$. If $x \in M$ then $x = \emptyset$ or $x = \{a, b\}$ for some a, b. If $x = \emptyset$ then $z = \emptyset \in M$. Otherwise by transitivity of M we have $a, b \in M$ and $z = \emptyset \vee z = \{a\} \vee z = \{b\} \vee z = x$. In all cases $z \in M$.

Suppose $x \in M$, $\phi(t, a_1, \ldots, a_n)$ is a formula of LST and $x_1, \ldots, x_n \in M$. In U apply **Separation** to get $z = \left\{t \in x : \phi^M(t, x_1, \ldots, x_n)\right\}$.

Note that $z \in M$ since $z \subseteq x$. By 3 and the construction of z (in U) it satifies

$$(z = \{t \in x : \phi(t, x_1, \ldots, x_n)\})^M.$$

Replacement True. Let $\phi(v_1, \ldots, v_n, x, y)$ for a formula of LST, $a_1, \ldots, a_n, d \in M$ and assume

$$\left[\forall x \in d \ \exists! y \ \phi(a_1, \ldots, a_n, x, y)\right]^M \equiv \forall x \in d \cap M \exists! y \in M \phi^M(a_1, \ldots, a_n, x, y).$$

Let

$$\psi(v_1, \ldots, v_n, x, y) \equiv y \in M \wedge \phi^M(v_1, \ldots, v_n, x, y).$$

We claim $\forall x \in d \ \exists! y \ \psi(a_1, \ldots, a_n, x, y)$: fix $x \in d$. Since $d \in M$ and M is transitive we have $x \in d \cap M$ so $\exists! y \in M \ \phi^M(a_1, \ldots, a_n, x, y)$. Pick $y \in M \subseteq U$ such that

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$\phi^M(a_1, \dots, a_n, x, y)$ to witness $\exists y\ \psi(a_1, \dots, a_n, x, y)$. If y_1, y_2 satisfy $\psi(a_1, \dots, x, y_i)$ then $y_1, y_2 \in M$ and $\phi^M(a_1, \dots, a_n, x, y)$ so that $y_1 = y_2$ by assumption.

Thus we can apply **Replacement** in U to obtain

$$z = \{y : \exists x \in d\ \psi(a_1, \dots, a_n, x, y)\}\,.$$

We note that $z \in M$ (because d in M, z contains at most two elements and by construction both of them are in M).

Thus by question3 we have

$$[z = \{y : \exists x \in d\ \phi(a_1, \dots, a_n, x, y)\}]^M\,.$$

7. Work in **ZF**.

   Show that if a is a non-empty transitive set then $\emptyset \in a$.

   Explain why the following sketch proof is not correct: Suppose $\emptyset \notin a$. By recursion on $n \in \omega$ find $x_n \in a$ such that $x_{n+1} \in x_n$ (in the inductive step we use that $x_n \in a$ means that $x_n \neq \emptyset$ and then transitivity of a to get $x_{n+1} \in a$ as well). Then $\{x_n : n \in \omega\}$ is a subset of a with no minimal element, contradicting **Foundation**.

   Solution: By **Foundation** find $m \in a$ such that $\forall t \in a\ t \notin m$. We claim $m = \emptyset$: if there is $t \in m \in a$ then by transitivity of a we have $t \in a$ contradicting minimality of $\in$.

   The sketch proof uses (countable dependent) **Choice** to choose the x_n.

8. Explain how you would formally express the statement of the following informal metatheorem (we will take it as a fact and use it freely):

   If $A \subseteq B$ are non-empty transitive classes satisfying (enough of) **ZF**, F is a class function that is absolute for A, B and $a \in A$ then the class function G given by recursion on On is absolute for A, B.

   Solution: First we think about the assumptions:

   - A is a non-empty transitive class means

     $$\tau_A \equiv (\exists x\ \in A\ x = x) \wedge \forall x \in A\ \forall t \in x\ t \in A$$

     and similarly for B.

   - $A \subseteq B \equiv \forall x \in A\ x \in B$.

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- 'satisfying (enough of) **ZF**': here we list the relevant axioms we care about and relativize these to A, i.e. $\phi_1^A \wedge \phi_2^A \cdots \wedge \phi_n^A$ where $\phi_1, \ldots, \phi_n$ are axioms of **ZF** (and similarly for B). The axioms we care about include those that we use in our discussion of ordinals and the proof of the Recursion Theorem (for F) on ordinals (this includes an instance of **Replacement** for a suitable formula ϕ given by F).

- F is really some formula $\phi(p)$ with one free variable p. We assert that $A \models \phi$ is a class function on $U = \{x : x = x\}$ (and the same for B) which becomes something like

$$\begin{aligned} &\forall p \in A\ (\phi^A(p) \to (p \text{ is an ordered pair})) \\ \wedge &\forall x \in A\ \exists y \in A\ \phi^A(\langle x, y\rangle) \\ \wedge &\forall x \in A\ \forall y_1, y_2 \in A\ (\phi^A(\langle x, y_1\rangle) \wedge \phi^A(\langle x, y_2\rangle) \to y_1 = y_2). \end{aligned}$$

  Note that since we may insist that A (and B) satisfy **Pairing** and **Extensionality** and are transitive so that being an ordered pair and $z = \langle x, y\rangle$ are absolute for A, U (and B, U resp.) we don't have to relativize 'is an ordered pair' and $\langle x, y\rangle$ and also don't need to worry too much what $\phi^A(\langle x, y\rangle)$ actually means (e.g. you could take it to mean $\exists p \in A\ p = \langle x, y\rangle \wedge \phi^A(p)$ or $\forall p \in A\ p = \langle x, y\rangle \to \phi^A(p)$ or the conjuction of the two).

- F is absolute for A, B should be (with F represented by ϕ as above)

$$\forall p \in A\ \phi^A(p) \leftrightarrow \phi^B(p)$$

  (noting that this says nothing about $p \in B \setminus A$).

- we of course need $a \in A$;

We then take a conjunction over all these as our premise and now think about the conclusion. Recall that in the Recursion Theorem we had a formula for G, namely

$$G_{F,a} \equiv \{\langle \alpha, y\rangle : \alpha \in \mathrm{On} \wedge [\exists g\ \ [\psi_{F,a}(\alpha, g) \wedge \langle \alpha, y\rangle \in g]]\}.$$

where

$$\begin{aligned} \psi_{F,a}(\alpha, g) \equiv &\alpha \in \mathrm{On} \wedge g \text{ is a function on } \alpha + 1 \wedge \\ &g(0) = a \wedge \\ &\forall \beta \in \alpha\ \ [g(\beta + 1) = F(g(\beta))] \wedge \\ &\forall \gamma \in \mathrm{Lim} \cap \alpha + 1\ \ \left[g(\gamma) = \bigcup \{g(\beta) : \beta \in \gamma\}\right] \end{aligned}$$

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Thus our conclusion is

$$\forall p \in A((p \in G_{F,a})^A \leftrightarrow (p \in G_{F,a})^B).$$

A couple of points are important: in ψ we need to relativize F to A and B respectively; we should also relativize On, $\alpha + 1$, 'being a function on ', 0, $\beta + 1$, Lim and $\bigcup$ to A (resp. B) and of course the $\langle .,. \rangle$ should also be relativized. Finally don't forget to bound all quantified variables to A (resp. B)! However, because of our insistence that A, B satisfy enough of **ZF** we have

- being an ordinal is absolute for A, U so the $(\alpha \in \mathrm{On})^A$ can be replaced by $\alpha \in \mathrm{On}$;
- ordered pairs exist and are absolute for A, U so all the $\langle .,. \rangle$ do not in fact need to be relativized; for the same reason we don't need to worry about all the $g(.) = .$ (which is really $\langle .,. \rangle \in g$) provided both the . in LHS and RHS exist (in A) so that we really should insist on $\mathbf{Union}^A$ in our premises;
- similarly $\alpha + 1$, being a function, 0, being a limit ordinal and $\bigcup$ are absolute for A, U so don't need relativization.

So after all these simplifications our $(p \in G_{F,a})^A$ becomes (with the remaining relativizations underlined)

$$\exists \alpha, y\ p = \langle \alpha, y \rangle \land \exists g \underline{\in A}\ \psi^A_{F,a}(\alpha, g) \land p \in g$$

(note that we don't need to insist that $\alpha, y \in A$ since $p \in A$, transitivity of A and $p = \langle \alpha, y \rangle$ gives this automatically) where

$$\begin{aligned}\psi^A_{F,a}(\alpha, g) \equiv & \alpha \in \mathrm{On} \land g \text{ is a function on } \alpha + 1 \land \\ & g(0) = a \land \\ & \forall \beta \in \alpha\ \left[g(\beta + 1) = \underline{F^A}(g(\beta))\right] \land \\ & \forall \gamma \in \mathrm{Lim} \cap \alpha + 1\ \left[g(\gamma) = \bigcup \{g(\beta) : \beta \in \gamma\}\right]\end{aligned}$$

Of course the same will work for B (since $p \in A$ implies $p \in B$).

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9. Work in **ZF**.

   We define 'x is an ordinal' to mean 'x is a transitive set well-ordered by $\in$'.

   (a) Show that 'x is an ordinal' is equivalent to 'x is a transitive set totally ordered by $\in$'.

   (b) Deduce that 'x is an ordinal' is absolute for non-empty transitive classes $A \subseteq B$ satisfying (enough of) **ZF**. Does this imply that $\mathrm{On}^A = \mathrm{On}^B$ (no full proof or counterexample expected)?

   Update: for a formula $\phi(x)$ coding a class C here I intended $C^A = \{x \in A : \phi^A(x)\}$ and not the (more?) reasonable $\{x : \phi^A(x)\}$. It turns out that for both interpretations there are counterexamples.

   (c) Assuming that it is consistent with $\mathbf{ZF}^-$ that there is x with $x = \{x\}$, show that the equivalence in part (a) requires **Foundation**.

   Update: As has been pointed out in some solutions, the existence of a set $x = \{x\}$ is not enough because we use **strict** well-orders (and total orders).

Solution:

(a) The forwards implication is trivial. For the backwards implication note that **Foundation** implies that $\in$ is well-founded.

(b) We note that 'x is a transitive set totally ordered by *in*' can be expressed as a Δ_0 formula and hence is absolute for A, B. Because A, B satisfy **Foundation** and the previous part we have for $x \in A$

$$\begin{aligned}[x \in \mathrm{On}]^A &\leftrightarrow [x \text{ is a transitive set totally ordered by } \in]^A \\ &\leftrightarrow [x \text{ is a transitive set totally ordered by } \in]^B \\ &\leftrightarrow [x \in \mathrm{On}]^B .\end{aligned}$$

(The first and third equivalence are the previous part, the middle one is the absoluteness noted above.)

This only shows that $\mathrm{On}^A \subseteq \mathrm{On}^B$ and that $\mathrm{On}^B \cap A = \mathrm{On}^A$, but we might have ordinals in B which are not in A.

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Section C

10. In your answers to question 1:

    Is the Δ_0-formula you gave still equivalent without assuming **ZF**? Which 'bit' of **ZF** is sufficient to give an equivalent Δ_0-formula.

    Is the concept (downward resp. upward) absolute for any classes $A \subseteq B$? For any transitive classes $A \subseteq B$? For any transitive classes $A \subseteq B$ satisfying enough of **ZF**?

11. For an arbitrary strict partial order $(P, <)$, find the order theoretic interpretations of $\emptyset$, $\{x, y\}$, $\bigcup x$, $\mathcal{P}(x)$ and conditions for their existence.

    Given a substructure $(Q, <)$ of $(P, <)$, can you find general conditions on Q (and P) are these absolute?

12. Show that in the theory of weak partial orders $(P, \leq)$ the concepts of minimum, maximum, greatest element and least element are not absolute.

    Note that in the theory of lattices $(L, \leq, \wedge, \vee)$ where $\wedge$ and $\vee$ are binary functions the concepts of minimum and maximum are absolute (by definition of what a substructure is).

13. In your favourite area of mathematics, think about different ways to axiomatize the theory and how these may give rise to different notions of 'substructure' and 'absoluteness' of important concepts.

14. In $(M, \in)$ as constructed in question 6, which of the other axioms of **ZFC** hold? Does it depend on the precise formulation of the axiom?

15. Prove the theorem in question 8.

16. Suppose $V \models$ **ZF**.

    Suppose $F : V \to V$ is a bijective class function.

    Define the relation E by xEy if and only if $x \in F^{-1}(y)$ and consider the structure (V, E) and we write ϕ^E for the formula ϕ where $\in$ is replaced by E (so we interpret ϕ in the structure (V, E)).

    (a) Show that $(V, E) \models$ **Extensionality**.

    (b) Compute $\emptyset^E$, $\{x, y\}^E$, $(\bigcup x)^E$, $\mathcal{P}(x)^E$, ω^E.

    (c) Show that (V, E) satisfies $\mathbf{ZF}^-$.

    (d) Find a concrete F so that there is x such that $(V, E) \models x = \{x\}$.

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