Axiomatic Set Theory

Sheet 2 — TT21

Section A

1. Ensure that you can show the facts about ordinals that we use (section 5 in the Lecture Notes). **Solution: lecture notes**

2. Complete the proof that $(V, \in) \models \mathbf{ZF}$, i.e. make sure you can prove the axioms which were skipped in lectures - this will probably be **Union** and **Infinity**. **Solution: lecture notes**

3. Work in **BST** = **ZF-Powerset**.

   Recall that for sets a, b we write $a^b = \{f : f : b \to a\}$ and we say that 'a is finite' if and only if there is $n \in \omega$ and a surjection $f : n \to a$.

   For a set x we write

$$
\begin{aligned}
[x]^{<\omega} &= \{y : y \subseteq x \wedge y \text{ is finite}\} \\
x^{<\omega} &= x^{[\omega]^{<\omega}} = \{f : y \to x : y \in [\omega]^{<\omega}\} \\
x^{[\omega]} &= \bigcup_{n \in \omega} x^n = \{f : \exists n \in \omega \; f : n \to x\} .
\end{aligned}
$$

   (a) For sets a, b define $a \times b$ and show that this is absolute for non-empty transitive classes satisfying (enough of) **BST**.

   (b) Show that $x^{[\omega]}$, $[x]^{<\omega}$ and $x^{<\omega}$ exist (as sets).

   (c) Note that $x^{[\omega]} \subseteq x^{<\omega}$ and that each $f \in x^{<\omega}$ is a restriction of some (non-unique) $\hat{f} \in x^{<\omega}$.

   (d) Show that $y \in [x]^{<\omega}$ if and only if there is $f \in x^{<\omega}$ such that $ran(f) = y$.

4. Convince yourself that you can prove that V_ω satisfies **ZF-Infinity**.

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Section B

5. Show that $x^{[\omega]}$ defined in question 3 is absolute for non-empty transitive classes satisfying (enough of) **BST** (and note but don't prove that similar proofs give absoluteness for $[x]^{<\omega}$ and $x^{<\omega}$).

6. Work in $\mathbf{ZF}^-$. Suppose $F : \mathrm{On} \to \mathrm{On}$ is a class function such that

$$F \text{ is strictly increasing, i.e. } \alpha < \beta \to F(\alpha) < F(\beta)$$
$$F \text{ is continuous, i.e. } \forall \gamma \in \mathrm{Lim}\ F(\gamma) = \bigcup_{\alpha<\gamma} F(\alpha)$$

   Prove that F has arbitrarily large fixed points, i.e. for all $\alpha \in \mathrm{On}$ there is $\beta \in \mathrm{On}$ such that $\alpha < \beta$ and $F(\beta) = \beta$.

   What is the smallest non-zero fixed point of the class function $F : \mathrm{On} \to \mathrm{On}; F(x) = \omega.x$?

7. Work in $\mathbf{ZF}^-$.
   (a) Prove that **Foundation** is equivalent to $\forall x\ x \in V$ (which we may write as $U = V$).
   (b) What does $V^V = V$ mean?
   (c) Show that $V^V = V$.

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8. Work in $\mathbf{ZF}^-$.

   Suppose R is a (class) relation on a class A.

   Define the 'transitive closure of R', $R^\star$, by

$$aR^\star b \equiv \exists n \in \omega\ \exists f : n+2 \to A\ f(0) = a \wedge f(n+1) = b \wedge \forall k \in n+1\ f(k)Rf(k+1).$$

   (a) Informally (no need to use axioms here) show that $R^\star$ is transitive, i.e. that $aR^\star b \wedge bR^\star c \to aR^\star c$.

   (b) Show that for any (class) relation S that is transitive (wrt R) and contains R (in the sense that $aRb \to aSb$) S contains $R^\star$.

   (c) Show that if $R = \in$ (formally $aRb \equiv a \in b$ so that as a class $R = \{\langle a, b\rangle : a \in b\}$) then for every set x the class $TC(x) = \{y : y \in^\star x\}$ is the smallest transitive (wrt $\in$) class containing x, i.e. a transitive class containing x (as a subset) such that whenever C is a transitive class and $x \subseteq C$ then $TC(x) \subseteq C$.

   (d) Show that we can define the class function TC by parametrized recursion (on $\omega+1$) as $G(x, \omega)$ where

$$\begin{aligned} G(x,0) &= x \\ \forall n \in \omega\ G(x, n+1) &= G(x,n) \cup \bigcup G(x,n) \\ G(x,\omega) &= \bigcup_{n\in\omega} G(x,n). \end{aligned}$$

   (e) Show that the class function TC is absolute for transitive classes satisfying (enough of) $\mathbf{ZF}$.

   (f) Show that in $\mathbf{ZF}$ we can define $TC(x)$ as the smallest transitive set containing x as a subset and argue why we do not use this definition in $\mathbf{ZF}^-$.

9. Work in $\mathbf{ZF}$.

   The class $H_\omega = \{x : TC(x) \text{ is finite}\}$ is the class of hereditarily finite sets.

   (a) Prove that $H_\omega = V_\omega$ (and hence that H_ω is a set).

   (b) Prove that $(V_\omega, \in) \models \neg\mathbf{Infinity}$.

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Section C

10. This question extends question 3 and 5.

    What is the problem with trying to naively define (in **BST**) 'a is finite' by recursion saying that $\emptyset$ is finite and that if a is finite and there is t such that $a = a' \cup \{t\}$ then a is finite.

    Is it possible to fix this if we want to define the finite subsets of x?

    Explore different notion of 'a is finite' (which are equivalent under **ZFC**) and what happens to the definitions and results in questions 3 if we start dropping axioms.

11. This question extends question 6.

    Formulate and prove an analogue for question 6 for class functions $F : U \to U$.

    Can you find an analogue so that the existence of arbitrarily large common fixed points for finitely many class functions $F_0, \ldots, F_n : U \to U$ is guaranteed?

    If we consider infinitely many functions $\{f_i : i \in I\}$ on some V_α (or generally some transitive set v) what extra assumptions do we need so that there are arbitrarily large common fixed points? Can we do this with class functions?

12. This extends question 8.

    We say that a relation R is set-like if and only if $\text{pred}_R(x) = \{y : yRx\}$ is a set.

    We say that a relation R is well-founded if and only if every non-empty set has an R-minimal element, i.e. $\forall x \ [x \neq \emptyset \to \exists m \in x \ \forall t \in x \ \neg tRm]$.

    (a) Show that if R is set-like and well-founded then so is $R^\star$ and we can define $TC_R(x)$ as the smallest set containing x as a subset and being downwards closed under R.

    (b) Prove the generalized recursion theorem: Suppose R is a well-founded, set-like relation on a class A and B is a class.

    If $F : A \times U \to B$ is a class function then there is a (essentially unique) class function $G : A \to B$ such that for all

    $$\forall a \in A \ G(a) = F(a, G|_{\text{pred}(a)}).$$

    (c) Deduce the usual Recursion Theorem on On from the Generalized Recursion Theorem.

    (d) Show that if R and F are absolute for non-empty transitive classes $A \subseteq B$ satisfying (enough of) $\mathbf{ZF}^-$, then G is absolute for A, B.

    (e) What happens to the results in question 8 if we work in weaker set theories than $\mathbf{ZF}^-$?

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