Axiomatic Set Theory

Sheet 2 — TT21

Section A

1. Ensure that you can show the facts about ordinals that we use (section 5 in the Lecture Notes). **Solution: lecture notes**

2. Complete the proof that $(V, \in) \models \mathbf{ZF}$, i.e. make sure you can prove the axioms which were skipped in lectures - this will probably be **Union** and **Infinity**. **Solution: lecture notes**

3. Work in **BST** = **ZF-Powerset**.

   Recall that for sets a, b we write $a^b = \{f : f : b \to a\}$ and we say that 'a is finite' if and only if there is $n \in \omega$ and a surjection $f : n \to a$.

   For a set x we write

$$
\begin{aligned}
[x]^{<\omega} &= \{y : y \subseteq x \wedge y \text{ is finite}\} \\
x^{<\omega} &= x^{[\omega]^{<\omega}} = \{f : y \to x : y \in [\omega]^{<\omega}\} \\
x^{[\omega]} &= \bigcup_{n \in \omega} x^n = \{f : \exists n \in \omega \ f : n \to x\}.
\end{aligned}
$$

   (a) For sets a, b define $a \times b$ and show that this is absolute for non-empty transitive classes satisfying (enough of) **BST**.

   (b) Show that $x^{[\omega]}$, $[x]^{<\omega}$ and $x^{<\omega}$ exist (as sets).

   (c) Note that $x^{[\omega]} \subseteq x^{<\omega}$ and that each $f \in x^{<\omega}$ is a restriction of some (non-unique) $\hat{f} \in x^{<\omega}$.

   (d) Show that $y \in [x]^{<\omega}$ if and only if there is $f \in x^{<\omega}$ such that $ran(f) = y$.

   Solution:

   (a) Assume **BST**. We define $a \times b = \bigcup_{u \in a} \bigcup_{v \in b} \{\langle u, v \rangle\}$ and argue that this exists as a set. Firstly consider the formula (with parameter u) $\phi(u, v, y) \equiv y = \{\langle u, v \rangle\}$. By **BST** this is a codes a function $v \mapsto y$ on b so $\{\{\langle u, v \rangle\} : v \in b\}$ exists by **Replacement** and thus by **Union** so does $\bigcup \{\{\langle u, v \rangle\} : v \in b\}$.

   Now we take $\psi(u, z) \equiv z = \bigcup \{\{\langle u, v \rangle\} : v \in b\}$ and note that this codes a function $u \mapsto z$ on a, so that as above $a \times b$ exists.

   For absoluteness note that everything can be phrased in a Δ_0 way, so is absolute for non-empty transitive classes (provided they satisfy **BST**).

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(b) We first show that for each $n \in \omega$ x^n is a set (i.e. exists):

For $n = \emptyset$ we have $x^0 = \{\emptyset\}$ exists.

Now suppose x^n exists. Then by part (a) we have that $x^n \times x$ exists and we can code the function on $x^n \times x$ such that $(g, y) \mapsto g \cup \{\langle n+1, y\rangle\}$ by a formula so that by **Replacement** x^{n+1} exists. Thus by induction on $n \in \omega$ each x^n exists.

By **Replacement** and **Union** $x^{[\omega]}$ exists.

Next

$$[x]^{<\omega} = \left\{ran(f) : f \in x^{[\omega]}\right\}$$

exists by **Replacement**.

Finally we code a function $x^{[w]} \times [x]^{<\omega} \to x^{<\omega}; \langle f, y\rangle \mapsto f|_y$ and apply **Replacement** (noting that $x^{[w]} \times [x]^{<\omega}$ exists) to get that $x^{<\omega}$ exists.

We should argue that it is onto: assume $y \subseteq \omega$ is finite and $f : y \to x$. If $y = \emptyset$ then $\emptyset = f : 0 \to x$. Otherwise let $M = \max y$, $m = \min y$ and define $\hat{f} : M \to x$ by $\hat{f}(t) = f(t)$ if $t \in y$ and $\hat{f}(t) = f(m)$ if $t \notin y$. Then $f = \hat{f}|_y$.

(c) If $n \in \omega$ then $n \in [\omega]^{<\omega}$ (via the identity), so $x^{[\omega]} \subseteq x^{<\omega}$. The other part was done above.

(d) This follows from the previous part: if $y \subseteq x$ and y is finite then we can find $f : n \to y$ surjective, $n \in \omega$ and this witnesses the RHS (since $f \in x^{[\omega]} \subseteq x^{<\omega}$). Conversely if $f \in x^{<\omega}$ with $y = ran(f)$ then extend f to $\hat{f} \in x^{[\omega]}$ as described above and note that $ran(f) = ran(\hat{f})$ so that $\hat{f}$ (and $M + 1$) witnesses y is finite.

4. Convince yourself that you can prove that V_ω satisfies **ZF-Infinity**.

Solution: The proof is as for V (except without **Infinity**).

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Section B

5. Show that $x^{[\omega]}$ defined in question 3 is absolute for non-empty transitive classes satisfying (enough of) **BST** (and note but don't prove that similar proofs give absoluteness for $[x]^{<\omega}$ and $x^{<\omega}$).

   Solution: Assume A is a non-empty transitive class satisfying **BST**.

   Because $\omega^A = \omega$ and 'being a surjective function $b \to a$' is Δ_0 so absolute for A, U we have $(a^n)^A = a^n \cap A$ and it is enough to show that

   $$\forall n \in \omega \; a^n \cap A = a^n.$$

   Assume not and let $n \in \omega$ be minimal such that $a^n \cap A \neq a^n$.

   If $n = 0$ then $a^n = \{f : \emptyset \to a\} = \{\emptyset\}$ and $\emptyset \in A$ (by **BST**) a contradiction. Thus $n > 0$, say $n = m + 1$.

   Since we have $a^n \cap A \subseteq a^n$ we can find $f \in a^n \setminus A$. Then $f|_m \in a^m = a^m \cap A$ by minimality of n and hence $f|_m \in A$. But also $m \in A$ and $f(m) \in a \in A$ and A is transitive so $f(m) \in A$. By **BST** (and absoluteness of the RHS for A, U) we get $f = f|_m \cup \{\langle m, f(m)\rangle\} \in A$ a contradiction.

6. Work in $\mathbf{ZF}^-$. Suppose $F : \mathrm{On} \to \mathrm{On}$ is a class function such that

   $$F \text{ is strictly increasing, i.e. } \alpha < \beta \to F(\alpha) < F(\beta)$$
   $$F \text{ is continuous, i.e. } \forall \gamma \in \mathrm{Lim} \; F(\gamma) = \bigcup_{\alpha<\gamma} F(\alpha)$$

   Prove that F has arbitrarily large fixed points, i.e. for all $\alpha \in \mathrm{On}$ there is $\beta \in \mathrm{On}$ such that $\alpha < \beta$ and $F(\beta) = \beta$.

   What is the smallest non-zero fixed point of the class function $F : \mathrm{On} \to \mathrm{On}; F(x) = \omega.x$?

   Solution: Note that since F is increasing by induction on α we have $\forall \alpha \in \mathrm{On} \; \alpha \leq F(\alpha)$.

   Also since F is continuous (and increasing) we have for any $s \subseteq \mathrm{On}$ that $\bigcup F[s] = F(\bigcup s)$ (here $F[s] = \{F(\alpha) : \alpha \in s\}$): for $\alpha \in s$ we have $\alpha \leq \bigcup s$ and hence $F(\alpha) \leq F(\bigcup s)$; conversely let $\beta = \bigcup s$: if $\beta \in s$ then $F(\beta) \subseteq \bigcup F[s]$; otherwise $\beta \in \mathrm{Lim}$ and for $\alpha < \beta$ there is $\alpha' \in s$ with $\alpha < \alpha'$ (since $\beta = \sup s$). Thus for $\alpha < \beta$ we have $F(\alpha) \subseteq F(\alpha') \subseteq \bigcup F[s]$.

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By recursion on $\omega+1$ with parameter α we define $G : \mathrm{On} \times \omega + 1 \to \mathrm{On}$

$$\begin{aligned}
\alpha_0 &= G(\alpha, 0) = \alpha + 1; \\
\alpha_{n+1} &= G(\alpha, n+1) = F(G(\alpha, n)) = F(\alpha_n); \\
\beta = G(\alpha, \omega) &= \bigcup_{n\in\omega} G(\alpha, n) = \bigcup \{\alpha_n : n \in \omega\}.
\end{aligned}$$

Let $s = \{\alpha_n : n \in \omega\}$ so that $\beta = \bigcup s$ and

$$F(\beta) = \bigcup F[s] = \bigcup \{F(\alpha_n) : n \in \omega\} = \bigcup \{\alpha_{n+1} : n \in \omega\} = \bigcup_{k=1}^{\infty} \alpha_k.$$

Note that the sequence α_n is non-decreasing (by the first comment) and thus $\bigcup_{k=1}^{\infty} \alpha_k = \bigcup_{k=0}^{\infty} \alpha_k = \beta$ giving $F(\beta) = \beta$ as required.

Finally $\alpha < \alpha + 1 = \alpha_0 \leq \beta$.

We also observe that the β we find is minimal: suppose $\alpha < \delta$ with $F(\delta) = \delta$. Either $\alpha + 1 = \alpha_0 = \delta$ in which case for all $n \in \omega$, $\alpha_n = \delta$ (by induction on n) and thus $\delta = \beta$ or $\alpha_0 < \delta$ in which case inductively for all $n \in \omega$ $\alpha_n < \delta$ and so $\beta \leq \delta$.

To find the smallest non-zero fixed point, we apply the above process: we let $\alpha_0 = 1$ and $\alpha_{n+1} = \omega.\alpha_n$ which gives $\alpha_n = \omega^n$. Then $\beta = \bigcup_n \omega^n = \omega^\omega$.

Comment: In the second paragraph I prove that $\bigcup F[s] = F(\bigcup s)$.

If you don't do this, then you need to distinguish between the cases $F(\alpha_0) = \alpha_0$ (so α_0 is the required fixed point and we are done) and $\alpha_0 \in F(\alpha_0)$ in which case β is a limit ordinal so we can use continuity of F as stated for β.

Because I prove abstractly that the β obtained is minimal and a fixed point, I don't actually need to do the computation for ω^ω any more.

7. Work in $\mathbf{ZF}^-$.
   (a) Prove that **Foundation** is equivalent to $\forall x\, x \in V$ (which we may write as $U = V$).
   (b) What does $V^V = V$ mean?
   (c) Show that $V^V = V$.

Solution:

(a) If $\forall x\, x \in V$ then since $\mathbf{Foundation}^V$ we obtain **Foundation**.

Conversely, assume **Foundation** and that $\neg\forall x\, x \in V$. Pick x_0 such that $x_0 \notin V$ and let $s = \{t \in TC(\{x_0\}) : t \notin V\} \ni x_0$. Apply **Foundation** to s to obtain $m \in s$ such that $\forall x \in m\; x \in V$ (since $TC(\{x_0\})$ is transitive $x \in m \to x \in TC(\{x_0\})$).

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But then $m \subseteq V$ and so in particular $m \subseteq V_\alpha$ for some α (formally we use **Replacement** with the function that sends $x \in V$ to the minimal $\alpha_x \in \mathrm{On}$ such that $x \in V_{\alpha_x}$ and **Union** to take $\alpha = \sup\{\alpha_x : x \in m\}$. But then $m \in V_{\alpha+1} \subseteq V$ a contradiction.

(b) For two classes A, B we have $A = B \equiv \forall x\ (x \in A \leftrightarrow x \in B)$.

But $x \in V$ means $\exists \alpha \in \mathrm{On}\ \phi(\langle \alpha, x\rangle)$ (where $\phi(\langle \alpha, x\rangle)$ expresses $x \in V_\alpha$) for the formula ϕ (for G) we get from the Recursion Theorem.

Then $(x \in V)^V$ means $\exists \alpha \in \mathrm{On}^V\ \phi^V(\langle \alpha, x\rangle^V)$ (it makes sense to argue that $(x \in V)^V$ also implies $x \in V$ but in this case it will be automatic).

(c) We observe that $\forall x \in V\ \mathcal{P}^V(x) = \mathcal{P}(x)$, $\emptyset^V = \emptyset \in V$ and appeal to absoluteness of recursion (for transitive non-empty classes satisfying enough of **ZF**, Sheet 1, Q7) to get $\forall \alpha \in \mathrm{On}^V\ V_\alpha^V = V_\alpha$. Then we note that $\mathrm{On}^V = \mathrm{On}$ so that $\forall \alpha \in \mathrm{On}\ V_\alpha^V = V_\alpha$ and hence $V^V = V$.

Alternatively (a clever solution by a student) we observe that (in $\mathbf{ZF}^-$) we have shown **Foundation** $\leftrightarrow \forall x\ x \in V$.

Since $V \models \mathbf{ZF}^-$, we relativize this proof to V to get a proof $\mathbf{Foundation}^V \leftrightarrow (\forall x\ x \in V)^V$ and since we can prove $\mathbf{Foundation}^V$ we get $(\forall x\ x \in V)^V \equiv \forall x \in V\ x \in V^V$ so $V \subseteq V^V$.

Since $V \subseteq U$ we must have $V^V \subseteq V$ as well.

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8. Work in $\mathbf{ZF}^-$.

   Suppose R is a (class) relation on a class A.

   Define the 'transitive closure of R', $R^\star$, by

   $aR^\star b \equiv \exists n \in \omega\ \exists f : n+2 \to A\ f(0) = a \wedge f(n+1) = b \wedge \forall k \in n+1\ f(k)Rf(k+1).$

   (a) Informally (no need to use axioms here) show that $R^\star$ is transitive, i.e. that $aR^\star b \wedge bR^\star c \to aR^\star c$.

   (b) Show that for any (class) relation S that is transitive (wrt R) and contains R (in the sense that $aRb \to aSb$) S contains $R^\star$.

   (c) Show that if $R = \in$ (formally $aRb \equiv a \in b$ so that as a class $R = \{\langle a, b\rangle : a \in b\}$) then for every set x the class $TC(x) = \{y : y \in^\star x\}$ is the smallest transitive (wrt $\in$) class containing x, i.e. a transitive class containing x (as a subset) such that whenever C is a transitive class and $x \subseteq C$ then $TC(x) \subseteq C$.

   (d) Show that we can define the class function TC by parametrized recursion (on $\omega+1$) as $G(x, \omega)$ where

   $$\begin{aligned} G(x,0) &= x \\ \forall n \in \omega\ G(x, n+1) &= G(x,n) \cup \bigcup G(x,n) \\ G(x,\omega) &= \bigcup_{n\in\omega} G(x,n). \end{aligned}$$

   (e) Show that the class function TC is absolute for transitive classes satisfying (enough of) $\mathbf{ZF}$.

   (f) Show that in $\mathbf{ZF}$ we can define $TC(x)$ as the smallest transitive set containing x as a subset and argue why we do not use this definition in $\mathbf{ZF}^-$.

Solution:

(a) Suppose $aR^\star b, bR^\star c$. Find $n, m \in \omega, f : n+2 \to A, g : m+2 \to A$ witnessing this and construct $h : n+m+2 \to A$ by $h(k) = f(k)$ if $k \leq n+1$ and $h(k) = g(k-n-1)$ if $n+1 < k < n+m+2$ to see that $aR^\star c$.

(b) Suppose there are $a, b \in A$ such that $aR^\star b$ but $\neg aSb$. Pick $n \in \omega$ minimal such that there are such a, b for which n is a witness that $aR^\star b$ and let $f : n+2 \to A$ be a witnessing function.

   If $n = 0$ then $a = f(0)Rf(1) = b$ contradicting $R \subseteq S$.

   If $n = m+1$ then restricting f to $m+2$ gives $aR^\star f(n)$ and by minimality of n we must have $aSf(n)$. But also $f(n)Rb$ so that $f(n)Sb$ (since $R \subseteq S$) and transitivity of S gives aSb, a contradiction.

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(c) Firstly $TC(x)$ is a transitive class: if $z \in y \in TC(x)$ then $z \in y \in^\star x$ so $z \in^\star y \in^\star x$ so by (a) $z \in^\star x$.

Next it contains x as a subset: if $y \in x$ then $y \in^\star x$ so $y \in TC(x)$.

Finally assume that C is a transitive class and $x \subseteq z$. We can argue as in the previous part to see that $TC(x) \subseteq C$: otherwise there is a minimal n such that there are $x_0, \ldots, x_{n+1} = x$ with $x_k \in x_{k+1}$ and $x_0 \notin C$. Since $x \subseteq C$ we must have $n > 0$. By minimality of n we get that $x_1 \in C$ and $x_0 \in x_1$ so that transitivity of C gives $x_0 \in C$ a contradiction.

(d) We check that $G(x, \omega)$ is transitive: if $y \in G(x, \omega)$ then $y \in G(x, n)$ for some n and thus $y \subseteq \bigcup G(x, n) \subseteq G(x, n+1) \subseteq G(x, \omega)$.

Also $x \subseteq G(x, 0) \subseteq G(x, \omega)$.

So by the previous part $TC(x) \subseteq G(x, \omega)$ (and hence $TC(x)$ is a set). Suppose $TC(x) \neq G(x, \omega) = \bigcup_{n \in \omega} G(x, n)$. Find $n \in \omega$ minimal such that $G(x, n) \not\subseteq TC(x)$. If $n = 0$ then $G(x, n) = x \subseteq TC(x)$ a contradiction. Otherwise $n = m + 1$ and $G(x, n) = G(x, m) \cup \bigcup G(x, m)$. By minimality $G(x, m) \subseteq TC(x)$ and by transitivity of $TC(x)$ we then get $\bigcup G(x, m) \subseteq TC(x)$ so that $G(x, m) \cup \bigcup G(x, m) \subseteq TC(x)$ a contradiction.

(e) all the concepts in the previous part are absolute (for transitive non-empty classes satisfying enough of **ZF**) and hence the result of the recursion is also absolute.

(f) In **ZF**: let $x \in V$ (recall that **ZF** implies that $\forall x\, x \in V$) and find $rk_V(x) = \alpha_x \in \mathrm{On}$ minimal such that $x \subseteq V_{\alpha_x}$ (this exists since there is β with $x \in V_\beta$ and transitivity then gives $x \subseteq V_\beta$). Note that V_{α_x} is a transitive set containing x as a subset and thus we could define $TC(x) = \bigcap \{z \in \mathcal{P}(V_{\alpha_x}) : z \text{ is transitive and } x \subseteq z\}$ (this makes sense as it is an intersection over a non-empty set).

In **ZF**$^-$: we do not have a ready-made transitive set that contains x as a subset (of course we could use (d) to construct one, but then we might as well use (d) to define $TC(x)$) and we cannot take the intersection over all transitive classes that contain x (we are only allowed finitely many classes outside in the theory).

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9. Work in **ZF**.

   The class $H_\omega = \{x : TC(x) \text{ is finite}\}$ is the class of hereditarily finite sets.

   (a) Prove that $H_\omega = V_\omega$ (and hence that H_ω is a set).

   (b) Prove that $(V_\omega, \in) \models \neg$**Infinity**.

   Solution:

   (a) If $x \in V_\omega$ then $x \in V_n$ for some $n \in \omega$ and so $x \subseteq V_n$ and V_n is transitive. Hence $TC(x) \subseteq V_n$ and V_n is finite (it has size 2^{n-1}) so $TC(x)$ is finite and so $x \in H_\omega$.

   Conversely assume $H_\omega \not\subseteq V_\omega$. Firstly note that since $y \in x \to TC(y) \subseteq TC(x)$ we have that H_ω is transitive.

   Now pick $x \in H_\omega \setminus V_\omega$. Noting that $TC(TC(\{x\})) = TC(\{x\}) = \{x\} \cup TC(x)$ is finite we see that $S = \{y \in TC(\{x\}) : y \notin V_\omega\} \subseteq H_\omega \setminus V_\omega$ and contains x. We can thus find $\in$-minimal m in S. Since $m \in H_\omega$ we have $m \subseteq H_\omega$ (by transitivity) and thus $m \subseteq V_\omega$ by $\in$-minimality of m in S. For each $t \in m$ let $n_t \in \omega$ be minimal such that $t \in V_{n_t}$. Since m is finite ($m \subseteq TC(m)$ finite) we have $N = \max_{t \in m} n_t \in \omega$ and $m \subseteq V_N$ so $m \in V_{N+1} \subseteq V_\omega$, a contradiction.

   (b) Note that $H_\omega = V_\omega \models$ **ZF-Infinity** (see Q4). If $(V_\omega, \in) \models$ **Infinity**, let $w \in V_\omega$ such that $Ind(w)$, i.e. $\emptyset \in w \wedge \forall t \in w\ t+1 \in w$. By induction on n we then get $\omega \subseteq w$ and hence $\omega = TC(\omega) \subseteq TC(w)$ contradicting finiteness of $TC(w)$.

   Technically we should argue that if $f : n+1 \to \omega$ is surjective then composing $f|_n$ with a surjection $\omega \setminus \{f(n)\} \to \omega$ gives a surjection from n to ω. Now if $TC(w)$ were finite there is a surjection from some n onto $TC(w)$. Compose with a surjection $TC(w) \to \omega$ (send elements of $TC(w) \setminus \omega$ to 0 and be the identity on the others) to get a surjection n onto ω and then take the smallest such n. It must be 0 by the above argument but then $\omega = \emptyset$ a contradiction.

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Section C

10. This question extends question 3 and 5.

    What is the problem with trying to naively define (in **BST**) 'a is finite' by recursion saying that $\emptyset$ is finite and that if a is finite and there is t such that $a = a' \cup \{t\}$ then a is finite.

    Is it possible to fix this if we want to define the finite subsets of x?

    Explore different notion of 'a is finite' (which are equivalent under **ZFC**) and what happens to the definitions and results in questions 3 if we start dropping axioms.

    Solution: The problem is that this is a recursive definition. If you try to get around this by some sort recursion on ω then you have the problem that there are class many t to consider. E.g. if you set $F_0 = \{\emptyset\}$ and $F_{n+1} = F_n \cup \{a' \cup \{t\} : a' \in F_n, t \in U\}$ then already F_1 is a class and thus we cannot apply the recursion theorem.

    However, if we want to define the finite subsets of x we can do recursion on $\omega + 1$ (parametrized by x) to define

    $$F_0(x) = \{\emptyset\}\,;\ F_{n+1}(x) = F_n(x) \cup \{a' \cup \{t\} : a' \in F_n(x), t \in x\}\,.$$

    Note that if F_n is a set then so is F_{n+1} by **Replacement** on $F_n(x) \times x$) and set $[x]^{<\omega} = \bigcup_{n\in\omega} F_n(x)$.

    Two typical notions to explore are 'every injection is a surjection' and 'every surjection is an injection'. (or look up 'Dedekind finite'.)

11. This question extends question 6.

    Formulate and prove an analogue for question 6 for class functions $F : U \to U$.

    Can you find an analogue so that the existence of arbitrarily large common fixed points for finitely many class functions $F_0, \ldots, F_n : U \to U$ is guaranteed?

    If we consider infinitely many functions $\{f_i : i \in I\}$ on some V_α (or generally some transitive set v) what extra assumptions do we need so that there are arbitrarily large common fixed points? Can we do this with class functions?

    Solution: There are multiple ways to make progress:

    Invariant sets First we will drop all conditions on F (except it still should be a class function) and ask for an invariant (large) set, i.e. for every $y \in U$ there is $x \in U$ such that $y \subseteq x$ and $F[x] = \{F(t) : t \in x\} \subseteq x$.

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We define this by recursion on $\omega + 1$ (with parameter y) as $x_0 = y$ and $x_{n+1} = x_n \cup \{F(t) : t \in x_n\}$ and note that $x = x_\omega = \bigcup_{n\in\omega} x_n$ works.

We get common fixed points of multiple functions $F_0, \ldots, F_k$ by taking a bijection $c : \omega \to \omega \times k+1$, $x_0 = y$ and $x_{n+1} = x_n \cup F_{c_2(n)}[x_n]$ ($c_2(n) = \pi_2(c(n))$, so the second co-ordinate) and $x = x_\omega$. This works because each $c^{-1}(\pi_2^{-1}(j))$ is unbounded in ω (i.e. unboundedly often we close under F_j).

If we have a set $\{f_i : i \in I\}$ of functions on some V_α we can (using **Choice**) find some limit ordinal γ such that there is a bijection $c : \gamma \to \omega \times I$ with each $c^{-1}(\pi_2^{-1}(i))$ unbounded in γ (well-order I and use the lexicographic well-order on $\omega \times I$). We then use $x_{\alpha+1} = x_\alpha \cup f_{c_2(\alpha)}[x_\alpha]$ and take unions at limits. Our candidate fixed point is then $x = \bigcup_{\beta\in\gamma} x_\beta$.

The key result we need is that if $x \in V_\alpha$ then $f_i[x] \in V_\alpha$ (so we can carry out the successor steps in V_α) and that unions of at most γ many elements of V_α are in V_α (so that we can carry out the limit step). **ZF-Powerset** together with $\gamma \in V_\alpha$ would ensure both happen.

For class functions we cannot do this because it makes no sense to talk about infinitely many formulae at once - we wouldn't even be able to state in LST what it means to be invariant under infinitely many class functions.

Fixed points Alternatively we can demand that we get a proper fixed point, i.e. $x \in U$ such that $F(x) = x$.

In this case we want F to be continuous in the sense that $F(\bigcup x) = \bigcup\{F(y) : y \in x\}$ and non-decreasing in the sense that $x \subseteq F(x)$.

As before we do $x_0 = x$, for $n \in \omega$ we take $x_{n+1} = F(x_n)$ and we use $x_\omega = \bigcup_{n\in\omega} x_n$ and claim that $F(x_\omega) = x_\omega$:

By continuity
$$F(x_\omega) = F(\bigcup_{n\in\omega} x_n) = \bigcup_{n\in\omega} F(x_n) = \bigcup_{n\in\omega} x_{n+1} \subseteq x_\omega.$$

By non-decreasingness, if $t \in x_\omega$ then $t \in x_n$ for some n and hence $t \in F(x_n)$. Thus find $y \in x_n \subseteq x_\omega$ with $F(y) = t$ to see that $t = F(y) \in F(x_\omega)$.

Finally note that $x = x_0 \subseteq x_\omega$ so we can get arbitrarily large fixed points (if you want $x \in x_\omega$, start with $x_0 = \{x\}$ instead).

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12. This extends question 8.

We say that a relation R is set-like if and only if $\mathrm{pred}_R(x) = \{y : yRx\}$ is a set.

We say that a relation R is well-founded if and only if every non-empty set has an R-minimal element, i.e. $\forall x\ [x \neq \emptyset \to \exists m \in x\ \forall t \in x\ \neg tRm]$.

(a) Show that if R is set-like and well-founded then so is $R^\star$ and we can define $TC_R(x)$ as the smallest set containing x as a subset and being downwards closed under R.

(b) Prove the generalized recursion theorem: Suppose R is a well-founded, set-like relation on a class A and B is a class.

If $F : A \times U \to B$ is a class function then there is a (essentially unique) class function $G : A \to B$ such that for all

$$\forall a \in A\ G(a) = F(a, G|_{\mathrm{pred}(a)}).$$

(c) Deduce the usual Recursion Theorem on On from the Generalized Recursion Theorem.

(d) Show that if R and F are absolute for non-empty transitive classes $A \subseteq B$ satisfying (enough of) $\mathbf{ZF}^-$, then G is absolute for A, B.

(e) What happens to the results in question 8 if we work in weaker set theories than $\mathbf{ZF}^-$?

Solution:

(a) The following proof hopefully avoids any explicit recursion. (It is easier if you use Recursion on $\omega + 1$ but the point is to proof General Recursion once and then deduce Recursion on On - note however that the first bit is essentially manual recursion on $\omega+1$. If you can find a truly recursion free proof, please let me know.)

By induction on n we show that for each $n \in \omega$:

$$\forall x \in A \exists! z\ z = \{f : n+2 \to A : f(n+1) = x\}.$$

(We only do existence, uniqueness is straightforward.)

For $n = 0$: fix $x \in A$, note that $\mathrm{pred}_R(x)$ is a set and so

$$z_0 = \{\{\langle 0, y\rangle, \langle 1, x\rangle\} : y \in \mathrm{pred}_R(x)\}$$

is a set (by **Replacement** and others) and is as required.

Assume the claim is true for n: fix $x \in A$ and take $z_n = \{f : n+2 \to A : f(n+1) = x\}$ which exists by IH. First note that $f : n+2 \to A \mapsto \hat{f} : n+3 \setminus \{0\} \to A$

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given by $\hat{f}(k+1) = f(k)$ codes a function and thus $\hat{z}_n = \left\{\hat{f} : f \in z_n\right\}$ is a set. Also by **Replacement** $p = \{y : \exists f \in z_n\ f(0) = p\}$ is a set and hence $q = \bigcup\{\text{pred}_R(y) : y \in p\}$ is a set (**Replacement**, **Union**). Now let

$$z_{n+1} = \left\{\hat{f} \cup \{\langle 0, y\rangle\} : \hat{f} \in \hat{z}_{n+1}, y \in q, \hat{f}(1) = y\right\}$$

and observe that this is as required.

Now we can use **Replacement** and **Union** to get

$$z_\omega = \bigcup\{z_n : n \in \omega\}$$

and observe that

$$\text{pred}_{R^\star}(x) = \{f(0) : f \in z_\omega\}$$

and obtain that $R^\star$ is set-like.

For well-foundedness, let $s \neq \emptyset$, $s \subseteq A$. For each $y \in s$ we let

$$b_y = \{b \in \text{pred}_{R^\star}(y) : \exists c \in s\ cR^\star b\}.$$

(Each of these are sets by **Separation** and we use the same instance for each y, so this is **one** formula that codes a function $y \mapsto b_y$ on s.)

Then $\emptyset \neq s \subseteq s \cup \bigcup\{b_y : y \in s\} =: s'$ so that s' has an R-minimal element, m say. (If you draw s' in A with arrows for R then s' is s together with the collection of all points between two points of s.)

We claim that m is $R^\star$-minimal in s:

firstly $m \in s$: if $m \notin s$ then find $y \in s$ with $m \in b_y$ and thus $c \in s$ with $cRmR^\star y$. But then $c \in s'$ contradicts R-minimality of m in s'.

secondly if $tR^\star m$ and $t \in s$ then tRm (contradicting R-minimality of m in s') or $t = t_0 R \ldots Rt_n Rm$ for some $t_1, \ldots, t_n \in A$. But then $t_1, \ldots, t_n \in b_m$ (as witnessed by t) and in particular $t_n \in s'$ and $t_n Rm$ contradicting R-minimality of m in s'.

Finally we define $TC_R(x) = \bigcup\{\text{pred}_{R^\star}(y) : y \in x\}$.

(b) We define $\psi_F(a, g)$ as

$$g : \text{pred}_{R^\star}(a) \cup \{a\} \to U \wedge \forall b \in \text{pred}_{R^\star}(a) \cup \{a\}\ \ g(b) = F(b, g|_{\text{pred}_R(b)})$$

and

$$G = \{\langle a, b\rangle \in A \times B : \exists g\ \psi_F(a, g) \wedge g(a) = b\}.$$

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We proceed by showing that

$$\forall a \in A\ \exists! g\ \psi_F(a, g).$$

Suppose not. Let $a' \in A$ be such that $\neg\exists! g\ \psi_F(a', g)$ and let a be $R^\star$ minimal in

$$s = \{b \in \text{pred}_{R^\star}(a') \cup \{a'\} : \neg\exists! g\ \psi_F(b, g)\}.$$

Note $a' \in s$ so $s \neq \emptyset$ as required.

Thus $\forall b \in \text{pred}_{R^\star}(a)$ there is a unique g_b with $\psi_F(b, g_b)$. We set

$$g' = \bigcup \{g_b : b \in \text{pred}_{R^\star}(a)\}$$

and claim that g' is a function on $\text{pred}_{R^\star}(a)$ such that $g(b) = F(b, g'|_{\text{pred}_R(b)})$ for all $b \in \text{pred}_{R^\star}(a)$.

It is a function by uniqueness of the g_b, i.e. if two of them are defined at some c then they are defined on $\text{pred}_{R^\star}(c) \cup \{c\}$ as well (since $cR^\star b \to \text{pred}_{R^\star}(c) \subseteq \text{pred}_{R^\star}(b)$) and hence their restrictions to this set are equal (by $\exists! g \psi_F(c, g)$) and hence they are equal at c.

By existence g' is defined at each $b \in \text{pred}_{R^\star}(a)$ and in particular on $\text{pred}_R(a)$.

Now we set

$$g_a = g' \cup \left\{\langle a, F(a, g'|_{\text{pred}_R(a)})\rangle\right\}$$

and note that this is as required (a few technical steps as above).

Hence the only way in which $\neg\exists! g\ \psi_F(a, g)$ can happen is by non-uniqueness. But if g_1, g_2 are two sets with $\psi_F(a, g_j), j = 1, 2$ then as in the argument above $g_1|_{\text{pred}_{R^\star}(a)} = g_2|_{\text{pred}_{R^\star}(a)}$ and so also $g_1|_{\text{pred}_R(a)} = g_2|_{\text{pred}_R(a)}$ so that $g_1(a) = F(a, g_1|_{\text{pred}_R(a)}) = F(a, g_2|_{\text{pred}_R(a)}) = g_2(a)$ and thus $g_1 = g_2$.

Finally, essentially repeating the construction of g' from above and the proof that it is a function with the required properties, we can show that G is a class function $A \to B$ as required.

(c) Our relation $R = \in$ which is set-like and well-founded on On. We note that for $\forall \alpha \in \text{On}\ \text{pred}_\in(\alpha) = \alpha$. So we set

$$\hat{F}(\alpha, g) = \begin{cases} a; & \alpha = \emptyset \\ F(g(\beta)); & \alpha = \beta + 1 \in \text{On} \\ \bigcup \{g(\beta) : \beta \in \alpha\}; & \alpha \in \text{Lim} \end{cases}.$$

(d) Absoluteness is as absoluteness for Recursion on On.

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(e) There are a couple of issues:

- The first is that our set theory might be too weak to sensibly talk about functions on finite sets (e.g. we don't have ordered pairs - although there could be alternatives to the ordered pair as we've defined it) or we don't even have each natural number (again, there could be alternative definitions of natural numbers). Here we can do very little.
- Next, we might not have ω (**Infinity** might fail). This isn't a problem in defining $R^\star$ (we can replace $n\ in \omega$ by 'n is a finite ordinal'. Of course we might not be able to form the *set* $TC(x)$ but it still is fine as a (parametrized) class.
- The same issue may arise if we don't have a suitable instance of **Replacement** available.

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