Axiomatic Set Theory

Sheet 3 — TT21

Section A

On this sheet all questions can be done in **ZFC** (unless otherwise indicated) but **Choice** is not needed in several and should be avoided if possible.

1. Complete the proof that L satisfies **ZF** (again, probably **Union** and **Infinity**). **Solution: lecture notes**

2. Work in $\mathbf{ZF}^-$.

   Show that if A is a transitive non-empty class such that

   $$\forall z\ [z \subseteq A \to z \in A]$$

   and such that A satisfies **Separation** then A satisfies $\mathbf{ZF}^-$.

3. The rank of a set a, $\mathrm{rk}(a)$, is the least $\alpha \in \mathrm{On}$ such that $a \subseteq V_\alpha$.

   (a) Show that $\mathrm{rk}(a)$ is the least $\alpha \in \mathrm{On}$ such that $a \in V_{\alpha+1}$.

   (b) Show that $\forall \alpha \in \mathrm{On}\ \ \mathrm{rk}(\alpha) = \alpha$.

   (c) Show that $\forall \alpha \in \mathrm{On}\ \ \mathrm{rk}(L_\alpha) = \alpha$.

   (d) Compute $\mathrm{rk}(\{x, y\})$ in terms of $\mathrm{rk}(x), \mathrm{rk}(y)$.

   (e) Compute $\mathrm{rk}(\bigcup x)$ in terms of $\mathrm{rk}(x)$.

   (f) Compute $\mathrm{rk}(\mathcal{P}\,(x))$ in terms of $\mathrm{rk}(x)$.

   (g) Why do we not define $\mathrm{rk}(a)$ as the least $\alpha \in \mathrm{On}$ such that $\mathrm{rk}(a) \in V_\alpha$?

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Section B

4. A club (in On) is a closed unbounded class of ordinals, i.e. a class $C \subseteq \mathrm{On}$ such that $\forall x\ [x \subseteq C \to \sup x \in C]$ (closedness) and $\forall \alpha \in \mathrm{On}\ \exists \beta \in C\ \alpha \in \beta$ (unboundedness).
   (a) Prove that is C_1 and C_2 are clubs then so is $C_1 \cap C_2$.
   (b) Suppose that $X \subseteq \omega \times \mathrm{On}$ is a class and for each $i \in \omega$ we write $X_i = \{\alpha \in \mathrm{On} : \langle i, \alpha \rangle \in X\}$. Carefully write down **one** formula expressing that for all $i \in \omega$, X_i is a club. Carefully define $\bigcap_{i \in \omega} X_i$ and prove that it is a club.

5. We use the following fact: there is a formula $\phi(x)$ of LST (with all free variables shown) such that (in **ZF** one can prove that) for any set a, '$\phi(a)$ if and only if a is transitive and $(a, \in) \models$ **ZF**'. Further this formula is absolute for any non-empty transitive classes $A \subseteq B$ satisfying enough of **ZF**.
   (a) Show that if **ZF** $\vdash \exists x\ \phi(x)$ then **ZF** is inconsistent. *[Consider the least* $\alpha \in \mathrm{On}$ *such that* $\exists x \in V_\alpha\ \phi(x)$*.]*
   (b) Show that if **ZF** is consistent then there is no finite collection **T** of axioms of **ZF** such that **T** $\vdash$ **ZF**. (Note that axiom schemes like **Separation** and **Replacement** count as infinitely many axioms, one for each formula.)
   (c) Give a formula ϕ of LST such that the class $A_\phi = \{\alpha \in \mathrm{On} : \phi \text{ is absolute for } V_\alpha, V\}$ is not a club.
   (d) (Difficult) If ϕ is a formula of LST, show that the class A_ϕ contains a club C_ϕ.

6. Let e denote the set of even natural numbers. Prove that $e \in L_{\omega+1}$.

7. Suppose $F : V \to V$ is a class function (without parameters, i.e the formula defining the class F has one free variable) which is an elementary map, i.e. for every formula $\phi(v_0, \ldots, v_n)$ of LST (with all free variables shown) we have

$$\forall a_0, \ldots, a_n\ \phi(a_0, \ldots, a_n) \leftrightarrow \phi(F(a_0), \ldots, F(a_n)).$$

   Prove that F is the identity.

   [You may want to show that for all ordinals α*,* $F(\alpha) = \alpha$ *by considering the least failure, but other (quicker) methods are available.]*

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8. The collection of Σ_1 formulae are defined (recursively in the meta-theory) as follows:

   - Δ_0 formulae are Σ_1;
   - if ϕ, ψ are Σ_1 then so are $\phi \vee \psi, \phi \wedge \psi, \forall x \in y\ \phi$ and $\exists x\ \phi$;
   - nothing else is a Σ_1 formula.

   (a) Show that for every Σ_1 formula $\phi(v_1, \ldots, v_n)$ there is a corresponding Δ_0 formula $\psi(v_1, \ldots, v_n, w_1, \ldots, w_m)$ such that

   $$\mathbf{ZF} \vdash \forall x_1, \ldots, x_n \left[\phi(x_1, \ldots, x_n) \leftrightarrow \exists y_1, \ldots, y_m\ \psi(x_1, \ldots, x_n, y_1, \ldots, y_m)\right].$$

   (b) Show that Σ_1 formulae are upwards absolute for non-empty transitive classes $A \subseteq B$, i.e. if $\phi(v_1, \ldots, v_n)$ is Σ_1 then

   $$\forall a_1, \ldots, a_n \in A\ \left[\phi(a_1, \ldots, a_n)^A \rightarrow \phi(a_1, \ldots, a_n)^B\right].$$

   (c) Give an example of a Σ_1 formula that is not absolute for non-empty transitive classes.

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Section C

9. This question extends question 4.

   A club on ω_1 is a closed unbounded subset of ω_1, i.e. a set $c \subseteq \omega_1$ such that $x \subseteq c \to \sup x \in c$ and $\forall \alpha \in \omega_1 \exists \beta \in c\ \alpha \in \beta$.

   (a) Show that a club on ω_1 is a club in On relativized to V_{ω_1}.

   (b) Show that the collection of clubs on ω_1 form a countably complete filter, i.e. that the intersection of countably many clubs is club.

   (c) Suppose that $c_\alpha, \alpha \in \omega_1$ is an uncountable family of clubs indexed by ω_1. Show that the diagonal intersection

   $$\Delta_{\alpha \in \omega_1} c_\alpha = \{\beta \in \omega_1 : \forall \delta \in \beta\ \beta \in c_\delta\}$$

   is a club.

   (d) *[Difficult]* A set $s \subseteq \omega_1$ is stationary if and only if it intersects every club. Show that there is a stationary, non-club subset.

   (i) Note that $\mathrm{Lim} \cap \omega_1$ is club.

   (ii) For each $\alpha \in \mathrm{Lim}$ let $(a_n^\alpha)_{n \in \omega}$ be a strictly increasing sequence with $\sup_{n \in \omega} a_n^\alpha = \alpha$.
   Show that

   $$\exists n \in \omega\ \forall \xi \in \omega_1\ S_\xi = \{\alpha \in \mathrm{Lim} : a_n^\alpha \geq \xi\}$$

   is stationary.

   (iii) Fix some $n \in \omega$ as above and show that for each $\xi \in \omega_1$ there is $\eta \in \omega_1$ such that $\xi \leq \eta$ and $\{\alpha \in \mathrm{Lim} : a_n^\alpha = \eta\}$ is stationary.

   (iv) Deduce that there are ω_1 many disjoint stationary sets (none of which can be club).

10. This question extends question 5.

    (a) Indicate how to write down ϕ.

    (b) What is wrong with the following argument: let $\phi_i, i \in \omega$ be an enumeration of all the axioms of **ZF**. For each $i \in \omega$, let $C_i = C_\phi$ be a club such that for $\alpha \in C_i$, ϕ_i is absolute for V_α, V, so that $\phi_i^{V_\alpha}$ holds (because $V \models \phi_i$). Then $\bigcap_i C_i$ is a club and so non-empty. Let $\beta \in \bigcap_i C_i$ so that $(V_\beta, \in) \models \phi_i$ for every i and hence $(V_\beta, \in)$ is a model of **ZF**. Thus $\exists x\ \phi(x)$ and so **ZF** is inconsistent.

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11. Work in **ZF+Global Choice** which means that there is a (defined) well-order of V (this follows for example from **V=L**).

    An ultrafilter on ω is a collection p of subsets of $\mathbb{N}$ such that $\emptyset \notin p$, $a, b \in p \to a \cap b \in p$, $a \in p \wedge a \subseteq b \to b \in p$ and $\forall a \ [a \in p \vee \omega \setminus a \in p]$.

    Assume that p is an ultrafilter on ω.

    Let $P = \{f : \mathbb{N} \to V\}$ and for $f, g \in P$ define $f \equiv g$ if and only if $\{n \in \omega : f(n) = g(n)\} \in p$ and fEg if and only if $\{n \in \omega : f(n) \in g(n)\} \in p$.

    Write W for the quotient of P by $\equiv$ (strictly speaking this could be P' from part (b)) and $\in_W$ for the relation induced by E on W (strictly speaking this could be the restriction of E to P').

    Identify elements x of V with the equivalence class of the constant function with value x.

    (a) Show that $\equiv$ defines an equivalence relation on P and show that E is invariant on equivalence classes. We write $[f]$ for the equivalence class of f (this is a proper class).

    (b) By considering minimal elements of $[f]$ (using **Global Choice**) find a class $P' \subseteq P$ such that $\forall f \in P\ \exists! f' \in P'[f] = [f']$.

    (c) Show that every formula is absolute for $(V, \in), (W, \in_W)$ (they are elementarily equivalent) and hence that W satisfies **ZF** and $\omega^W = \omega^V$ (under the identification).

    (d) Let $f_n : \omega \to \omega$ be given by $f_n(m) = \max\{0, m - n\}$. Show that if $n < m$ then $[f_m] \in_W [f_n] \in_W \omega$ (so each $[f_n]$ is an 'infinite' natural number).
    Deduce that $\{[f_n] : n \in \omega\} \notin W$.

    (e) Think about what that means for internalizing formulae, the satisfaction relation and proofs.

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12. This question extends question 7.

    (a) Does your proof also work for class functions F which may depend on a parameter? I.e. if $\phi(a, z)$ is a formula with two free variables a, z such that there is a parameter a such that $\phi(a, .)$ codes an elementary map $V \to V$, must this map be the identity?

    (b) Now assume that M is a transitive class and $j : V \to M$ an elementary map (a class function possibly with parameters), i.e. such that for every formula $\phi(v_1, \ldots, v_n)$

    $$\forall a_1, \ldots, a_n \in V \ \phi^V(a_1, \ldots, a_n) \leftrightarrow \phi^M(j(a_1), \ldots, j(a_n)).$$

    Show that j maps ordinals to ordinals, is strictly increasing on the ordinals, $j(\omega) = \omega$.

    Show that if V satisfies **ZFC** and j is the identity on On then j is the identity (and $M = V$).

    (c) Continuing from the last part, assume $M \models$ **ZFC**, that $j : V \to M$ is a non-identity elementary map and let κ be the least ordinal such that $j(\kappa) \neq \kappa$ (this is called the critical point of j).

    Show that $\{A \subseteq \kappa : \kappa \in j(A)\}$ is a countably complete, non-principal ultrafilter on κ.

13. Work in **ZFC**.

    This question extends question 8.

    The Π_1 formulae are the negations of the Σ_1 formulae. Derive the analogous results for Π_1 formulae as for Σ_1 formulae.

    Show that 'r is a well-order on x' is equivalent (in **ZFC**) to both a Σ_1 formula and to a (different) Π_1 formula and deduce that it is absolute for non-empty transitive classes satisfying enough of **ZFC**.

    Prove by hand (i.e. without using the previous part) that 'r is a well-order on x' is absolute for transitive classes satisfying enough of **ZFC**.

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