Axiomatic Set Theory

Sheet 3 — TT21

Section A

On this sheet all questions can be done in **ZFC** (unless otherwise indicated) but **Choice** is not needed in several and should be avoided if possible.

1. Complete the proof that L satisfies **ZF** (again, probably **Union** and **Infinity**). **Solution: lecture notes**

2. Work in $\mathbf{ZF}^-$.

   Show that if A is a transitive non-empty class such that

   $$\forall z \ \left[z \subseteq A \rightarrow z \in A\right]$$

   and such that A satisfies **Separation** then A satisfies $\mathbf{ZF}^-$.

   Solution:

   - **Extensionality** follows from transitivity of A;
   - **Emptyset**: $\emptyset \subseteq A$ so $\emptyset \in A$ and $\emptyset$ is absolute;
   - **Pairing**: if $x, y \in A$ then $\{x, y\} \subseteq A$ so $\{x, y\} \in A$ and $\{x, y\}$ is absolute;
   - **Union**: if $x \in A$ then $\bigcup x \subseteq A$ (by transitivity of A: if $t \in y \in x \in A$ then $t \in y \in A$ and so $t \in A$) giving $\bigcup x \in A$ and $\bigcup x$ is absolute;
   - **Powerset**: if $x \in A$ then $z = \mathcal{P}(x) \cap A \subseteq A$ so $z \in A$ and $A \models z = \mathcal{P}(A)$ (by transitivity of A);
   - **Replacement**: follow the proof that $V \models$ **Replacement** to get

     $$z = \left\{y : \exists x \in d \ y \in A \wedge \phi^A(a_1, \ldots, a_n, x, y)\right\} \subseteq A.$$

     Thus $z \in A$ and as for V this is the correct z.
   - **Infinity**: by induction on n, for each $n \in \omega$, $n \in A$ ($\emptyset \in A$ by **Emptyset** and absoluteness; **Pairing**, **Union** and absoluteness of $a + 1$ for inductive step); thus $\omega \subseteq A$ so $\omega \in A$ and $Ind(x)$ being absolute.

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3. The rank of a set a, $\mathrm{rk}(a)$, is the least $\alpha \in \mathrm{On}$ such that $a \subseteq V_\alpha$.
   (a) Show that $\mathrm{rk}(a)$ is the least $\alpha \in \mathrm{On}$ such that $a \in V_{\alpha+1}$.
   (b) Show that $\forall \alpha \in \mathrm{On}\ \ \mathrm{rk}(\alpha) = \alpha$.
   (c) Show that $\forall \alpha \in \mathrm{On}\ \ \mathrm{rk}(L_\alpha) = \alpha$.
   (d) Compute $\mathrm{rk}(\{x, y\})$ in terms of $\mathrm{rk}(x), \mathrm{rk}(y)$.
   (e) Compute $\mathrm{rk}(\bigcup x)$ in terms of $\mathrm{rk}(x)$.
   (f) Compute $\mathrm{rk}(\mathcal{P}(x))$ in terms of $\mathrm{rk}(x)$.
   (g) Why do we not define $\mathrm{rk}(a)$ as the least $\alpha \in \mathrm{On}$ such that $\mathrm{rk}(a) \in V_\alpha$?

Solution: Note that clearly $a \subseteq b \to \mathrm{rk}(a) \leq \mathrm{rk}(b)$ and also $a \in b \to \mathrm{rk}(a) < \mathrm{rk}(b)$: if $a \in b \subseteq V_\alpha$ then $\alpha \neq 0$; if $\alpha = \beta + 1$ then $a \in V_{\beta+1} = \mathcal{P}(V_\beta)$ gives $a \subseteq V_\beta$; if $\alpha \in \mathrm{Lim}$ then $a \in V_\alpha = \bigcup_{\beta<\alpha} V_\beta$ give $a \in V_\beta$ for some $\beta < \alpha$.

(a) if $a \subseteq V_\alpha$ then $a \in \mathcal{P}(V_\alpha) = V_{\alpha+1}$; conversely if $a \in V_{\alpha+1} = \mathcal{P}(V_\alpha)$ then $a \subseteq V_\alpha$. Hence $a \subseteq V_\alpha \leftrightarrow a \in V_{\alpha+1}$.

(b) This follows immediately from $V_\alpha \cap \mathrm{On} = \alpha$.

(c) Inductively $L_\alpha \subseteq V_\alpha$ gives $\mathrm{rk}(L_\alpha) \leq \alpha$. Conversely $\alpha = L_\alpha \cap \mathrm{On}$ so $rk(L_\alpha) \geq \alpha$.

(d) $\mathrm{rk}(\{x, y\}) = \max(\mathrm{rk}(x) + 1, \mathrm{rk}(y) + 1)$ (by the two properties at the beginning)

(e) if $\mathrm{rk}(x) = \alpha + 1$ then $\mathrm{rk}(\bigcup x) = \alpha$: firstly if $t \in \bigcup x$ then find $y \in x$ with $t \in y$; then $\mathrm{rk}(y) < \mathrm{rk}(x)$ so $\mathrm{rk}(y) \leq \alpha$ and hence $t \in V_\alpha$; next if $\bigcup x \subseteq V_\beta$ for $\beta < \alpha$ then every $y \in x$ is contained in V_β (as a subset) and hence every $y \in x$ is in $V_{\beta+1}$ as an element so $x \subseteq V_{\beta+1}$ a contradiction;

   if $\mathrm{rk}(x) = 0$ then $x = \emptyset$ and $\bigcup x = \emptyset$ so $\mathrm{rk}(\bigcup x) = 0$;

   if $\mathrm{rk}(x) \in \mathrm{Lim}$ then $\mathrm{rk}(\bigcup x) = \mathrm{rk}(x)$ as for the successor case.

(f) $\mathrm{rk}(\mathcal{P}(x)) = \mathrm{rk}(x) + 1$: since $x \in \mathcal{P}(x)$ we have $\geq$. But if $x \subseteq V_\alpha$ and $y \subseteq x$ then $y \subseteq V_\alpha$, so $y \in V_{\alpha+1}$. Thus $\mathrm{rk}(\mathcal{P}(x)) \leq \alpha + 1$.

(g) In this case we would have no elements with a limit rank (at limit stages we don't get any new elements).

   With our current definition we have the pleasing recursive formula

$$\mathrm{rk}(x) = \sup \{\mathrm{rk}(y) + 1 : y \in x\}.$$

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Section B

4. A club (in On) is a closed unbounded class of ordinals, i.e. a class $C \subseteq \mathrm{On}$ such that $\forall x\ [x \subseteq C \to \sup x \in C]$ (closedness) and $\forall \alpha \in \mathrm{On}\ \exists \beta \in C\ \alpha \in \beta$ (unboundedness).
   (a) Prove that is C_1 and C_2 are clubs then so is $C_1 \cap C_2$.
   (b) Suppose that $X \subseteq \omega \times \mathrm{On}$ is a class and for each $i \in \omega$ we write $X_i = \{\alpha \in \mathrm{On} : \langle i, \alpha\rangle \in X\}$. Carefully write down **one** formula expressing that for all $i \in \omega$, X_i is a club. Carefully define $\bigcap_{i\in\omega} X_i$ and prove that it is a club.

Solution:

(a) For unboundedness, recursively (on ω, with parameter $\alpha \in \mathrm{On}$) define

$$\alpha_0 = \alpha + 1$$
$$\alpha_{n+1} = \begin{cases} \text{least } \delta \in C_1\ \delta > \alpha_n; & n \text{ odd} \\ \text{least } \delta \in C_2\ \delta > \alpha_n; & n \text{ even} \end{cases}$$

and set $\beta = \sup_n \alpha_n$. Since the sequence of α_n are strictly increasing we have $\beta = \sup_k \alpha_{2k+1} \in C_1$ and $\beta = \sup_k \alpha_{2k} \in C_2$ and clearly $\alpha \in \alpha_0 \subseteq \beta$.

Closedness is straightforward: if $s \subseteq C_1 \cap C_2$ then $s \subseteq C_i, i = 1, 2$ so $\sup s \in C_i, i = 1, 2$ as each C_i is closed.

(b) The formula is

$$\begin{aligned}\phi \equiv &\forall i \in \omega\ \forall \alpha \in \mathrm{On}\ \exists \beta \in \mathrm{On}\ \langle i, \beta\rangle \in X \wedge \alpha \in \beta \\ &\wedge \forall s\ \forall i \in \omega\ \left[(\forall \alpha \in s\ \langle i, \alpha\rangle \in X) \to \langle i, \sup s\rangle \in X\right].\end{aligned}$$

We similarly define '$\bigcap_{i\in\omega} X_i$' by

$$\{\beta : \forall i \in \omega\ \langle i, \beta\rangle \in X\}.$$

To prove unboundedness we take a function $f : \omega \to \omega$ which hits every $k \in \omega$ infinitely often (e.g. take a bijection $f : \omega \to \omega \times \omega$ and take $\pi_1 \circ f$) and by recursion on ω with parameter α define $\alpha_0 = \alpha + 1$ and

$$\alpha_{n+1} = \text{least } \delta \in \mathrm{On}\ \delta > \alpha_n \wedge \langle f(n+1), \delta\rangle \in X.$$

As before α_n is a strictly increasing sequence and since each $f^{-1}(k)$ is unbounded in ω we get that

$$\beta := \sup\{\alpha_n : n \in \omega\} = \sup\left\{\alpha_n : n \in f^{-1}(k)\right\}, k \in \omega$$

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so that

$$\forall i \in \omega \ \langle i, \beta \rangle \in X.$$

Closedness is the same as in the previous part.

Comment: It is important that we have one (or finitely many) formulae and use the parameter i to 'split' these into 'infinitely many' classes, i.e. our classes have a uniform formula. Of course in the meta-theory (depending on whether you have an axiom of infinity) we can imagine infinitely many formulae defining infinitely many classes, but we have no way of writing a formula involving all of them in the theory.

5. We use the following fact: there is a formula $\phi(x)$ of LST (with all free variables shown) such that (in **ZF** one can prove that) for any set a, '$\phi(a)$ if and only if a is transitive and $(a, \in) \models$ **ZF**'. Further this formula is absolute for any non-empty transitive classes $A \subseteq B$ satisfying enough of **ZF**.
   (a) Show that if **ZF** $\vdash \exists x\ \phi(x)$ then **ZF** is inconsistent. *[Consider the least* $\alpha \in \mathrm{On}$ *such that* $\exists x \in V_\alpha\ \phi(x)$*.]*
   (b) Show that if **ZF** is consistent then there is no finite collection **T** of axioms of **ZF** such that **T** $\vdash$ **ZF**. (Note that axiom schemes like **Separation** and **Replacement** count as infinitely many axioms, one for each formula.)
   (c) Give a formula ϕ of LST such that the class $A_\phi = \{\alpha \in \mathrm{On} : \phi \text{ is absolute for } V_\alpha, V\}$ is not a club.
   (d) (Difficult) If ϕ is a formula of LST, show that the class A_ϕ contains a club C_ϕ.

Solution:

(a) Suppose **ZF** $\vdash \exists x\ \phi(x)$. Let $\alpha \in \mathrm{On}$ be minimal s.t. $\exists x \in V_\alpha\ \phi(x)$ (since **ZF** implies $\forall x\ x \in V$). Then $x \models$ **ZF** and hence $x \models \exists y\ \phi(y)$. Pick some $y \in x$ such that $\phi^x(y)$. By absoluteness of ϕ for x, V (x is transitive) we have $\phi(y)$. But because $y \in x$ we have $\mathrm{rk}(y) < \mathrm{rk}(x)$ contradicting minimality of α.

(b) Let **T** be a finite collection of axioms of **ZF** such that **T** $\vdash$ **ZF**.

Work in **ZF** (i.e. the following is always '**ZF** proves ...'): by the Levy Reflection Principle there is α such that $V_\alpha \models$ **T**. But then V_α is a transitive set satisfying **T** and hence **ZF**. Thus $\exists x\ \phi(x)$.

Hence **ZF** $\vdash \exists x\ \phi(x)$ and thus **ZF** is inconsistent.

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(c) We can take ϕ to be the disjunction of 'there is a maximal ordinal' and **Infinity**: since $\textbf{Infinity}^V$ we have ϕ^V. For $\alpha \in \omega, \alpha \neq 0$ we have ϕ^{V_α} (since there is a maximal ordinal in V_α and this is absolute) and for $\alpha > \omega$ we have $\textbf{Infinity}^{V_\alpha}$. But (again absoluteness of both disjunction for V_ω, V) $\neg\phi^{V_\omega}$ and also $\neg\phi^{\emptyset}$. Thus $A_\phi = \mathrm{On} \setminus \{\emptyset, \omega\}$ which is not closed.

(d) In the proof of the LRP we define (implicitly) a class function $F : \mathrm{On} \to \mathrm{On}$ which gives $\alpha_{m+1} = F(\alpha_m)$ and then use recursion on $\omega + 1$. We could use F and recursion on On to get a class function $G : \mathrm{On} \to \mathrm{On}$ in the usual way (starting with $G(0) = 1$ for example) and then check that $G[\mathrm{Lim}]$ is club which follows from noting that G is continuous and increasing. Also the Tarski-Vaught criterion shows that for every $\alpha \in G[\mathrm{Lim}]$ we have that ϕ is absolute for V_α, V (in the same way as in the proof of the LRP).

6. Let e denote the set of even natural numbers. Prove that $e \in L_{\omega+1}$.

Solution: Let $\phi(n) \equiv n \in \mathrm{On} \wedge (n = \emptyset \vee \exists m \in n\ n = m + m)$. Note that $\mathrm{On}^{L_\omega} = \omega$, $(n = \emptyset)^{L_\omega} \leftrightarrow n = \emptyset$ and (for $n, m \in \omega$) $(n = m + m)^{L_\omega} \leftrightarrow n = m + m$. Thus

$$e = z = \{n \in L_\omega : L_\omega \models \phi(n)\} \in L_{\omega+1}.$$

Note that we are not allowed to use ω as a parameter (since $\omega \notin L_\omega$) but instead use On (this works because $(x \in \mathrm{On})^{L_\omega} \leftrightarrow x \in \omega$).

7. Suppose $F : V \to V$ is a class function (without parameters, i.e the formula defining the class F has one free variable) which is an elementary map, i.e. for every formula $\phi(v_0, \ldots, v_n)$ of LST (with all free variables shown) we have

$$\forall a_0, \ldots, a_n\ \phi(a_0, \ldots, a_n) \leftrightarrow \phi(F(a_0), \ldots, F(a_n)).$$

Prove that F is the identity.

[You may want to show that for all ordinals α, $F(\alpha) = \alpha$ by considering the least failure, but other (quicker) methods are available.]

Solution: F is injective by considering $\phi(a_0, a_1) \equiv a_0 = a_1$: this gives $F(a_0) = F(a_1) \to a_0 = a_1$.

F is surjective by considering $\phi(a_0) \equiv \exists a_1\ F(a_1) = a_0$: suppose $x \in V$ and let $y = F(x)$. Then $\phi(y)$ (as witnessed by x) and hence $\phi(F(x))$ (since $y = F(x)$ and thus by elementariness of F we have $\phi(x)$ as required.

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F is the identity: suppose not and let x be $\in$-minimal (e.g. take x of minimal rank) such that $F(x) \neq x$. Let $\phi(a_0, a_1) \equiv a_0 \in a_1$. If $t \in x$ then $F(t) = t$ and so by elementariness $t = F(t) \in F(x)$. Conversely if $t \in F(x)$ then $t = F(u)$ for some u by surjectivity and thus $u \in x$ meaning $u = F(u) = t$.

8. The collection of Σ_1 formulae are defined (recursively in the meta-theory) as follows:

   - Δ_0 formulae are Σ_1;
   - if ϕ, ψ are Σ_1 then so are $\phi \vee \psi, \phi \wedge \psi, \forall x \in y\ \phi$ and $\exists x\ \phi$;
   - nothing else is a Σ_1 formula.

   (a) Show that for every Σ_1 formula $\phi(v_1, \ldots, v_n)$ there is a corresponding Δ_0 formula $\psi(v_1, \ldots, v_n, w_1, \ldots, w_m)$ such that

   $$\mathbf{ZF} \vdash \forall x_1, \ldots, x_n \left[\phi(x_1, \ldots, x_n) \leftrightarrow \exists y_1, \ldots, y_m\ \psi(x_1, \ldots, x_n, y_1, \ldots, y_m)\right].$$

   (b) Show that Σ_1 formulae are upwards absolute for non-empty transitive classes $A \subseteq B$, i.e. if $\phi(v_1, \ldots, v_n)$ is Σ_1 then

   $$\forall a_1, \ldots, a_n \in A\ \left[\phi(a_1, \ldots, a_n)^A \to \phi(a_1, \ldots, a_n)^B\right].$$

   (c) Give an example of a Σ_1 formula that is not absolute for non-empty transitive classes.

Solution:

(a) By induction on the complexity of ϕ:

For a Δ_0 formula ϕ we take $\psi = \phi$ and $m = 0$.

If ϕ_1, ϕ_2 are equivalent to $\exists y_1, \ldots, y_{m_i}\ \psi_i$ with $\psi_i\ \Delta_0$ then relabel the free occurrences of $y_1, \ldots, y_{m_2}$ in ψ_2 to $y_{m_1+1}, \ldots, y_{m_1+m_2}$ in ψ_2 and note that $\phi_1 \wedge \phi_2 \leftrightarrow \exists y_1, \ldots, y_{m_1+m_2} \psi_1 \wedge \psi_2$ and similarly for $\vee$.

If ϕ is equivalent to $\exists y_1, \ldots, y_m\ \psi$ with $\psi\ \Delta_0$ then $\exists x \phi \leftrightarrow \exists x \exists y_1, \ldots, y_m\ \psi$.

Now suppose ϕ is equivalent to $\exists y_1, \ldots, y_m\ \psi$ with $\psi\ \Delta_0$. We let

$$\psi' \equiv \forall x \in y \exists y_1, \ldots, y_m \in y_{m+1}\ \psi$$

and claim this works for $\forall x \in y\ \phi$, i.e. that

$$\forall x \in y\ \phi \leftrightarrow \exists y_{m+1}\ \psi'$$

(or more precisely that $\mathbf{ZF}$ proves its universal closure).

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Firstly $\forall x \in y \exists y_1, \ldots, y_m \in y_{m+1}\ \psi$ is clearly Δ_0 provided ψ is.

Now for the equivalence fix the free variables (in particular y):

Assume $\forall x \in y \phi$; for each $x \in y$ we have $\exists y_1, \ldots, y_m\ \psi$ and because we assume **ZF** this means that there is $\alpha_x \in \mathrm{On}$ (minimal) such that $\exists y_1, \ldots, y_m \in V_{\alpha_x}\ \psi$. So we can take $\alpha = \sup_{x \in y} \alpha_x$ to get

$$\forall x \in y\ \exists y_1, \ldots, y_m \in V_\alpha\ \psi.$$

Thus $y_{m+1} = V_\alpha$ witnesses $\exists y_{m+1} \psi'$ (for these free variables).

Conversely assume that $\exists y_{m+1} \psi'$ and fix some such. Fix $x \in y$. Then $\exists y_1, \ldots, y_m \in y_{m+1} \psi$ and thus $\exists y_1, \ldots, y_m\ \psi$. Thus by I.H. we have ϕ. But since $x \in y$ was arbitrary we actually have $\forall x \in y\ \phi$ as required.

(b) An easy induction on the complexity of the formula. Everything proceeds as for the Δ_0 case except that when we come to a $\exists x \phi$ we can only prove upwards absoluteness.

(c) A straightforward example is **Infinity**. This is Σ_1 but not absolute for V_1, V.

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Section C

9. This question extends question 4.

   A club on ω_1 is a closed unbounded subset of ω_1, i.e. a set $c \subseteq \omega_1$ such that $x \subseteq c \rightarrow \sup x \in c$ and $\forall \alpha \in \omega_1 \exists \beta \in c\ \alpha \in \beta$.

   (a) Show that a club on ω_1 is a club in On relativized to V_{ω_1}.

   (b) Show that the collection of clubs on ω_1 form a countably complete filter, i.e. that the intersection of countably many clubs is club.

   (c) Suppose that $c_\alpha, \alpha \in \omega_1$ is an uncountable family of clubs indexed by ω_1. Show that the diagonal intersection

   $$\Delta_{\alpha\in\omega_1} c_\alpha = \{\beta \in \omega_1 : \forall \delta \in \beta\ \beta \in c_\delta\}$$

   is a club.

   (d) *[Difficult]* A set $s \subseteq \omega_1$ is stationary if and only if it intersects every club. Show that there is a stationary, non-club subset.

      (i) Note that $\mathrm{Lim} \cap \omega_1$ is club.

      (ii) For each $\alpha \in \mathrm{Lim}$ let $(a_n^\alpha)_{n\in\omega}$ be a strictly increasing sequence with $\sup_{n\in\omega} a_n^\alpha = \alpha$.
      Show that

      $$\exists n \in \omega\ \forall \xi \in \omega_1\ S_\xi = \{\alpha \in \mathrm{Lim} : a_n^\alpha \geq \xi\} \text{ is stationary.}$$

      (iii) Fix some $n \in \omega$ as above and show that for each $\xi \in \omega_1$ there is $\eta \in \omega_1$ such that $\xi \leq \eta$ and $\{\alpha \in \mathrm{Lim} : a_n^\alpha = \eta\}$ is stationary.

      (iv) Deduce that there are ω_1 many disjoint stationary sets (none of which can be club).

Solution:

(a) Since V_{ω_1} satisfies **ZF − Replacement** we get that $x \in \mathrm{On}$ is absolute for V_{ω_1}, V so $\mathrm{On}^{V_{\omega_1}} = \mathrm{On} \cap V_{\omega_1} = \omega_1$.

A subtlety is that (possibly) not every club subset of ω_1 is a class (given by a formula).

(b) The same proof as in question 4 works (this time instead of a formula whose slices are the club classes, we have a function $f : \omega \rightarrow V$ such that each $f(n)$ is a club on ω_1 (so again, we have only one uniform description in terms of f). We need to observe that using **Choice** we can show that a countable union of countable sets is countable, i.e. the sup we construct is in ω_1.

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(c) For closedness, let $c'_\alpha = c_\alpha \cup [\alpha, \omega_1)$ (closed as a finite union of closed sets) and note that

$$\Delta_{\alpha\in\omega_1} c_\alpha = \bigcap_{\alpha\in\omega_1} c'_\alpha.$$

For unboundedness, let $\alpha \in \omega_1$ and set $\beta_0 = \alpha$. By the previous part we can find $\beta_1 \in \omega$ such that $\beta_1 \in \bigcap_{\delta\in\beta_0} c_\delta$ with $\beta_0 < \beta_1$. Continue recursively, i.e. find

$$\beta_{n+1} \in (\beta_n, \omega_1) \cap \bigcap_{\delta\in\beta_1} c_\delta$$

and let $\beta = \sup\{\beta_n : n \in \omega\} = \bigcup_{n\in\omega} \beta_n$.

Note that $n \mapsto \beta_n$ is strictly increasing so $\beta \in \mathrm{Lim}$.

Also if $\delta \in \beta$ then $\delta \in \beta_n$ for some $n \in \omega$ and thus $\delta \in \beta_{n+k}$ for all $k \in \omega$. Thus $\beta_{n+1} \in c_\delta$ and continuing inductively $\beta_{n+1+k} \in c_\delta$ for $k \in \omega$. But then as c_δ is closed we must have $\beta = \sup_{k\in\omega} \beta_{n+1+k} \in c_\delta$.

Thus $\forall \delta \in \beta\ \beta \in c_\delta$ as required.

(d) (i) Straightforward (the limit limits is a limit and for each $\alpha \in \omega_1$ we have $\alpha + \omega \in \mathrm{Lim} \cap \omega_1$).

(ii) Suppose not: for each $n \in \omega$ fix $\xi_n \in \omega_1$ such that S_{ξ_n} is not stationary as witnessed by some club C_n that doesn't meet S_{ξ_n}.

Let $\xi = \sup_{n\in\omega} \xi_n \in \omega_1$ (a countable sup of countable ordinals is countable). Then for each $n \in \omega$ the set $\{\alpha \in \mathrm{Lim} : a^\alpha_n \geq \xi\} \subseteq S_{\xi_n}$ misses C_n and hence their union, $S = \{\alpha \in \mathrm{Lim} : \exists n \in \omega\ a^\alpha_n \geq \xi\}$ misses $C = \bigcap_{n\in\omega} C_n$. But C is club (a countable intersection of clubs) and $S \supset \{\alpha \in \mathrm{Lim} : \alpha > \xi\}$ contains a club and hence must intersect C, a contradiction.

(iii) Fix $n \in \omega$ and $\xi \in \omega_1$ and assume $\forall \eta \in [\xi, \omega_1)$ there is a club C_η which misses $\{\alpha \in \mathrm{Lim} : a^\alpha_n = \eta\}$. Fix such C_η and let $C = \Delta_{\xi\leq\eta\in\omega_1} C_\eta$ which is club by (c). Then C intersects S_ξ in some α (as S_ξ) is stationary.

We claim that this contradicts the choice of a^α_n: firstly $\alpha \in S_\xi$ gives $\xi \leq a^\alpha_n$.

Next since $(a^\alpha_k)_k$ is strictly increasing with supremum α we must have $a^\alpha_n < \alpha$. But then since $\alpha \in C = \Delta_{\xi\leq\eta<\omega_1} C_\eta$ we must have $\alpha \in C_{a^\alpha_n}$ (taking $\delta = a^\alpha_n$ in the definition of the diagonal intersection) contradicting the choice of $C_{a^\alpha_n}$.

(iv) Fix an n from (ii). Now take $\xi = 0$ and use (iii) to get a $\eta_0 \in \omega_1$ such that $S_0 = \{\alpha \in \mathrm{Lim} : a^\alpha_n = \eta_0\}$ is stationary. Then take $\xi = \eta_0 + 1$ and use (iii) to get $\eta_1 > \eta_0$ $S_1 = \{\alpha \in \mathrm{Lim} : a^\alpha_n = \eta_1\}$ is stationary. Clearly S_0 and S_1 are disjoint (since a^α_n can only be one of η_0 or η_1). Neither can be club since if one of them is the other can't be stationary (as they are disjoint).

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In fact continuing recursively we get ω_1 many disjoint stationary sets (none of which can be club).

10. This question extends question 5.

   (a) Indicate how to write down ϕ.

   (b) What is wrong with the following argument: let $\phi_i, i \in \omega$ be an enumeration of all the axioms of **ZF**. For each $i \in \omega$, let $C_i = C_\phi$ be a club such that for $\alpha \in C_i$, ϕ_i is absolute for V_α, V, so that $\phi_i^{V_\alpha}$ holds (because $V \models \phi_i$). Then $\bigcap_i C_i$ is a club and so non-empty. Let $\beta \in \bigcap_i C_i$ so that $(V_\beta, \in) \models \phi_i$ for every i and hence $(V_\beta, \in)$ is a model of **ZF**. Thus $\exists x\ \phi(x)$ and so **ZF** is inconsistent.

Solution:

(a) First we have to agree what $(a, \in) \models$ **ZF** means **internally**: for this we can write down an condition $Ax(n)$ on Gödel codes n in the meta-theory which express that it is the Gödel code for some axiom and we then take this formula and say $(a, \in) \models$ **ZF** if and only if $\forall n \in \omega\ Ax(n) \to val(a, 0, n) = 1$.

Note that this may be different from the 'intended' meaning because there might be (additional) non-standard natural numbers which satisfy $Ax(n)$. We then only have to check that we can write down $Ax(n)$ as an absolute formula for transitive classes (and this is fine because it will only involve finite ordinal arithmetic).

$\phi(a)$ will then be that a is transitive and that $\forall n \in \omega\ Ax(n) \to val(a, 0, n) = 1$.

(b) The problem with the argument is that we cannot reference all the C_i at once. There is no one (uniform) formula which given n (such that $Ax(n)$) spits out a slice C_n which is a club. In particular our proof of LRP gets longer the more complicated the formula is and thus we can't prove in the theory that $\forall n \in Ax\ C_n$ is club although for each individual axiom of **ZF** we could write down a club C and a proof that C is a club.

Trying to internalize doesn't help: we cannot even express what it means that $V \models \phi_n$ for infinitely many ϕ_n (since we can't define $val(V, 0, \lceil n \rceil)$) so even if we code up axioms by integers we can't even express that 'all axioms are absolute for V_β, V'.

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11. Work in **ZF+Global Choice** which means that there is a (defined) well-order of V (this follows for example from **V=L**).

    An ultrafilter on ω is a collection p of subsets of $\mathbb{N}$ such that $\emptyset \notin p$, $a, b \in p \to a \cap b \in p$, $a \in p \wedge a \subseteq b \to b \in p$ and $\forall a \ [a \in p \vee \omega \setminus a \in p]$.

    Assume that p is an ultrafilter on ω.

    Let $P = \{f : \mathbb{N} \to V\}$ and for $f, g \in P$ define $f \equiv g$ if and only if $\{n \in \omega : f(n) = g(n)\} \in p$ and fEg if and only if $\{n \in \omega : f(n) \in g(n)\} \in p$.

    Write W for the quotient of P by $\equiv$ (strictly speaking this could be P' from part (b)) and $\in_W$ for the relation induced by E on W (strictly speaking this could be the restriction of E to P').

    Identify elements x of V with the equivalence class of the constant function with value x.

    (a) Show that $\equiv$ defines an equivalence relation on P and show that E is invariant on equivalence classes. We write $[f]$ for the equivalence class of f (this is a proper class).

    (b) By considering minimal elements of $[f]$ (using **Global Choice**) find a class $P' \subseteq P$ such that $\forall f \in P \ \exists! f' \in P'[f] = [f']$.

    (c) Show that every formula is absolute for $(V, \in), (W, \in_W)$ (they are elementarily equivalent) and hence that W satisfies **ZF** and $\omega^W = \omega^V$ (under the identification).

    (d) Let $f_n : \omega \to \omega$ be given by $f_n(m) = \max\{0, m - n\}$. Show that if $n < m$ then $[f_m] \in_W [f_n] \in_W \omega$ (so each $[f_n]$ is an 'infinite' natural number).
    Deduce that $\{[f_n] : n \in \omega\} \notin W$.

    (e) Think about what that means for internalizing formulae, the satisfaction relation and proofs.

Solution:

(a) Straightforward checks using that p is a filter. Symmetry is immediate, reflexivity follows from $\omega \in p$ and transitivity follows from $a, b \in p \to a \cap b \in p$.

(b) This is Scott's trick which enables us to pretend to talk about 'collections of classes'.

(c) This is essentially Łoś's Theorem on ultraproducts.

    To see that it is an elementary embedding we prove by induction on the complexity of the formula $\phi(v_1, \ldots, v_k)$ that

$$(W, \in_W) \models \phi([f_1], \ldots, [f_k]) \leftrightarrow \{n \in \omega : (V, \in) \models \phi(f_1(n), \ldots, f_k(n))\} \in p.$$

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This is true at atomic formulae by definition of $\in_W$.

For conjunctions use that p is closed under finite intersections (for one direction) and supersets (for the other direction).

For negation it follows by p being an ultrafilter (i.e. that for $s \subseteq \omega$ we have $\omega \setminus s \in p \leftrightarrow s \notin p$).

Other logical connectives can be defined in terms of negation and conjunction (or do them directly).

For existential quantifiers, suppose $(W, \in_W) \models \exists v_0\ \phi$. Find such an $[f_0]$ and note that $\{n \in \omega : V \models \phi(f_0(n), \dots, f_k(n)\} \in p$. For each such n we have that $f_0(n)$ witnesses $V \models \exists v_n\ \phi$ so $\{n \in \omega : V \models \exists v_0\ \phi\} \supseteq \{n \in \omega : V \models \phi(f_0(n), \dots, f_k(n)\}$ and hence is in p.

Conversely suppose $T = \{n \in \omega : \exists v_0\ \phi(v_0, f_1(n), \dots, f_k(n)\} \in u$. By **Global Choice** there is $f_0 : T \to V$ such that for $n \in T$ $V \models \phi(f_0(n), \dots, f_k(n))$. Extend f_0 to ω arbitrarily (e.g. define $f_0(n) = 0$ for $n \notin T$) and observe that $[f_0]$ then witnesses $W \models \exists v_0\ \phi$.

Universal quantifiers can be defined in terms of existential quantifiers and negation.

Elementariness then follows since if $V \models \phi(x_1, \dots, x_n)$ and we write f_i for the constant function on ω with value x_i then $\{n \in \omega : \phi(f_1(n), \dots, f_k(n)\} = \omega \in p$.

Since we have defined ω by a formula we get $\omega^W = \omega^V$.

(d) If we write out $f_n(m)$ as a sequence we can see that it is

$$(0, \dots, 0, 1, 2, 3, \dots)$$

with $n+1$ many zeros at the start. Thus for every m we have $f_n(m) \in f_{n+1}(m) \in \omega$ and hence $[f_m] \in_W [f_n] \in_W \omega^W = \omega^V$.

If $z = \{[f_n] : n \in \omega\} \in W$ then z would contradict **Foundation**^{W}.

Note that this only says that we can't come up with a function $f : \omega \to V$ in V such that $[f] = \{[f_n] : n \in \omega\}$. If we try to ensure that each $[f_n] in [f]$ then we will have 'accidentally' added extra elements to $[f]$. In particular a minimal element, namely if m_n is $\in$-minimal in each $f(n) \neq \emptyset$ then $[n \mapsto m_n]$ will be minimal in $[f]$. Note that for p-many n we must have $f(n) \neq \emptyset$ since we will need $[f] \neq \emptyset$.

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12. This question extends question 7.

    (a) Does your proof also work for class functions F which may depend on a parameter? I.e. if $\phi(a, z)$ is a formula with two free variables a, z such that there is a parameter a such that $\phi(a, .)$ codes an elementary map $V \to V$, must this map be the identity?

    (b) Now assume that M is a transitive class and $j : V \to M$ an elementary map (a class function possibly with parameters), i.e. such that for every formula $\phi(v_1, \ldots, v_n)$

    $$\forall a_1, \ldots, a_n \in V \ \phi^V(a_1, \ldots, a_n) \leftrightarrow \phi^M(j(a_1), \ldots, j(a_n)).$$

    Show that j maps ordinals to ordinals, is strictly increasing on the ordinals, $j(\omega) = \omega$.

    Show that if V satisfies **ZFC** and j is the identity on On then j is the identity (and $M = V$).

    (c) Continuing from the last part, assume $M \models$ **ZFC**, that $j : V \to M$ is a non-identity elementary map and let κ be the least ordinal such that $j(\kappa) \neq \kappa$ (this is called the critical point of j).

    Show that $\{A \subseteq \kappa : \kappa \in j(A)\}$ is a countably complete, non-principal ultrafilter on κ.

Solution:

(a) My proof does not work: write $F_a(x)$ for the coded elementary map. Elementariness now means that for every formula $\phi(v_1, \ldots, v_n)$ we have

$$\forall a_1, \ldots, a_n \phi(a_1, \ldots, a_n) \leftrightarrow \phi(F_a(a_1), \ldots, F_a(a_n)).$$

The surjectivity of F_a fails: the formula $\exists a_1 F_a(a_1) = a_0$ has two free variables, a_0 and a, so elementariness now says

$$\exists a_1 F_a(a_1) = a_0 \leftrightarrow \exists a_1 F_{F_a(a)}(a_1) = F_a(a_0).$$

In the proof we take $x \in V$, let $y = F_a(x)$ and observe $\exists a_1 \ F_a(a_1) = F_a(x)$ which doesn't match the RHS.

For more information look up ‘Kunen Inconsistency’ (which essentially says that if you assume **Choice** then there is no definable elementary map with parameters either).

(b) First note that M satsifies the same axioms as V (since these are sentences, i.e. formulae without free variables so are preserved).

‘being an ordinal’ is preserved by j, so j maps ordinals to ordinals.

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Also $\alpha \in \beta$ implies $j(\alpha) \in j(\beta)$ so j is strictly increasing on the ordinals.

Next by induction on $n \in \omega^V$ we get $j(n) = n$: since 0 is defined by $\forall t \in z\ t \neq t$ we have $j(0^V) = 0^M = 0^V$ and then successor steps work as well since the formula $\phi(m, n) \equiv n = m + 1$ is preserved. Thus j is the identity on ω^V (and in particular $\omega^V \subseteq M$).

Let $z = j(\omega^V)$. By elementariness, $M \models z = \omega$ and because M satisfies **Foundation** this is absolute for M, V, so $V \models z = \omega$ giving $j(\omega^V) = \omega^V$.

Now assume that **Choice** (whether in M or V doesn't matter) and that $\forall \alpha \in \mathrm{On}\ j(\alpha) = \alpha$. Assume j is not the identity and find $x \in V$ of minimal rank such that $j(x) \neq x$. In V find $\kappa \in \mathrm{On}$ and $f : \kappa \to x$ surjective. Then $M \models j(f) : j(\kappa) \to j(x)$ surjective and $j(\kappa) = \kappa$ so $M \models j(f) : \kappa \to j(x)$ surjective. But being a surjective function from κ to $j(x)$ is absolute for M, V so $j(f) : \kappa \to j(x)$ is surjective (in V).

Fix $t \in j(x)$ and find $\alpha \in j(\kappa) = \kappa$ such that $j(f)(\alpha) = t$. Note that $\alpha = j(\alpha)$ so $j(f)(j(\alpha)) = t$. Let $y = f(\alpha)$. Then $j(y) = j(f)(j(\alpha)) = j(f)(\alpha)$ (in M so in V by absoluteness) and since $j(f)$ is a function (in M and so in V) $j(y) = t$. Thus j is onto $j(x)$ and as in the proof of question 7 this gives $j(x) = x$.

(c) Let $u = \{A \subseteq \kappa : \kappa \in j(A)\}$.

By the previous part we know that $j(\kappa) \in \mathrm{On}$ and be minimality of κ we must have $\kappa < j(\kappa)$. Thus $\kappa \in u$.

Clearly $\emptyset \notin u$.

Clearly u is closed under superset.

Note that $z = a \cap b \leftrightarrow j(z) = j(a) \cap j(b)$ (via the formula $z = a \cap b \equiv \forall t\ (t \in z \leftrightarrow t \in a \wedge t \in b)$) and then observing that this is absolute for M, V. More concisely we can say that $j(a \cap b) = j(a) \cap j(b)$ and thus that u is indeed a filter.

Similarly we can show countable completeness (in fact $< \kappa$-completeness): if $f : \omega \to u$ then $j(\bigcap_{n\in\omega} f(n)) = \bigcap_{n\in\omega} j(f(n))$ since $j(\omega) = \omega$. Technically we note that

$$z = \bigcap_{n \in y} f(n) \equiv \forall t\ (t \in z \leftrightarrow \forall n \in y\ t \in f(n))$$

(with free variables z, y, f) and using $y = \omega = j(\omega)$ and $j(f(n)) = j(f)(j(n)) = j(f)(n)$.

Finally to see that u is an ultrafilter: if $a \cup b = \kappa$ then $j(a) \cup j(b) = j(\kappa) \ni \kappa$ so that one of $\kappa \in j(a)$ or $\kappa \in j(b)$ holds as required.

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13. Work in **ZFC**.

    This question extends question 8.

    The Π_1 formulae are the negations of the Σ_1 formulae. Derive the analogous results for Π_1 formulae as for Σ_1 formulae.

    Show that 'r is a well-order on x' is equivalent (in **ZFC**) to both a Σ_1 formula and to a (different) Π_1 formula and deduce that it is absolute for non-empty transitive classes satisfying enough of **ZFC**.

    Prove by hand (i.e. without using the previous part) that 'r is a well-order on x' is absolute for transitive classes satisfying enough of **ZFC**.

    Solution: We can express 'r is a well-order on x' as 'r is an order on x and $\forall s\ (s \subseteq x \wedge s \neq \emptyset \rightarrow \exists m \in s\ \forall t \in m\ t \notin s)$' which is Π_1 (the only unbound quantifier is that $\forall s$).

    Using **ZFC** we can also express this as $\exists \alpha \in \mathrm{On}\ \exists f : x \to \alpha\ f$ is an $\in -r$ isomorphism which is Σ_1 (when writing $\alpha \in \mathrm{On}$ as α transitive and totally ordered by $\in$ using **Foundation**).

    That these are equivalent is witnessed by the Mostowski collapse along r in one direction and by defining $trs \leftrightarrow f(t) \in f(s)$ in the other direction.

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