Axiomatic Set Theory

Sheet 4 — TT21

Section A

1. Prove that for any infinite cardial κ, $cf(\kappa)$ is a regular cardinal and show that every infinite successor cardinal κ^+ is regular.

2. Suppose κ is an uncountable regular cardinal and let $g : \kappa \to \kappa$ be any function.

   Show that for any $\alpha \in \kappa$ there is $\beta \in \kappa$ with $\alpha \subseteq \beta$ and such that β is closed under g, i.e. $\gamma \in \beta \to g(\gamma) \in \beta$.

Rolf Suabedissen: rolf.suabedissen@maths.ox.ac.uk

Section B

3. Work in **ZF+V=L**.
   (a) Show that for ordinals $\alpha > \omega$, $L_\alpha = V_\alpha$ if and only if $\alpha = \aleph_\alpha$.
   (b) Show that there are ordinals α with $\alpha = \aleph_\alpha$.
   (c) Indicate briefly why the existence of regular ordinals α with $\alpha = \aleph_\alpha$ implies the consistency of **ZF**.

4. Suppose κ, λ are infinite cardinals.
   (a) Show that if $cf(\kappa) \leq \lambda \leq \kappa$ then $\kappa < \kappa^\lambda$.
   (b) Show that if $\lambda < cf(\kappa)$ and for every cardinal $\mu < \kappa$ we have $2^\mu \leq \kappa$ then $\kappa^\lambda = \kappa$.
   (c) If **GCH** is assumed, give a simple formula (with three non-trivial cases) for computing κ^λ.

5. Work in **ZFC**.

   Suppose κ is an uncountable regular cardinal such that for every $\mu < \kappa$ we have $2^\mu < \kappa$ (this is called strongly inaccessible).

   Note that ω is strongly inaccessible and regular (but of course not uncountable).
   (a) Show that if $\alpha < \kappa$ then $|V_\alpha| < \kappa$.
   (b) Show that $|V_\kappa| = \kappa$.
   (c) Indicate why $(V_\kappa, \in) \models$ **ZFC**.
   (d) Deduce that if ZFC is consistent then it can't prove the existence of a strongly inaccessible cardinal.
   (e) Show that if κ is an uncountable regular cardinal such that for every $\mu < \kappa$ we have $\mu^+ < \kappa$ (in V) then κ is strongly inaccessible in L.

Mathematical Institute, University of Oxford
Rolf Suabedissen: rolf.suabedissen@maths.ox.ac.uk

Section C

6. Assume **ZF**.

   For a set a, define $L[a]$ by recursion (with parameter a) on On by

$$L_0 = TC(\{a\})$$
$$\forall\alpha \in \text{On } L[a]_{\alpha+1} = Def(L[a]_\alpha)$$
$$\forall\gamma \in \text{Lim } L[a]_\gamma = \bigcup_{\beta\in\gamma} L[a]_beta$$

   and set

$$L[a] = \bigcup_{\alpha\in\text{On}} L[a]_\alpha.$$

   (a) Show that $L[a] \models$ **ZF**.

   (b) Show that if $a \subseteq \text{On}$ then $L[a] \models$ **ZF**.

   (c) Show that if $a \subseteq \omega$ then $L[a] \models$ **GCH**.

   (d) (very difficult) Show that if $a \subseteq \omega_1$ and **V=L[a]** then $L[a] \models$ **CH**

Rolf Suabedissen: rolf.suabedissen@maths.ox.ac.uk