Axiomatic Set Theory

Sheet 4 — TT21

Section A

1. Prove that for any infinite cardial κ, $cf(\kappa)$ is a regular cardinal and show that every infinite successor cardinal κ^+ is regular.

   Solution: The first bit follows from $cf(cf(\kappa)) = cf(\kappa)$ (lecture notes).

   The second one follows because a κ-union of κ-sized sets has size (at most κ). More formally, assume that $\alpha < \kappa^+$ and $f : \alpha \to \kappa^+$ unbounded, meaning

   $$\kappa^+ = \sup f[\alpha] = \bigcup_{\beta \in \alpha} f(\beta).$$

   Then $|\alpha| \leq \kappa$ and by assumption each $|f(\beta)| \leq \kappa$ so that $\left|\bigcup_{\beta\in\alpha} f(\beta)\right| \leq \kappa \times \kappa = \kappa$ a contradiction.

2. Suppose κ is an uncountable regular cardinal and let $g : \kappa \to \kappa$ be any function.

   Show that for any $\alpha \in \kappa$ there is $\beta \in \kappa$ with $\alpha \subseteq \beta$ and such that β is closed under g, i.e. $\gamma \in \beta \to g(\gamma) \in \beta$.

   Solution: Recursively define $\alpha_0 = \alpha$, $\alpha_{n+1} = \alpha_n \cup \sup g[\alpha_n]$ and $\beta = \sup_{n\in\omega} \alpha_n = \bigcup_{n\in\omega} \alpha_n$. The second step makes sense since $g[\alpha_n]$ is bounded in κ by regularity of κ and hence $\sup g[\alpha_n] \in \kappa$. Since κ is uncountable and regular (in particular it has uncountable cofinality) we also have $\beta \in \kappa$.

   Finally, if $\gamma \in \beta$ then $\gamma \in \alpha_n$ for some $n \in \omega$ and so $g(\gamma) \in \alpha_{n+1} \subseteq \beta$.

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Section B

3. Work in **ZF+V=L**.
   (a) Show that for ordinals $\alpha > \omega$, $L_\alpha = V_\alpha$ if and only if $\alpha = \aleph_\alpha$.
   (b) Show that there are ordinals α with $\alpha = \aleph_\alpha$.
   (c) Indicate briefly why the existence of regular ordinals α with $\alpha = \aleph_\alpha$ implies the consistency of **ZF**.

Solution:

(a) Suppose $L_\alpha = V_\alpha$. Then $|\alpha| = |L_\alpha| = |V_\alpha| > \aleph_0$ (since $\omega < \alpha$) and thus $|V_\alpha| \geq \aleph_\alpha$. (Note that for $\alpha > \omega^2$ we have $\alpha = \omega + \alpha$ and inductively $|V_{\omega+\alpha}| = \beth_\alpha \geq \aleph_\alpha$.) Hence $\alpha \geq |\alpha| \geq \aleph_\alpha$. Of course $\alpha \leq \aleph_\alpha$ is a straightforward induction.

Now suppose $\alpha = \aleph_\alpha$. We always have $L_\alpha \subseteq V_\alpha$.

Now α is an infinite limit ordinal and a cardinal (as it equals $\aleph_\alpha$) and is uncountable (since $\omega \neq \aleph_\omega$). Now let $x \in V_\alpha$. As α is a limit ordinal there is $\beta < \alpha$ with $\omega^2 < \beta$ and $x \in V_\beta$ and hence $|TC(x)| \leq |V_\beta| = \beth_\beta = \aleph_\beta < \aleph_\alpha$ (by **GCH** from **V=L**). Thus $x \in H_{\aleph_\alpha}$ and since **ZF+V=L** implies $H_\kappa = L_\kappa$ for every cardinal κ we get $x \in L_{\aleph_\alpha} = L_\alpha$.

(b) We construct fixed points of $\alpha \mapsto \aleph_\alpha$ as usual (this is a continuous, strictly increasing function so has arbitrarily large fixed points by an earlier sheet).

(c) First assume $L \models \alpha = \aleph_\alpha$, i.e. work inside L: If α is regular uncountable cardinal then L_α satisfies **ZF-Powerset** and V_α satisfies **Powerset**. If $\alpha = \aleph_\alpha$ then $L_\alpha = V_\alpha$ so that $V_\alpha \models$ **ZF**.

Now assume that $V \models \alpha = \aleph_\alpha$ and don't assume $V = L$. Being a cardinal is downwards absolute (it says something like $\forall f\ \forall \beta \in \kappa\ f$ is not a surjection $\beta \to \kappa$) so $\aleph_\alpha^L \leq \aleph_\alpha^V$ (inductively). Hence we have $\alpha \geq \aleph_\alpha^L$ or equivalently $L \models \alpha \geq \aleph_\alpha$. As before we always have $L \models \alpha \leq \aleph_\alpha$, so $L \models \alpha = \aleph_\alpha$ and we can do the above argument inside L.

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4. Suppose κ, λ are infinite cardinals.

   (a) Show that if $cf(\kappa) \leq \lambda \leq \kappa$ then $\kappa < \kappa^\lambda$.

   (b) Show that if $\lambda < cf(\kappa)$ and for every cardinal $\mu < \kappa$ we have $2^\mu \leq \kappa$ then $\kappa^\lambda = \kappa$.

   (c) If **GCH** is assumed, give a simple formula (with three non-trivial cases) for computing κ^λ.

Solution:

(a) Let $f : \lambda \to \kappa$ be unbounded. Then $\forall \alpha \in \lambda$, $|f(\alpha)| < \kappa$ so that by König's Lemma

$$|\kappa| = \left|\bigcup_{\alpha\in\lambda} f(\alpha)\right| \leq \sum_{\alpha\in\lambda} |f(\alpha)| < \prod_{\alpha\in\lambda} \kappa = \kappa^\lambda.$$

(b) Note that if $\lambda < cf(\kappa)$ then for every $f : \lambda \to \kappa$ there is $\alpha \in \kappa$ with $f : \lambda \to \alpha$ (α being an upper bound for $ran(f)$ which is bounded in κ). Thus

$$\begin{aligned}
\kappa^\lambda &= |\{f : \lambda \to \kappa\}| \\
&= \left|\bigcup_{\alpha\in\kappa} \{f : \lambda \to \alpha\}\right| \\
&\leq \sum_{\alpha\in\kappa} \lambda^{|\alpha|} \\
&\leq \sum_{\alpha\in\kappa} (2^\lambda)^{|\alpha|} \\
&= \sum_{\alpha\in\kappa} 2^{\lambda\otimes|\alpha|} \\
&\leq \sum_{\alpha\in\kappa} \kappa = \kappa \otimes \kappa = \kappa
\end{aligned}$$

(since $\mu = \lambda \otimes |\alpha| < \kappa$ and the assumption $2^\mu \leq \kappa$).

Of course since $1 \leq \lambda$ we have $\kappa \leq \kappa^\lambda$.

(c) We note that **GCH** implies the condition in part (b) (since $\mu < \kappa$ gives $2^\mu = \mu^+ \leq \kappa$).

We claim:

$$\kappa^\lambda = \begin{cases} \kappa; & \lambda < cf(\kappa) \\ \kappa^+; & cf(\kappa) \leq \lambda \leq \kappa \\ \lambda^+; & \kappa < \lambda. \end{cases}$$

Most of the work was done before. It remains to observe that

$$\kappa^+ = 2^\kappa \leq \kappa^\kappa \leq 2^{\kappa\otimes\kappa} = 2^\kappa = \kappa^+$$

and if $\kappa < \lambda$ then

$$\lambda^+ = 2^\lambda \leq \kappa^\lambda \leq 2^{\kappa\otimes\lambda} = 2^\lambda = \lambda^+$$

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5. Work in **ZFC**.

   Suppose κ is an uncountable regular cardinal such that for every $\mu < \kappa$ we have $2^\mu < \kappa$ (this is called strongly inaccessible).

   Note that ω is strongly inaccessible and regular (but of course not uncountable).

   (a) Show that if $\alpha < \kappa$ then $|V_\alpha| < \kappa$.

   (b) Show that $|V_\kappa| = \kappa$.

   (c) Indicate why $(V_\kappa, \in) \models$ **ZFC**.

   (d) Deduce that if ZFC is consistent then it can't prove the existence of a strongly inaccessible cardinal.

   (e) Show that if κ is an uncountable regular cardinal such that for every $\mu < \kappa$ we have $\mu^+ < \kappa$ (in V) then κ is strongly inaccessible in L.

Solution:

(a) Induction on α with successor step $|V_{\alpha+1}| = 2^{|V_\alpha|} < \kappa$ if $|V_\alpha| < \kappa$ and at limit steps (for $\alpha \in \kappa$ so that $|\alpha| < \kappa$) regularity.

(b) $\leq$ follows from (a) and $\geq$ follows from $\kappa \subseteq V_\kappa$.

(c) Only **Replacement** needs checking since κ is an infinite limit ordinal. For this follow the usual proof to construct

$$z = \{y : y \in V_\kappa \wedge \exists x \in d\ \psi^{V_\kappa}(a_1, \ldots, a_n, x, y)\}$$

which will witness **Replacement** provided $z \in V_\kappa$.

But since $d \in V_\kappa$ we have $d \subseteq V_\alpha$ for some $\alpha \in \kappa$ so that $|d| < \kappa$ and thus writing α_y for the least $\alpha \in \kappa$ with $y \in V_\alpha$ we get that $z \subseteq V_{\sup_{y\in\alpha} \alpha_y}$ and by regularity $\hat{\alpha} = \sup_{y\in\alpha} \alpha_y < \kappa$. Thus $z \in V_{\hat{\alpha}+1} \subseteq V_\kappa$.

(d) If **ZFC** is consistent then by a previous sheet it cannot prove the existence of a transitive set satisfying **ZFC** and hence the existence of a strongly inaccessible κ.

(e) We note that being an uncountable regular cardinal is downwards absolute for non-empty transitive classes satisfying **ZFC** (because we can write it as a Π_1 formula). Now assume that we have κ in V with the given properties (relative to V).

If $\alpha \in$ On then $|\alpha|^L \leq |\alpha|^V$ (since every V-cardinal is an L-cardinal) and so $(|\alpha|^+)^L \leq (|\alpha|^+)^V$ (previous fact and the same reason again) and hence $(2^{|\alpha|})^L = (|\alpha|^+)^L \leq (|\alpha|^+)^V$. Thus if $\alpha \in \kappa$ then $|\alpha| < \kappa$ and so by assumption $(2^{|\alpha|})^L < \kappa$.

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Section C

6. Assume **ZF**.

   For a set a, define $L[a]$ by recursion (with parameter a) on On by

$$L_0 = TC(\{a\})$$
$$\forall\alpha \in \text{On } L[a]_{\alpha+1} = Def(L[a]_\alpha)$$
$$\forall\gamma \in \text{Lim } L[a]_\gamma = \bigcup_{\beta\in\gamma} L[a]_beta$$

   and set

$$L[a] = \bigcup_{\alpha\in\text{On}} L[a]_\alpha.$$

   (a) Show that $L[a] \models$ **ZF**.

   (b) Show that if $a \subseteq \text{On}$ then $L[a] \models$ **ZF**.

   (c) Show that if $a \subseteq \omega$ then $L[a] \models$ **GCH**.

   (d) (very difficult) Show that if $a \subseteq \omega_1$ and **V=L[a]** then $L[a] \models$ **CH**

7. Show that there are arbitrarily large $\alpha \in \omega_1$ such that $\mathcal{P}(\omega) \cap L_{\alpha+1} \setminus L_\alpha \neq \emptyset$.

   Solution: We start by working in L (i.e. under **ZFC** + **V=L** so that $H_{\omega_1} = L_{\omega_1}$.

   Since $\mathcal{P}(\omega)$ is uncountable and each $L_\alpha, \alpha \in \omega_1$ is countable (note that $L_\alpha^L = L_\alpha$) we cannot have $\mathcal{P}(\omega) \subseteq L_\alpha$ for any $\alpha \in \omega_1$. But $\mathcal{P}(\omega) \subseteq H_{\omega_1} = L_{\omega_1}$ so the result follows.

   Now without assuming $V = L$ we see that $\mathcal{P}(\omega)^L \subseteq \mathcal{P}(\omega)$ (and $L_\alpha^L = L_\alpha^V$).

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