1 Version Information

24.05.21: a lot more minor corrections (thanks to Joel Summerfield).
08.02.18: Corrections to the Levy Reflection Principle.
10.01.18: More corrections and some minor additions.
02.03.17: More corrections (thanks to Peter Neumann).
08.02.17: Added material up to $L \models V = L$ and fixed more errors.
27.01.17: Many thanks to Peter Neumann for the numerous little errors he spotted.
16.01.17: I have rewritten and restructured some of the introductory material. At the moment these notes cover the course up to and including the definition of the cumulative hierarchy.

2 How to use the Lecture Notes

These lecture notes start with a lot of technicalities the reason for which will become clearer (I hope) later on. In the lectures, I will thus start with Section 3 and then skip to Section 7, go back to Section 5 and then proceed from Section 8 onwards. I will not cover Section 4 systematically but as the need arises.

If you find any mistakes, please contact me.

3 Background

We work in first order logic with equality and are concerned with theories with one binary relation, written $\in$. The language is called the Language of Set Theory (LST).

Our objects are going to be called sets and denoted (generally) by small latin letters.

For quantifiers, we use the abbreviations

$$\forall x \in y \ \psi \equiv \forall x \, [x \in y \to \psi]$$

and

$$\exists x \in y \ \psi \equiv \exists x \, [x \in y \wedge \psi] \, .$$

Finally we single out a specific collection of formulae: we call a formula a Δ_0 formula if every quantifier is bounded. Formally, we define the collection of Δ_0 formula (in the metatheory) by recursion: it is the smallest collection of formulae that contains the atomic ones ($x \in y$ and $x = y$) is closed under the logical connectives $\wedge, \vee, \to, \leftrightarrow, \neg$ and if ψ is Δ_0 then so are $\forall x \in y \ \psi$ and $\exists x \in y \ \psi$ (y must be a variable).

Note that it might not make sense to talk about the 'collection' of Δ_0 formulae. This depends on your metatheory! But for any specific formula, it is trivial to verify (in any reasonable metatheory) whether or not this is Δ_0.

3.1 Mathematics and Metamathematics

Note that we are, mostly, studying $\in$-structures from the 'outside'. Of course, the question arises in which theory we do this, i.e. what our 'metatheory' is. We are purposely vague about this. Any reasonable finitistic metatheory should work. It is however important that we do not confuse the metatheory and the theory. We will, for example, prove theorems within our theory (see for example the section on Ordinals). We will also prove theorems which cannot even be stated about in our theory: for example if we say 'for every formula ϕ of LST ...', this is a theorem in the metatheory.

4 Defined Notions

4.1 Classes

A *class* C_ϕ is a formula $\phi(t)$ with one free variable (here t). Instead of writing $\phi(x)$ we write $x \in C_\phi$.

We frequently do not give an explicit formula for a class but simply denote it by a capital latin letter.

In the following we define various notions for sets. When these make sense for classes, we will use the same notation. As an example of this, if C_ϕ is a class given by the formula $\phi(t)$ we write

$$\forall x \in C_\phi \ \psi \equiv \forall x \, [x \in C_\phi \to \psi] \equiv \forall x \, [\phi(x) \to \psi] \, .$$

4.1.1 The Universe

We will write U for the class given by the formula $t = t$ (so that trivially $\forall x \ x \in U$).

Note however that the formula $\forall x \in U \ \psi$ is not Δ_0: it is after all just an abbreviation for the formula $\forall x \, [x = x \to \psi]$ and this is not Δ_0.

4.2 Defined Notions

To enhance readability of our formulae, we define abbreviations for formulae of LST.

The formulae given are somewhat arbitrary, there are plenty others which are equivalent. We try to give Δ_0-formulae - this is important later on when we talk about absoluteness.

We will often define something like

$$z = \{x, y\} \equiv \phi(x, y, z)$$

but then use $\{x, y\} \in w$ to mean $\exists t \in w \, [t = \{x, y\}]$ or more explicitly $\exists t \in w \ \phi(x, y, t)$. This is Δ_0 provided that the definition of $w = \{x, y\}$, namely ϕ, was Δ_0.

We will also write $w \in \{x, y\}$. This is harder to define abstractly, so we are (sometimes) explicit about it.

4.3 Comprehension

For a formula ϕ
$$z = \{y : \phi(y)\} \equiv \forall t \, [t \in z \leftrightarrow \phi(t)] \, .$$

4.4 Subset

$$x \subseteq y \equiv \forall t \in x \; t \in y.$$

4.5 Emptyset

$$\begin{aligned} z = \emptyset \equiv & \forall t \in z \, [t \neq t] \\ t \in \emptyset \equiv & t \neq t \\ \emptyset \in t \equiv & \exists z \in t \; z = \emptyset. \end{aligned}$$

4.6 Unordered Pair

$$z = \{x, y\} \equiv x \in z \wedge y \in z \wedge \forall t \in z \, [t = x \vee t = y]$$

and
$$t \in \{x, y\} \equiv t = x \vee t = y.$$

We use the shorthand
$$\{x\} \equiv \{x, x\} \, .$$

4.7 Ordered Pair

$$\begin{aligned} z = \langle x, y \rangle \equiv z = \{\{x\}, \{x, y\}\} \equiv & \exists t \in z \, [t = \{x, x\}] \wedge \\ & \exists t \in z \, [t = \{x, y\}] \wedge \\ & \forall t \in z \, [t = \{x, x\} \vee t = \{x, y\}] \, . \end{aligned}$$

Note that the first statement only makes sense if $\{x\}$ and $\{x, y\}$ are objects of our model, whereas the full formula is defined independently of the existence of $\{x\}$ and $\{x, y\}$.

$$z \text{ is an ordered pair} \equiv \exists a \in z \exists b \in z \exists x \in a \exists y \in b \; z = \langle x, y \rangle$$

We could of course simply write down the more obvious

$$z \text{ is an ordered pair} \equiv \exists x \exists y \; z = \langle x, y \rangle$$

but this is not Δ_0 and we prefer Δ_0 formulae for reasons which will become clear later (see Lemma 1).

We will frequently refer to the first and second coordinate of the ordered pair. To do so we define

$$x = \pi_1(z) \equiv z \text{ is an ordered pair } \wedge \exists b \in z \exists y \in b \, [z = \langle x, y \rangle]$$

and

$$y = \pi_2(z) \equiv z \text{ is an ordered pair } \wedge \exists a \in z \exists x \in a \, [z = \langle x, y \rangle] \, .$$

We will also use the fact that

$$z = z' \text{ are ordered pairs } \leftrightarrow \exists x, y \ \ [\pi_1(z) = x = \pi_1(z') \wedge \pi_2(z) = y = \pi_2(z')]$$

4.8 Relations

We are only interested in binary relations, so simply use the word 'relation' for binary relations.

$$r \text{ is a relation } \equiv \forall t \in r \ t \text{ is an ordered pair.}$$

$$r \text{ is a transitive relation } \equiv r \text{ is a relation } \wedge \forall u \in r \forall v \in r \, [\pi_2(u) = \pi_1(v) \to \langle \pi_1(u), \pi_2(v) \rangle \in z] \, .$$

It is somewhat unclear how the above is translated into a formula of LST (i.e. how to eliminate the defined notions). Here is one way to do that:

$$\begin{aligned} r \text{ is a transitive relation } \equiv & r \text{ is a relation } \wedge \\ & \forall u \in r \forall v \in r \exists x, y, z \\ & \quad [[x = \pi_1(u) \wedge y = \pi_2(u) \wedge y = \pi_1(v) \wedge z = \pi_2(v)] \to \exists w \in r \, [w = \langle x, z \rangle]] \, . \end{aligned}$$

In this formula, not every quantifier is bounded, but we can write one down where every quantifier is indeed bounded: replace the last bit by

$$\exists a \in u \exists x \in a \exists b \in u \exists y \in b \exists c \in v \exists z \in c \, [u = \langle x, y \rangle \wedge v = \langle y, z \rangle \to \exists w \in r \, [w = \langle x, z \rangle]] \, .$$

The more you do this, the more unwieldy the formula becomes. We will in the future hence avoid these complicated formulae, but you should always check that you can carry out this replacement.

We will need to consider relations on sets so we will define

$$x \in dom(r) \equiv r \text{ is a relation } \wedge \exists z \in r \, [x = \pi_1(z)]$$

and

$$y \in ran(r) \equiv r \text{ is a relation } \wedge \exists z \in r \, [y = \pi_2(z)]$$

as well as the classes (!)

$$dom(r) = \{x : x \in dom(r)\} \quad ran(r) = \{y : y \in ran(r)\} \, .$$

Although these definitions seem self-referential, they are not: the string '$x \in dom(r)$' is simply replaced by 'r is a relation$\wedge \exists z \in r \, [x = \pi_1(z)]$'. In particular, if r is not a relation then $dom(r) = \emptyset$.

Note that we seem to have defined infinitely many classes here, which is of course a bad thing. But in practice we will only ever use finitely many instances of this or insist on r being a set and having sufficiently many axioms available to show that this implies that $dom(r)$ and $ran(r)$ are sets.

So we can define

$$r \text{ is a relation on } x \equiv r \text{ is a relation } \wedge dom(r) \subseteq x \wedge ran(r) \subseteq x$$

and

$$r \text{ is a transitive relation on } x \equiv r \text{ is a relation on } x \wedge \forall u, v, w \in x\, [\langle u, v\rangle \in r \wedge \langle v, w\rangle \in r \to \langle u, w\rangle \in r]\,.$$

Note that this is Δ_0 (provided r and x are sets).

Similarly

$$r \text{ is a reflexive relation on } x \equiv r \text{ is a relation on } x \wedge \forall t \in x\, [\langle t, t\rangle \in r]$$

$$r \text{ is an irreflexive relation on } x \equiv r \text{ is a relation on } x \wedge \forall t \in x\, [\langle t, t\rangle \notin r]\,.$$

Of course, usually we write binary relations in infix notation, i.e. arb instead of $\langle a, b\rangle \in r$.

4.9 Orders

$$r \text{ is a partial strict order on } x \equiv r \text{ is a transitive, irreflexive relation on } x.$$

$$\begin{aligned} r \text{ is a total strict order on } x \equiv\, & r \text{ is a transitive, irreflexive relation on } x \wedge \\ & \forall u, v \in x\, [\langle u, v\rangle \in r \vee \langle v, u\rangle \in r \vee u = v]\,. \end{aligned}$$

$$\begin{aligned} r \text{ is a strict well-order on } x \equiv\, & r \text{ is a strict total order on } x \wedge \\ & \forall p\, [p \subseteq x \wedge p \neq \emptyset \to \exists m \in p\, [\forall t \in p\, \neg\, [t\ r\ m]]] \end{aligned}$$

Note that this latter definition is not a Δ_0-formula (and in fact cannot be replaced by one).

4.10 Functions

$$f \text{ is a function } \equiv f \text{ is a relation } \wedge \forall x \in dom(f) \forall y \in ran(f) \forall y' \in ran(f)\, [xfy \wedge xfy' \to y = y']$$

We can replace this by a Δ_0 formula (as it stands it is not since $dom(f)$ and $ran(f)$ are defined notions and not variables).

If we have shown that f is a function and $x \in dom(f)$, we will write $f(x)$ for the unique y such that xfy.

4.11 Class functions

Note that the notion of being a function makes sense for classes. In this case we talk about 'class functions'. We spell out what this means formally (because class functions are central to what we will be doing).

Definition 1. Suppose $\phi(t)$ is a formula of LST. We say that ϕ is a class function if and only if

$$\begin{gathered}\forall t\ \ [\phi(t) \rightarrow t \text{ is an ordered pair}] \wedge \\ \forall x \forall y \forall y'\ \ [\phi(\langle x, y\rangle) \wedge \phi(\langle x, y'\rangle) \rightarrow y = y'] \,.\end{gathered}$$

Usually we denote class functions by capital latin letters and write $F(x) = y$ for $\phi(\langle x, y\rangle)$ (where ϕ is the formula defining F).

If A, B are classes, we say that $F : A \rightarrow B$ is a class function if and only if

$$\begin{gathered}F \text{ is a class function } \wedge \\ dom(F) = A \wedge ran(F) \subseteq B\end{gathered}$$

4.12 Union

$$z = \bigcup x \equiv [\forall y \in x \forall t \in y\, [t \in z]] \wedge [\forall t \in z \exists y \in x\ \ [t \in y \wedge y \in x]]$$

and

$$z = x \cup y \equiv x \subseteq z \wedge y \subseteq z \wedge \forall t \in z\, [t \in x \vee t \in y] \,.$$

Again, you might prefer the more natural

$$z = \bigcup x \equiv \forall t\ \ [t \in z \leftrightarrow \exists y \in x\ t \in y]$$

but this is not Δ_0.

4.13 Powerset

$$z = \mathcal{P}\left(x\right) \equiv \forall t\, [t \in z \leftrightarrow t \subseteq x] \,.$$

Although we won't in fact prove it in this course (but I will frequently remark on it), there is no Δ_0 formula which we could use here.

4.14 Successor

$$z = x + 1 \equiv z = x \cup \{x, x\} \equiv x \subseteq z \wedge x \in z \wedge \forall t \in z\, [t \in x \vee t = x] \,.$$

4.15 Inductive Set

$$\mathrm{Ind}(x) \equiv \emptyset \in x \wedge \forall t \in x\, [t + 1 \in x] \,.$$

5 Axiom Summary

Recall the definitions:

$$\begin{aligned}
x \subseteq y \equiv & \forall t \in x \ t \in y \\
z = \emptyset \equiv & \forall t \in z \ t \neq t \\
z = \{x, y\} \equiv & x \in z \wedge y \in z \wedge \forall t \in z \ [t = x \vee t = y] \\
z = \bigcup x \equiv & \forall t \in z \exists y \in x \ [t \in y] \wedge \forall y \in x \forall t \in y \ [t \in z] \\
z = \mathcal{P}(x) \equiv & \forall t \ [t \subseteq x \rightarrow t \in z] \wedge \forall t \in z \, [t \subseteq x] \\
z = S(x) \equiv & x \in z \wedge \forall t \in x \ [t \in z] \wedge \forall t \in z \ [t = x \vee t \in x]
\end{aligned}$$

and then state (any free variables are implicitly universally quantified)

Extensionality

$$x \subseteq y \wedge y \subseteq x \rightarrow x = y$$

Separation For each formula $\phi(v_1, \ldots, v_n, v_{n+1})$ with all free variables shown

$$\forall x \exists y \ y = \{z \in x : \phi(a_1, \ldots, a_n, z)\}.$$

Emptyset

$$\exists z \ z = \emptyset$$

Pairing

$$\forall x \forall y \exists z \ z = \{x, y\}$$

Union

$$\forall x \exists y \ y = \bigcup x$$

Powerset

$$\forall x \exists y \ y = \mathcal{P}(x)$$

Replacement For each formula $\phi(p_1, \ldots, p_n, t, u)$ (with all free variables displayed)

$$\begin{aligned}
&\forall a_1, \ldots, a_n \forall d \\
&\qquad \forall x \in d \exists ! y \ \phi(a_1, \ldots, a_n, x, y) \\
&\qquad\qquad \rightarrow \\
&\qquad\qquad\qquad \exists z \ z = \{y : \exists x \in d \ \phi(a_1, \ldots, a_n, x, y)\}
\end{aligned}$$

Infinity

$$\exists z \, [\exists x \in z \ x = \emptyset \wedge \forall y \in z \exists w \in z \ w = S(y)]$$

Foundation

$$\forall x \, [x \neq \emptyset \rightarrow \exists y \in x \, [\forall z \in x \, [z \notin y]]]$$

Choice We will eventually state the **Well-ordering Principle** which is equivalent to the usual Axiom of Choice (see Part b, Set Theory).

5.1 Axiom Systems

We will sometimes work in axiom systems which do not include all of the above axioms. Standard abbreviations are

- **ZF**$^-$ (sometimes denoted by ZF*): **Extensionality** + **Separation** + **Emptyset** + **Pairing** + **Union** + **Powerset** + **Replacement** + **Infinity**.
- **ZF**: **ZF**$^-$ + **Foundation**.
- **ZFC**: **ZF** + **Choice**.

6 Alternative Axiom Summary

If you would like to remove the defined notions above, you may end up with something like the following:

Extensionality

$$\forall x \forall y \ \ [\forall z \ \ [z \in x \leftrightarrow z \in y] \to x = y]$$

Separation For each formula $\phi(p_1, \ldots, p_n, t)$ (with all free variables displayed)

$$\forall a_1, \ldots, a_n \forall x \exists y \forall z \ \ [z \in y \leftrightarrow z \in x \wedge \phi\left(a_1, \ldots, a_n, z\right)]$$

This is also called the **Comprehension Scheme** and is often written as

$$\forall a_1, \ldots, a_n \forall x \exists y \ y = \left\{z \in x : \phi\left(a_1, \ldots, a_n, z\right)\right\}.$$

Emptyset

$$\exists x \forall y \ y \notin x$$

Note that **Emptyset** in fact follows form **Separation** with the formula $\phi(t) \equiv t \neq t$ and the existence of any x (or by assuming that all models are non-empty).

Pairing

$$\forall x \forall y \exists z \forall t \left[t \in z \leftrightarrow [t = x \vee t = y]\right]$$

Note that this definition of $z = \{x, y\} \equiv \forall t \left[t \in z \leftrightarrow [t = x \vee t = y]\right]$ is not Δ_0, so we would have to prove manually that it is absolute for transitive non-empty classes $A \subseteq B$ (assuming 'enough' of **ZF**$^-$). With Comprehension this is equivalent to

$$\forall x \forall y \exists z \left[x \in z \wedge y \in z\right]$$

Union

$$\forall x \exists y \forall w \left[w \in y \leftrightarrow \exists z \left[w \in z \wedge z \in y\right]\right]$$

With Comprehension this is equivalent to

$$\forall x \exists y \forall z \forall w \left[w \in z \wedge z \in x \to w \in y\right]$$

Powerset

$$\forall x \exists y \forall z \left[z \in y \leftrightarrow \forall w \left[w \in z \to w \in x \right] \right]$$

With Comprehension this is equivalent to

$$\forall x \exists y \forall z \left[\forall w \left[w \in z \to w \in x \right] \to z \in y \right]$$

Replacement For each formula $\phi(p_1, \ldots, p_n, t, u)$ (with all free variables displayed)

$$\begin{aligned} &\forall a_1, \ldots, a_n \forall d \\ &\qquad \forall x \in d \exists ! y \; \phi(a_1, \ldots, a_n, x, y) \\ &\qquad\qquad \to \\ &\qquad\qquad\qquad \exists z \; z = \{ y : \exists x \in d \; \phi(a_1, \ldots, a_n, x, y) \} \end{aligned}$$

Infinity

$$\exists z \left[\exists x \in z \; x = \emptyset \wedge \forall y \in z \exists w \in z \; w = \bigcup \{ y, \{ y, y \} \} \right]$$

Foundation

$$\forall x \left[x \neq \emptyset \to \exists y \in x \left[\forall z \in x \left[z \notin y \right] \right] \right]$$

Choice We will eventually state the **Well-ordering Principle** which is equivalent to the usual Axiom of Choice (see Part b, Set Theory).

7 Relativization and Absoluteness

I will only define 'relativization' and 'absoluteness' for LST, but it can easily be defined for any theory. The examples in lectures are intended to be given in the relevant theories.

Definition 2. Given a class A and a formula ϕ of LST, the *relativization of* ϕ *to* A is the formula ϕ^A where each quantifier is bounded by A. Formally, by induction on the complexity of the formula we define

- $(x = y)^A \equiv (x = y)$;
- $(x \in y)^A \equiv (x \in y)$;
- $(\neg\phi)^A \equiv \neg\phi^A$;
- $(\phi \wedge \psi)^A \equiv (\phi^A \wedge \psi^A)$;
- $(\phi \vee \psi)^A \equiv (\phi^A \vee \psi^A)$;
- $(\phi \to \psi)^A \equiv (\phi^A \to \psi^A)$;
- $(\phi \leftrightarrow \psi)^A \equiv (\phi^A \leftrightarrow \psi^A)$;
- $(\exists x\ \phi)^A \equiv \exists x \in A\ \phi^A \equiv \exists x\ (x \in A \wedge \phi^A)$;
- $(\forall x\ \phi)^A \equiv \forall x \in A\ \phi^A \equiv \forall x\ (x \in A \to \phi^A)$.

If $a_1, \ldots, a_n \in A$ and ϕ free variables $x_1, \ldots, x_n$ then we also write

$$A \models \phi(a_1, \ldots, a_n) \equiv \phi(a_1, \ldots, a_n)^A.$$

This is just your standard 'interpretation of ϕ in the model A', except that of course A might not be a 'model' (it might not be a set but only a class) and the above is purely syntactic.

Remark 1 (Expanding on the previous sentence)**.** Intuitively, the relativization of a formula ϕ to a class A is simply the interpretation of of ϕ in the model $(A, \in)$. The problem we are facing is that if A is really a class and not a set (in our meta-theory), then it does not make sense to talk about the model $(A, \in)$. We cannot (or do not want to) use the semantic notions, so have to rely on purely syntactical defintions.

If you are willing to work in a meta-theory in which you can show that 'if a theory is consistent, then it has a model' this problem goes away. You will assume that $\mathbf{ZF}^-$ is consistent, and call its model $(U, \in)$ (so U **really** does exist as an object of study - just like you usually assume in ordinary mathematics that $\mathbb{R}$ really does exist as an object of study). You then work with a subobject A of U and can interpret formulae in $(A, \in)$ in the classical model theoretic sense.

If you do this, formally we will then be showing that 'if $\mathbf{ZF}^-$ is consistent, then so is $\mathbf{ZF}$ and $\mathbf{ZFC}$ and $\mathbf{ZFC} + \mathbf{CH}$'. But of course, it is very likely that if your meta-theory is strong enough to show that 'if a theory is consistent, then it has a model', then it will include some version of **Choice**, so that somewhat defeats the purpose.

If you are prepared to formally jump through the extra logical hoops, then you can get away with a much weaker meta-theory (some finitistic meta-theory) and prove the 'stronger' statement 'if **ZFC** + **CH** is inconsistent, then so is **ZF**$^-$'. For (most) practical purposes it is enough to think about the intuitive definition, but keep these comments in the back of your mind.

Definition 3. Suppose A is a class and $\phi(x_1, \dots, x_n)$ is a formula of LST (with all free variables shown) and Γ is a collection of sentences.

We say that A models ϕ (or A believes ϕ) in the context Γ, written $A \models_\Gamma \phi$ if and only if

$$\Gamma \vdash \forall a_1, \dots, a_n \in A \ \phi(a_1, \dots, a_n)^A.$$

If Δ is a collection of formulae of LST, we say that A models Δ (or A believes in Δ) in the context Γ, written $A \models_\Gamma \Delta$ if and only if for each ϕ from Δ, $A \models_\Gamma \phi$.

Usually we do not specify Γ explicitly and take it to be some suitable subcollection of **ZFC**.

Definition 4. Suppose $\phi(x_1, \dots, x_n)$ is a formula of LST (with all free variables shown), A, B are classes and Γ is a collection of sentences such that $\Gamma \vdash A \subseteq B$.

We say that ϕ is absolute for A, B in the context Γ if and only if

$$\Gamma \vdash \forall a_1, \dots, a_n \in A \ (\phi(a_1, \dots, a_n)^A \leftrightarrow \phi(a_1, \dots, a_n)^B).$$

In this case we write $A \preceq_\phi B$.

Note that often we do not mention Γ. For us, it will always be a subset of **ZFC** and we take it whatever subset we need in a proof.

Intuitively this says that the two models $(A, \in)$ and $(B, \in)$ have the same believe about some formula ϕ whenever that makes sense (i.e. all free variables are instantiated with elements of both A and B). Since formally, we will not be working with 'models' (of **ZF**$^-$ or **ZF** or **ZFC**), we have to give a syntactic definition which only relies on the existence of proofs (finite sequences of finite strings of symbols which follow some easily checkable rules).

Recall that a class A is transitive if and only if $\forall x \in A \ x \subseteq A$.

Definition 5. Suppose ϕ is a formula of LST.

We say that ϕ is absolute (for transitive classes satisfying Δ) [in the context Γ] if and only if for any (transitive) classes A, B (such that $\Gamma \vdash A \models \Delta$ and $\Gamma \vdash B \models \Delta$) and $\Gamma \vdash A \subseteq B$, ϕ is absolute for A, B in the context Γ.

Lemma 1. Δ_0 *formulae are absolute for transitive classes.*

Proof. Let A, B be transitive classes such that $A \subseteq B$.

We do this by induction of the complexity of the formula. By the definition of relativization, the only interesting steps are the quantifiers. We do the existential case. The universal case is similar or can be deduced by the replacement $\forall \equiv \neg\exists\neg$.

Case $\phi \equiv \exists x \in y\ \psi(x, y, x_1, \dots, x_n)$**:** Fix $x_1, \dots, x_n, y \in A$.

First assume $A \models \exists x \in y\ \psi(x, y, x_1, \dots, x_n)$, i.e. $\exists x \in A\ (x \in y \wedge \psi(x, y, x_1, \dots, x_n)^A)$. Find $x \in A$ such that $x \in y \wedge \psi(x, y, x_1, \dots, x_n)^A$. Since $A \subseteq B$ we have $x \in B$ and by absoluteness of ψ (inductive assumption) $\psi(x, y, x_1, \dots, x_n)^B$. Thus $B \models \exists x \in y\ \psi(x, y, x_1, \dots, x_n)$.

Conversely, assume that $B \models \exists x \in y\ \psi(x, y, x_1, \dots, x_n)$, i.e. $\exists x \in B\ (x \in y \wedge \psi(x, y, x_1, dots, x_n)^B)$. Find $x \in B$ such that $x \in y \in A$ and $\psi(x, y, x_1, \dots, x_n)^B$. By transitivity of A we have $x \in A$ and by absoluteness of ψ (inductive assumption) $\psi(x, y, x_1, \dots, x_n)^A$. Thus $A \models \exists x \in y\ \psi(x, y, x_1, \dots, x_n)$. $\square$

Remark 2. Note that the above in fact shows that if $\psi(x)$ is absolute for $A \subseteq B$ and $A \models \exists x\ \psi$ then $B \models \exists x\ \psi$. It is only for the reverse direction that we need transitivity and the fact that the quantifier is bounded.

Remark 3. This is a classic 'meta-theorem'. View it as a 'factory' that produces proofs: we will be interested (mostly) in specific applications of this to specific formulae. So for example we will have reason to show that 'pairing is absolute', i.e. that if (we can prove that) $A \subseteq B$ are non-empty transitive classes, then we can prove that $\forall x, y, z \in A\ [A \models z = \{x, y\} \leftrightarrow B \models z = \{x, y\}]$ (this is a sentence in LST). We could (and you should) go through these proofs by hand, the above lemma is simply an abbreviating step.

8 The Ordinals

In this section we develop just enough of the theory of ordinals to prove the results of the next section. You are strongly encouraged to consult the literature for more results about ordinals.

Unless otherwise specified, we work in $\mathbf{ZF}^- - \mathbf{Powerset}$ (it may be fun to figure out exactly which axioms are needed to prove the results below).

Definition 6.

$$\mathrm{On} = \{\alpha : \alpha \text{ is transitive} \wedge \in \text{ is a well-order on } \alpha\}$$

For the benefit of the reader we write out $\alpha \in \mathrm{On}$:

$$\begin{aligned} \forall x \in \alpha \forall y \in x\ y \in \alpha \quad & [\alpha \text{ is transitive}] \\ \wedge \forall x \in \alpha\ [x \notin x] \quad & \\ \wedge \forall x, y, z \in \alpha\, [x \in y \wedge y \in z \to x \in z] \quad & \\ \wedge \forall x, y \in \alpha\, [x \in y \vee y \in x \vee x = y] \quad & [\in \text{ is a strict total order on } \alpha] \\ \wedge \forall x\, [x \subseteq \alpha \wedge x \neq \emptyset \to \exists m \in x \forall z \in m\ z \notin x] \quad & [\in \text{ is well-founded on } \alpha] \end{aligned}$$

Lemma 2.

$$\boldsymbol{ZF} \vdash \alpha \in \mathrm{On} \leftrightarrow \alpha \textit{ is transitive and totally ordered by } \in$$

Thus if A, B are non-empty transitive classes satisfying (enough of) ***ZF*** *then $x \in \mathrm{On}$ is absolute.*

Lemma 3. *1.* $\forall \alpha \in \mathrm{On}\ \alpha \notin \alpha$*;*

2. $\emptyset \in \mathrm{On}$*;*

3. $\forall \alpha \in \mathrm{On}\ \alpha + 1 := \alpha \cup \{\alpha\} \in \mathrm{On}$*;*

4. $\forall \alpha, \beta \in \mathrm{On}\ \alpha \cap \beta \in \mathrm{On}$*;*

5. $\forall \alpha, \beta \in \mathrm{On}\ \ [\alpha \subseteq \beta \to \alpha \in \beta \vee \alpha = \beta]$*;*

6. $\forall \alpha \in \mathrm{On}\ \alpha \subseteq \mathrm{On}$*;*

Proof. 1. Suppose $\alpha \in \mathrm{On}$ and $\alpha \in \alpha$. Then by irreflexivity, $\alpha \notin \alpha$, a contradiction.

2. $\emptyset$ is vacuously transitive and $\in$ is vacuously a well-order on $\emptyset$.

3. Suppose $\alpha \in \mathrm{On}$.

   $\alpha + 1$ is transitive: Suppose $x \in y \in \alpha + 1$. Either $y \in \alpha$ and then by transitivity of α, $x \in \alpha$ or $y = \alpha$ and then $x \in \alpha$. In any case $x \in \alpha \subseteq \alpha+1$.

   $\in$ is irreflexive on $\alpha + 1$: Suppose $x \in \alpha + 1$. Then either $x \in \alpha$ so that by assumption $x \notin x$ or $x = \alpha$ and thus $\alpha \notin \alpha$ by (1).

   $\in$ is transitive on $\alpha + 1$: Suppose $x, y, z \in \alpha + 1$ and $x \in y \in z$. If $x, y, z \in \alpha$ then we get $x \in z$ from the assumption that $\alpha \in \mathrm{On}$. Next, assume $x = \alpha$. Then $y \neq \alpha$ by (1) so $y \in \alpha$. By transitivity of α we have $\alpha = x \in \alpha$ contradicting (1). Similarly if $y = \alpha$ (use $z \neq \alpha$). So, suppose $z = \alpha$. Then by transitivity of α we get $x \in \alpha = z$.

   $\in$ is a total order on $\alpha + 1$: We need to verify trichotomy, so suppose $x, y \in \alpha + 1$. If $x = y = \alpha$ we are done. If $x, y \in \alpha$ we are done by the assumption that $\alpha \in \mathrm{On}$. If (wlog) $x \in \alpha, y = \alpha$ then we are also done.

   $\in$ is well-founded on $\alpha + 1$: Now suppose $\emptyset \neq x \subseteq \alpha + 1$. If $x = \{\alpha\}$ then α is clearly the $\in$-minimal element of x (as $\alpha \notin \alpha$ by (1)). So assume $x \cap \alpha \neq \emptyset$: then $\emptyset \neq x \cap \alpha \subseteq \alpha$ so $x \cap \alpha$ has an $\in$-minimal element m. Now if $t \in x$ then either $t \in \alpha$ so that $t \notin m$ by construction of m or $t = \alpha$. But $\alpha = t \in x \in \alpha$ gives $\alpha \in \alpha$ by transitivity of α, a contradiction. Hence m is $\in$-minimal in x.

4. Suppose $\alpha, \beta \in \mathrm{On}$ and $x \in y \in \alpha \cap \beta$. By transitivity of α and β respectively we obtain that $x \in \alpha \cap \beta$. Now observe that the restriction of any well-founded strict total order to a subset (or subclass) is a well-founded strict total order.

5. Suppose $\alpha, \beta \in \mathrm{On}$ with $\alpha \subseteq \beta$ but $\alpha \neq \beta$. Then $\beta \setminus \alpha$ is a non-empty subset of β so has an $\in$-minimal element x.

   We claim $x = \alpha$: Firstly if $t \in x$ but $t \notin \alpha$ then by transitivity of β we have $t \in \beta \setminus \alpha$ contradicting $\in$-minimality of x. Thus $x \subseteq \alpha$.

   Secondly, if $t \in \alpha$ then by assumption (namely $\alpha \subseteq \beta$) $t \in \beta$. So one of $x \in t$ or $t \in x$ or $t = x$ must be true. If $x \in t$ then by transitivity of α we have $x \in \alpha$ contradicting $x \in \beta \setminus \alpha$. Similarly if $t = x$. Thus $t \in x$ giving $\alpha \subseteq x$, as required.

6. Suppose $\alpha \in \mathrm{On}$ and $x \in \alpha$. If $r \in t \in x$ then $r \in x$ by transitivity of $\in$ on α. Also, because α is transitive we have $x \subseteq \alpha$ so that $\in$ restricts to a well-founded strict total order on x. Hence $x \in \mathrm{On}$.

$\square$

Theorem 1. *$\in$ is a well-founded strict total order on* On*.*

Theorem 2. *If X is a non-empty subclass of* On*, then X has an $\in$-minimal element.*

Proof. We show:

$\in$ is a strict total order on On: By the lemma, $\in$ is irreflexive. If $\alpha, \beta, \gamma \in \mathrm{On}$ and $\alpha \in \beta \in \gamma$ then by transitivity of γ we have $\alpha \in \gamma$ as required. For totality, assume that $\alpha, \beta \in \mathrm{On}$. Let $x = \alpha \cap \beta$ and note that by the lemma $x \in \mathrm{On}$ and $x \subseteq \alpha$ and $x \subseteq \beta$. So by the lemma ($x \in \alpha$ or $x = \alpha$) and ($x \in \beta$ or $x = \beta$). Unless $x \in \alpha$ and $x \in \beta$ we have one of $\alpha \in \beta$, $\beta \in \alpha$ or $\alpha = \beta$. But if we assume $x \in \alpha$ and $x \in \beta$. Then $x \in \alpha \cap \beta = x$ contradicting the lemma.

$\in$ is well-founded on On: Suppose that x is a non-empty subset of On. Pick $\alpha \in x$ and let $y = \alpha \cap x$. If $y = \emptyset$ then α is $\in$-minimal in x. If on the other hand $y \neq \emptyset$ then y is a non-empty subset of α so has an $\in$-minimal element m. By construction $m \in x$. If $t \in x \cap m$ then as $m \in \alpha$ and α is transitive we have $t \in \alpha$ so $t \in x \cap \alpha = y$ contradicting minimality of y. Thus $x \cap m = \emptyset$ and m is in fact $\in$-minimal in x.

Note that this works independent of whether x is a class or a set. $\square$

Remark 4. Again, note that the first theorem is in fact a theorem of $\mathbf{ZF}^- -$ **Powerset**. The second theorem is a result in our meta-theory. It cannot even be stated in the theory, since we don't have classes in our theory.

Corollary 1 (Induction on On - another meta-theorem)**.** *Suppose that $\phi(t)$ is a formula. Then*

$$\mathbf{ZF}^- - \textbf{\textit{Powerset}} \vdash [\forall \alpha \in \mathrm{On}\, [\forall \beta \in \alpha\ \phi(\beta)] \to \phi(\alpha)] \to \forall \alpha \in \mathrm{On}\ \phi(\alpha).$$

Definition 7. Suppose that $\alpha \in \mathrm{On}$.

$$\begin{aligned}
\alpha \text{ is a successor} &\equiv \exists \beta \in \alpha \ \alpha = \beta + 1 \\
\alpha \text{ is a limit} &\equiv \alpha \neq \emptyset \wedge \alpha \text{ is not a successor} \\
\alpha \text{ is finite} &\equiv [\alpha \text{ is a successor} \vee \alpha = \emptyset] \wedge \forall t \in \alpha \ \ [t = \emptyset \vee t \text{ is a successor}] \\
x = \omega &\equiv x \in \mathrm{On} \wedge \mathit{Ind}(x) \wedge \forall t \in x \ \ [t \text{ is finite}] \\
\mathit{Lim} &= \{\gamma \in \mathrm{On} : \gamma \text{ is a limit}\}\,.
\end{aligned}$$

Proof. We need to check that $x = \omega$ is in fact a definition, i.e. that if x and y satsify the RHS then $x = y$: Note that $x, y \in \mathrm{On}$ gives wlog that $x \in y$ (or $x = y$ and we are done). Then x is 'finite' so in particular a successor, say $\beta + 1$. But then $\beta \in x$ and as x is inductive, $x = \beta + 1 \in x$, a contradiction. □

Lemma 4.

$$\forall z \ \ [\mathit{Ind}(z) \rightarrow \omega \subseteq z]$$

Proof. Suppose z is an inductive set. By induction on the elements of ω we show $\omega \subseteq z$. Formally, suppose $n \in \omega \setminus z$. As $\in$ is well-founded on ω, we may assume that n is $\in$-minimal in $\omega \setminus z$. As $\mathit{Ind}(z)$ we cannot have $n = \emptyset$. Thus $n = m + 1$ for some m and by transitivity of ω we have $m \in \omega$. By minimality of n, we must have $m \in z$. But $\mathit{Ind}(z)$ then gives $n = m + 1 \in z$, a contradiction. □

Remark 5. Often ω is defined as the 'smallest' inductive set containing $\emptyset$, i.e. $x = \omega \equiv \mathit{Ind}(x) \wedge \forall z\,[\mathit{Ind}(z) \rightarrow x \subseteq z]$ and then 'x is finiteq' as $x \in \omega$. The advantage of this is that it doesn't require the technology of the ordinals to make sense. The disadvantage is that it is less clear that ω is absolute for non-empty transitive classes satisfying enough of **ZF** (**Foundation** is crucial since otherwise 'being an ordinal' might not be absoulte).

Lemma 5. *Suppose $x \subseteq \mathrm{On}$ and x is a set. Then $\bigcup x \in \mathrm{On}$ and $\sup_{\in} x = \bigcup x$.*

Proof. If $r \in t \in \bigcup x$ then there is $\alpha \in x$ with $r \in t \in \alpha$ so $r \in \alpha \subseteq \bigcup x$ since α is transitive.

Next, if $t \in \bigcup x$ then $t \in \alpha$ for some $\alpha \in x$ so $t \in \mathrm{On}$. Hence $\bigcup x \subseteq \mathrm{On}$ and so $\bigcup x$ is well-ordered by $\in$.

Now let $\alpha_0 = \bigcup x$. If $\beta \in x$ and $t \in \beta$ then $t \in \alpha_0$ so that $\beta \subseteq \alpha_0$. Thus α_0 is an upper bound for x.

Finally, if $\alpha' \in \mathrm{On}$ is an upper bound for x and $t \in \alpha_0$ then find $\beta \in x$ with $t \in \beta \subseteq \alpha'$ (since α' is an upper bound for x). Thus $\alpha_0 \subseteq \alpha'$ as required. □

Lemma 6. *Suppose $\alpha \neq \emptyset$ is an ordinal.*
α is a limit ordinal if and only if $\forall \beta \in \alpha \ \beta + 1 \in \alpha$.

Proof. Suppose α is a non-empty limit ordinal and that $\beta \in \alpha$. By transitivity of α we have $\beta \subseteq \alpha$ so $\beta + 1 = \beta \cup \{\beta\} \subseteq \alpha$. Hence we must have $\beta + 1 \in \alpha$ or $\beta + 1 = \alpha$. Since α is a limit (not a successor), we can't have $\beta + 1 = \alpha$ (noting that β is an ordinal) and hence $\beta + 1 \in \alpha$ as claimed.

Conversely, suppose α is not a limit. Because $\alpha \neq \emptyset$, it is a successor, so $\alpha = \beta + 1$ for some ordinal β. Then $\beta \in \alpha$ and $\beta + 1 \notin \alpha$ as required. □

9 Recursion

A few more meta-theorems.

Theorem 3 (The informal Recursion Theorem)**.** *Suppose F is a class function on U and that $a \in U$. Then there is a 'unique' class function G on* On *such that:*

1. $G(0) = a$;
2. $\forall \alpha \in \mathrm{On}\ G(\alpha + 1) = F(G(\alpha))$;
3. $\forall \gamma \in \mathrm{Lim}\ G(\gamma) = \bigcup \{G(\alpha) : \alpha < \gamma\}$

We restate this more formally:

Definition 8. Suppose that F is a class function on U and that $a \in U$. We let

$$\begin{aligned}\psi_{F,a}(\alpha, g) \equiv & \alpha \in \mathrm{On} \wedge g \text{ is a function on } \alpha + 1 \wedge \\ & g(0) = a \wedge \\ & \forall \beta \in \alpha\ \ [g(\beta + 1) = F(g(\beta))] \wedge \\ & \forall \gamma \in \mathrm{Lim} \cap \alpha + 1\ \ \left[g(\gamma) = \bigcup \{g(\beta) : \beta \in \gamma\}\right]\end{aligned}$$

expressing that g is a function on $\alpha+1$ and that g satisfies the conditions above on its domain. We let

$$G_{F,a} \equiv \{\langle \alpha, y\rangle : \alpha \in \mathrm{On} \wedge [\exists g\ \ [\psi_{F,a}(\alpha, g) \wedge \langle \alpha, y\rangle \in g]]\}.$$

Theorem 4 (The Recursion Theorem)**.** *If* $\mathbf{ZF}^- - \boldsymbol{Powerset}$ *proves that F is a class function on U and $a \in U$, then* $\mathbf{ZF}^- - \boldsymbol{Powerset}$ *proves that $G_{F,a}$ is a class function on* On *and*

$$\begin{aligned}& G_{F,a}(0) = a \wedge \\ & \forall \beta \in \mathrm{On}\ G_{F,a}(\beta + 1) = F(G_{F,a}(\beta)) \wedge \\ & \forall \gamma \in \mathrm{Lim}\ G(\gamma) = \bigcup \{G(\beta) : \beta \in \gamma\}\end{aligned}$$

and $\mathbf{ZF}^- - \boldsymbol{Powerset}$ *proves that if G, H are class functions satisfying the displayed formula then $\forall \alpha \in \mathrm{On}\ H(\alpha) = G(\alpha)$.*

Even this is (technically speaking) not formal enough. For once, I give the formal version of this theorem without the uniqueness bit (with defined notions not yet eliminated):

Theorem 5 (The Formal Recursion Theorem)**.** *Suppose $\phi(t)$ is a formula with free variable displayed. As always we write $F = \{x : \phi(x)\}$ for the class defined by ϕ*

$$
\begin{aligned}
&\mathbf{ZF}^- - \textbf{\textit{Powerset}} \vdash \\
&[F \textit{ is a class function on } U] \rightarrow \\
&\quad \forall a \in U \\
&\qquad G_{F,a} \textit{ is a class function on } \mathrm{On} \wedge \\
&\qquad G_{F,a}(0) = a \wedge \\
&\qquad \forall \beta \in \mathrm{On}\ G_{F,a}(\beta+1) = F(G_{F,a}(\beta)) \wedge \\
&\qquad \forall \gamma \in \mathrm{Lim}\ G(\gamma) = \bigcup \{G(\beta) : \beta \in \gamma\}
\end{aligned}
$$

As an exercise, you can try to remove all the defined notions from this theorem.

Needless to say, for examination purposes the Informal Recursion Theorem is sufficient (though for the proof you will likely have to define $\psi_{F,a}$ and $G_{F,a}$).

Lemma 7. *Suppose* $\mathbf{ZF}^- -$ ***Powerset*** *proves that F is a class function on U and that $a \in U$.* $\mathbf{ZF}^- -$ ***Powerset*** *proves that for every $\alpha \in \mathrm{On}$ there is a unique g such that $\psi_{F,a}(\alpha, g)$.*

Proof. Induction on On:

For $\alpha = 0$, use $g = \{\langle 0, a\rangle\}$ (which exists by **Pairing**) and uniqueness follows from **Extensionality**.

For $\alpha = \beta + 1$: Let g' be the unique function for β which exists by the inductive hypothesis. Let $g = g' \cup \{\langle \alpha, g'(\beta)\rangle\}$ (this is a set by **Union**, **Pairing** and inductive hypothesis). It is clear that g is a function on $\alpha + 1$ which satsifies the conditions (noting that $\alpha+1 \setminus \beta+1 = \{\alpha\}$ so there are no new limit ordinals). Uniqueness follwos once more by the requirement of the conditions and **Extensionality**.

For $\gamma \in \mathrm{Lim}$: For each $\beta \in \gamma$, let g_β be the unique function given by the lemma for β and note that if $\beta < \beta'$ then $\hat{g} = g_{\beta'}|_{\beta+1}$ is a function on $\beta+1$ that satisfies $\psi_{F,a}(\beta', \hat{g})$, so must equal g_β. Next note that $\{g_\beta : \beta \in \gamma\} = \{g : \exists \beta \in \gamma\ \psi_{F,a}(\beta, g)\}$ is a set by **Replacement** . So $g = \{\langle \gamma, \bigcup\{g_\beta(\beta) : \beta \in \gamma\}\rangle\} \cup \bigcup\{g_\beta : \beta \in \gamma\}$ is a set by **Union** and **Pairing**. Now, by the note (that the g_β agree on the common elements of their domains) g is a function on $\gamma + 1$ and it is clear that g satisfies the conditions. □

Proof of the Recursion Theorem. Note that we do two proofs at once (for efficiency reasons): one proof giving the theorem as stated and one giving the theorem 'up to some limit ordinal γ'. The latter only uses the previous lemma for $\alpha < \gamma$, whereas the former uses the lemma for all ordinals.

We start by demonstrating that G is indeed a class function on On (resp. γ): it is clear from the definition of G that G is a relation from On to U. For $\alpha \in \mathrm{On}$ (resp. $\alpha < \gamma$), we apply the previous lemma to see that there is g such that $\psi_{F,a}(\alpha, g)$. Since $\alpha \in \alpha + 1$, there is $y \in U$ with $\langle \alpha, y\rangle \in g$. Hence the domain of G is all of On. Finally assume that there is α (resp. $\alpha < \gamma$) and

$y, y' \in U$ with $\langle \alpha, y \rangle, \langle \alpha, y' \rangle \in G$. Then we may take the witnessing g, g' and note that by the previous lemma $g = g'$ so that $y = y'$ as required.

Finally we need to check that G satisfies the formula (up to γ). But this follows directly from the definition of G and induction on On. $\square$

Remark 6. Let us note that if A, B are transitive non-empty classes satisfying (enough of) **ZF**, $\mathrm{On} \subseteq A \subseteq B$, F is absolute for A, B and $a \in A$ is given by a defined notion absolute for A, B then G is absolute for A, B.

This requires some careful checking (e.g. $\psi_{F,a}$ is absolute for A, B) but is essentially straightforward once we have that On is absolute for A, B.

There are some variations on recursion which we will use:

Theorem 6. *Suppose A, B are classes and that $F : A \times B \to B$ and $H : A \to B$ are class functions. Then there is a unique class function $G : A \times \mathrm{On} \to B$ such that:*

- $G(x, 0) = H(x)$;
- $\forall \alpha \in \mathrm{On}\ G(x, \alpha + 1) = F(x, G(x, \alpha))$;
- $\forall \gamma \in \mathrm{Lim}\ G(\gamma) = \bigcup_{\beta < \gamma} G(\beta)$.

Proof. As the proof of the Recursion Theorem, with ψ now being $\psi_{F,H}(x, \alpha, g)$ where $g(0) = a$ is replaced by $g(0) = H(x)$ and

$$G_{F,H} \equiv \{\langle \langle x, \alpha \rangle, y \rangle : \alpha \in \mathrm{On} \wedge x \in A \wedge [\exists g\, [\psi_{F,H}(x, \alpha, g) \wedge \langle \alpha, y \rangle \in g]]\}.$$

$\square$

Also, we may sometimes only define G up to some fixed ordinal α_0 (typically ω). The proof works as before, except that we insist on $\alpha + 1 < \alpha_0$ throughout.

Theorem 7. *Suppose A, B are classes and that $F : A \times B \to B$ and $H : A \to B$ are class functions and $0 < \alpha_0 \in \mathrm{On}$. Then there is a unique class function $G : A \times \alpha_0 \to B$ such that:*

- $G(x, 0) = H(x)$;
- $\forall \alpha \in \mathrm{On}\ [\alpha + 1 \in \alpha_0 \to G(x, \alpha + 1) = F(x, G(x, \alpha))]$;
- $\forall \gamma \in \mathrm{Lim} \cap \alpha_0\ G(\gamma) = \bigcup_{\beta < \gamma} G(\beta)$.

If $\alpha_0 = \omega$ then ***Replacement*** *is not needed.*

If A is a set (and ***Replacement*** *holds or A is a singleton (finite set)) then G is a set.*

10 The Cumulative (von Neumann) Hierarchy V

Definition 9. Let $F(x) = \mathcal{P}(x)$. Then $\mathbf{ZF}^-$ proves that F is a class function and we apply the Recursion Theorem with $a = \emptyset$ to obtain a class function V on On such that (writing V_α for $V(\alpha)$)

- $V_0 = \emptyset$;
- $\forall\alpha \in \mathrm{On}\ V_{\alpha+1} = \mathcal{P}(V_\alpha)$;
- $\forall\gamma \in \mathrm{Lim}\ V_\gamma = \bigcup_{\beta<\gamma} V_\beta$.

Abusing notation, we write $V = \bigcup_{\alpha\in\mathrm{On}} V_\alpha$.

V is called the Cumulative (von Neumann) Hierarchy.

Lemma 8. $\mathbf{ZF}^-$ *proves that for all* $\alpha \in \mathrm{On}$*:*

- V_α *is transitive;*
- $V_\alpha \subseteq V_{\alpha+1}$*;*
- $\alpha \in V_{\alpha+1}$*;*

Hence V *is a transitive non-empty class containing* On *and for* $\alpha, \beta \in \mathrm{On}$ *we have*

$$\alpha \subseteq \beta \to V_\alpha \subseteq V_\beta.$$

Proof. We prove this by simultaneous induction on α:

Base Case: $\alpha = \emptyset$: Vacuously $V_0 = \emptyset$ is transitive and contained in $V_1 = \mathcal{P}(\emptyset) = \{\emptyset\}$. By inspection $0 = \emptyset \in V_1$.

Successor Step: Suppose (1)-(3) hold for $\alpha \in \mathrm{On}$. We will show that they hold for $\alpha + 1$:

1. Let $r \in t \in V_{\alpha+1} = \mathcal{P}(V_\alpha)$. Then $t \subseteq V_\alpha$ (by construction) so $r \in V_\alpha \subseteq V_{\alpha+1}$ (by inductive hypothesis) as required.
2. Let $t \in V_{\alpha+1}$. Then $t \subseteq V_\alpha \subseteq V_{\alpha+1}$ (by inductive hypothesis) so that $t \in \mathcal{P}(V_{\alpha+1}) = V_{\alpha+1+1}$.
3. By inductive hyothesis $\alpha \subseteq V_\alpha \subseteq V_{\alpha+1}$ and $\alpha \in V_{\alpha+1}$ so that $\alpha + 1 \subseteq V_{\alpha+1}$ giving $\alpha + 1 \in \mathcal{P}(V_{\alpha+1}) = V_{\alpha+1+1}$.

Limit Step: Suppose $\gamma \in \mathrm{Lim}$ and (1)-(3) hold for $\alpha < \gamma$. Recall that $V_\gamma = \bigcup_{\alpha<\gamma} V_\alpha$.

1. If $r \in t \in V_\gamma$ then find $\alpha < \gamma$ with $r \in t \in V_\alpha$ and by transitivity of V_α we have $r \in V_\alpha \subseteq V_\gamma$.

2. Suppose $t \in V_\gamma$. Then $t \in V_\alpha$ for some $\alpha < \gamma$ so by transitivity of V_α, $t \subseteq V_\alpha \subseteq V_\gamma$, giving $t \in \mathcal{P}(V_\gamma) = V_{\gamma+1}$.

3. For each $\alpha < \gamma$ we have $\alpha + 1 < \gamma$ (since γ is not a successor) and thus $\alpha \in V_{\alpha+1} \subseteq V_\gamma$. Hence $\gamma \subseteq V_\gamma$ so $\gamma \in V_{\gamma+1}$.

The 'Hence' now follows from (3) and induction. □

We prove a little utitlity lemma (we only need this to show that $V \models$ **Replacement** but it makes the other axioms easier to check).

Lemma 9. $\mathbf{ZF}^-$ *proves that if z is a set and $z \subseteq V$ then $z \in V$.*

Proof. For $t \in z$ let α_t be the least ordinal with $t \in V_{\alpha_t}$ (formally, we write down a formula expressing this and check that this is a class function on z). Then $\beta = \sup\{\alpha_t + 1 : t \in z\}$ is a set by **Replacement** (and some others which we needed to establish that sets of ordinals have sups). By the previous lemma $z \subseteq V_\beta$ so $z \in V_{\beta+1}$ as required. □

Theorem 8. *For every $\phi \in$ **ZF**, $V \models \phi$, i.e. $\mathbf{ZF}^- \vdash \phi^V$.*

*In fact, if A is a transitive, non-empty class such that $A \models$ **Separation** and $\forall x \subseteq A \exists z \in A\ x \subseteq z$ then $A \models \mathbf{ZF}^-$.*

Proof. We prove this for V but remark that the proof also works for the A as specified.

Extensionality: Extensionality follows from transitivity of V.

Separation: We do **Separation** next (since then the weaker versions of the axioms stated in here imply the stronger versions with $\leftrightarrow$ in place of $\rightarrow$): so let $\phi(t; v_1, \ldots, v_n)$ be a formula of LST with all free variables shown and let $a_1, \ldots, a_n, y \in V$. Define

$$z = \left\{ t : t \in V \wedge t \in y \wedge \phi(t; a_1, \ldots, a_n)^V \right\}.$$

Then z is a set by **Separation** (in U) and by the little lemma $z \in V$ (in fact, if $y \in V_\alpha$ then $z \subseteq V_\alpha$ so $z \in V_{\alpha+1}$ avoids the lemma and hence the use of **Replacement**).

We need to check that

$$V \models \forall t\ \ [t \in z \leftrightarrow t \in y \wedge \phi(t; a_1, \ldots, a_n)]$$

or more explicitly (using the definition of $\models$) that

$$\mathbf{ZF}^- \vdash \forall t \in V\ \ \left[t \in z \leftrightarrow t \in y \wedge \phi(t; a_1, \ldots, a_n)^V\right].$$

So let $t \in V$ and assume $t \in z$. By definition of z (in U) $t \in y \wedge \phi(t; a_1, \ldots, a_n)^V$ as required. Now let $t \in V$ and assume $t \in y \wedge \phi(t; a_1, \ldots, a_n)^V$. Then by definition of z (in U) we have $t \in z$ as required.

Replacement: We follow the strategy from above: given a formula ϕ we cook up a formula ψ and apply **Replacement** with ψ in U to obtain some z. Then we check $z \in V$ and that V believes the right stuff about z. Since the format of the axiom is a little more complicated, the proof is slightly messier:

So suppose that $\phi(x, t, d; v_1, \ldots, v_n)$ is a formula of LST with all free variables shown. Let $a_1, \ldots, a_n, d \in V$ and assume that $V \models \forall x \in d \exists! y \phi(x, y, d)$. Since V is transitive and $d \in V$ this gives $\mathbf{ZF}^- \vdash \forall x \in d \exists! y \in V\ \phi(x, y, d)^V$. So let $\psi(x, y, d)$ be

$$y \in V \wedge \phi(x, y, d; a_1, \ldots, a_n)^V$$

and observe that we may apply **Replacement** in U to obtain c such that $\forall x \in d \exists y \in c \psi(x, y, d)$. Let $\hat{c} = c \cap V \subseteq V$ (we are using **Separation** here - this is unnecessary if we had used the stronger form of **Replacement**) so $\hat{c} \in V$. Now we show that

$$V \models \forall x \in d \exists y \in \hat{c} \phi(x, y, d; a_1, \ldots, a_n).$$

Assume $x \in d$ and obtain $y \in c$ (not $\hat{c}$!) such that $\psi(x, y, d)$. But then $y \in V$, so $y \in \hat{c}$ and $\phi(x, y, d, a_1, \ldots, a_n)^V$.

Pairing: The formula defining $x = \{y, z\}$ is Δ_0 so absolute for V, U and if $y, z \in V$ then $\{y, z\} \subseteq V$ so $\{y, z\} \in V$. (Again, strictly speaking we don't need the utitility lemma: we can choose α, β so that $y \in V_\alpha, z \in V_\beta$ and wlog $\alpha \leq \beta$. Then $\{y, z\} \subseteq V_\beta$ so $\{y, z\} \in V_{\beta+1}$.)

Union: Write it out in detail as an exercise. Essentially: the formula $z = \bigcup x$ is absolute for transitive classes and by transitivity $t \in x \in V \to t \subseteq V$ so $\bigcup x \in V$ (and again, we could choose minimal $\alpha_t \in \mathrm{On}$ and then observe that $\bigcup x \subseteq V_\beta$ where $\beta = \sup \{\alpha_t : t \in x\}$).

Powerset: Here we show that $\forall x \in V \exists z \in V\ \mathcal{P}(x) \cap V \subseteq z$: If $x \in V$, then let $\alpha \in \mathrm{On}$ (minimal or not) such that $x \in V_\alpha$ and let $z = V_\alpha + 1 \subseteq V$ so $z \in V$ (and again, the utility lemma is not really needed). If $t \in \mathcal{P}(x) \cap V$ then $t \in V_\alpha$ so $t \subseteq V_\alpha$ so $t \in z$.

Infinity: Exercise (go through the details which show that $V \models Ind(\omega)$ by absoluteness and observe $\omega \in V_{\omega+1}$).

Foundation: Let $x \in V$ and assume $x \neq \emptyset$. So there is $\alpha \in \mathrm{On}$ such that $x \cap V_\alpha \neq \emptyset$. Consider $\{\beta \in \alpha + 1 : x \cap V_\beta \neq \emptyset\}$ and note that this is a non-empty set of ordinals, so has an $\in$-minimal element μ. Note that μ cannot be a limit as $x \cap \bigcup_{\beta<\mu} V_\beta \neq \emptyset$ means that there is $\beta < \mu$ with $x \cap V_\beta \neq \emptyset$ contradicting minimality of μ. Also $\mu \neq \emptyset$ as $V_\emptyset = \emptyset$. So $\mu = \eta + 1$ for some ordinal η. Pick $m \in x \cap V_\mu$ (which exists by construction of μ). Then $m \subseteq V_\eta$ (by definition of $V_{\eta+1} = V_\mu$). So if $t \in m \cap x$ then $t \in V_\eta \cap x$ once again contradicting minimality of μ. Thus $m \cap x = \emptyset$ as required. $\square$

Theorem 9. *If **ZF** is inconsistent, then so is* $\mathbf{ZF}^-$.

Proof. Suppose there is are proofs P_1 giving $\mathbf{ZF} \vdash \psi$ and P_2 giving $\mathbf{ZF} \vdash \neg\psi$ for a sentence ψ.

Note that for every axiom ϕ used in the proofs above, by the previous theorem we can write down proofs of $\mathbf{ZF}^- \vdash \phi^V$ and follow them with P_1^V and P_2^V (i.e. where every line of P_i is relativized to V) to obtain a $\mathbf{ZF}^- \vdash \psi^V$ and $\mathbf{ZF}^- \vdash \neg\psi^V$ and thus $\mathbf{ZF}^-$ is inconsistent.

The precise details of course depend on your formal proof system, but it will be important that we have $\mathbf{ZF}^- \vdash \exists x\ x \in V$. $\square$

10.1 An alternative definition of V:

There is an alternative approach to defining V. For a set x, we define the transitive closure of x as the smallest transitive set containing x and denote it by $trcl(x)$. That a transitive set containing x exists follows from applying the Recursion Theorem (on ω) with $F = \bigcup x$ and $a = x$ and noting that $G(\omega) \cup \{x\}$ will then by transitive. We can then form the minimal one, just like the alternative definition of ω.

Now, we can define a set x to be hereditarily well-founded if $\in$ is well founded on $trcl(x)$ and let $V = \{x : x \text{ is hereditarily well-founded}\}$. We can then show that $\in$ is well founded on every subset of V (so $V \models$ **Foundation**) and also that applying any axiom of $\mathbf{ZF}^-$ to hereditarily well-founded sets gives new sets which are hereditarily well-founded. This will then (with a bit of extra work) show that $V \models \mathbf{ZF}$.

We can recover the V_α by observing that in fact $\in$ is well-founded on V and recursively defining

$$V_\alpha = \left\{ x \in V \setminus \bigcup_{\beta<\alpha} V_\beta : x \text{ is } \in\text{-minimal in } V \setminus \bigcup_{\beta<\alpha} V_\beta \right\}$$

although it is not clear that the V_α thus defined are sets! See Kunen for details on this approach.

It is a nice approach in that we explicitly construct the largest V which could be well-founded (and transitive) and then check that it works. Our approach is to 'discover' V and it is 'pure luck' that it does satisfy **ZF**.

11 Gödel's Constructible Universe L

We will now work in **ZF** (and note the instances of **Powerset** we will use) to define a 'smaller' universe L. Our surrounding universe is now $V = \{x : x = x\}$

(under **ZF** this is the same as V^U as defined in the previous section, but I want to emphasize that we assume **Foundation**).

11.1 The Definable Subsets

We want to define

$$Def(x) = \left\{ y : \begin{array}{r} \text{there is a formula } \phi(v_1, \ldots, v_n, v_{n+1}) \text{ and } a_1, \ldots, a_n \in x \text{ such that} \\ \forall t\, [t \in y \leftrightarrow t \in x \wedge (x, \in) \models \phi(a_1, \ldots, a_n, t)] \end{array} \right\}.$$

Of course, we cannot do this formally since we **cannot quantify over formulae**!

However, there is a workaround. We can internalize $(x, \in) \models \phi(a_1, \ldots, a_n)$ within any class A which satisfies enough of $\mathbf{ZF} - \mathbf{Powerset}$. More precisely there is a set $F \in A$ (in fact $F \subseteq \omega$), a class function $\mathrm{val}\colon A \times \omega \times A \to \{0, 1\}$ and a function $\mathrm{free}\colon \omega \to \omega^{<\omega}$, $\mathrm{free} \in A$, such that whenever $\phi(v_{k_1}, \ldots, v_{k_n})$ is a formula of LST with all free variables shown then there is $\lceil \phi \rceil \in F$ such that

$$\mathbf{ZF} - \mathbf{Powerset} \vdash \{k_1, \ldots, k_n\} = \mathrm{free}(\lceil \phi \rceil)$$
$$\mathbf{ZF} - \mathbf{Powerset} \vdash \forall x \in A\ \forall a \in x^{<\omega}\ \ [\mathrm{val}(x, \lceil \phi \rceil, a) = 1 \leftrightarrow [\mathrm{free}(\lceil \phi \rceil) \subseteq dom(a) \wedge \phi(a(k_1), \ldots, a(k_n))^x]].$$

Using this, we can define

$$y \in Def(x) \equiv \exists m \in F\ \exists a \in x^{\mathrm{free}(m) \setminus \{1\}}\, [1 \in \mathrm{free}(m) \wedge \forall t \in x\ \ [t \in y \leftrightarrow \mathrm{val}(x, m, a \cup \{\langle 1, t \rangle\}) = 1]].$$

For later, we note that if $A \subseteq B$ are non-empty transitive classes satisfying enough of $\mathbf{ZF} - \mathbf{Powerset}$ then F as well as val and hence $y \in Def(x)$ are absolute for A, B.

We can also see that $y \in Def(x) \to y \subseteq x$ so that if A satisfies additionally **Powerset** (and a further instance of **Separation**) then in fact the above defines the class function $Def\colon A \to A$ via

$$Def(x) = \{y \in \mathcal{P}(x) : y \in Def(x)\}.$$

In fact, there is no need for **Powerset**: you can use **Replacement** instead.

11.1.1 The details of defining Def - Non-Examinable

There are a lot of different ways of defining val, F and free as well as $\lceil \phi \rceil$. We present one of them, the details of which are not important.

We first note that we can define the usual arithmetic functions on ω (interpreted as $\mathbb{N}$) by the Recursion Theorem and that these are absolute.

In the meta-theory, we then use a 'nice' Gödel numbering of the formulae of LST (although this is only relevant at the very end - but it does help understanding). Of course this does depend on our language, so we need to fix it: The terms are $v' \ldots '$ (or more formally we define recursively $t_0 = \{v'\}$, $t_{n+1} = \{s' : s \in t_n\}$) and we code them by

$$\lceil t \rceil = \begin{cases} 2; & t = v' \\ 2 \lceil s \rceil; & t = s' \end{cases}$$

So 'terms' are powers of 2 and we write v_k instead of $v' \ldots '$ (k $'$s) (for sanity reasons) and we let $T = \left\{2^k : k \in \omega, k \geq 1\right\}$.

Next, the atomic formulae are (for t, s terms, so we can think $\ulcorner t \urcorner, \ulcorner s \urcorner \in T$)

$$t = s$$
$$t \in s$$

coded by

$$\ulcorner t = s \urcorner = 3^{\ulcorner t \urcorner} 5^{\ulcorner s \urcorner} 7^1$$
$$\ulcorner t \in s \urcorner = 3^{\ulcorner t \urcorner} 5^{\ulcorner s \urcorner} 7^2$$

and we let $A = \left\{3^t 5^s 7^k : t, s \in T, k \in \{1, 2\}\right\}$.

Finally, the formulae are

$$\begin{aligned} \phi; &\quad \phi \text{ an atomic formula} \\ \neg\phi; &\quad \phi \text{ a formula} \\ \phi \wedge \psi; &\quad \psi, \phi \text{ formulae} \\ \forall v_k \phi; &\quad v_k \text{ a term}, \phi \text{ a formula} \end{aligned}$$

coded by

$$\ulcorner \neg\phi \urcorner = 3^{\ulcorner \phi \urcorner} 7^3$$
$$\ulcorner \phi \wedge \psi \urcorner = 3^{\ulcorner \phi \urcorner} 5^{\ulcorner \psi \urcorner} 7^4$$
$$\ulcorner \forall v_k \phi \urcorner = 3^{\ulcorner \phi \urcorner} 5^{\ulcorner v_k \urcorner} 7^5$$

and we let

$$F = A \cup \left\{3^p 7^3 : p \in F\right\} \cup \left\{3^p 5^q 7^4 : p, q \in F\right\} \cup \left\{3^p 5^t 7^5 : p \in F, t \in T\right\}.$$

Of course, the definition of F doesn't seem to make sense, so we should (by recursion on ω) set

$$\begin{aligned} F_0 =& A \\ F_{n+1} =& F_0 \cup \left\{3^p 7^3 : p \in F_n\right\} \cup \left\{3^p 5^q 7^4 : p, q \in F_n\right\} \cup \left\{3^p 5^t 7^5 : p \in F_n, t \in T\right\} \\ F =& \bigcup_{n \in \omega} F_n. \end{aligned}$$

For convenience, we have chosen a minimal language. It is not difficult to see how to deal with a more complicated language having all logical connectives as well as existential quantifiers.

We note that T, A, F can be defined in a sufficiently large fragment of **ZF** $-$ **Powerset** and are absolute for transitive non-empty models of this fragment.

Now we define the function free on ω which takes values in $\omega^{<\omega}$ (finite functions into ω, formally I think I want $\omega^{<\omega} = \{f : F \to \omega : F \text{ finite } \subseteq \omega\}$) as follows (by recursion on ω):

$$\mathrm{free}(0) = \{0\}$$

$$\mathrm{free}(n+1) = \begin{cases} \{0\}\,; & n+1 \notin F \\ \mathrm{free}(k); & n+1 \in F \wedge n+1 = 3^k 7^3 \\ \mathrm{free}(k) \cup \mathrm{free}(l); & n+1 \in F \wedge n+1 = 3^k 5^l 7^4 \\ \mathrm{free}(k) \setminus \{l\}\,; & n+1 \in F \wedge n+1 = 3^k 5^l 7^5. \end{cases}$$

You should convince yourself that free gives $\{0\}$ if the input is not (the code for) a formula and otherwise the set of free variables in the formula.

(Note that I have made sure that $0 \notin T$ so that $0 \in \mathrm{free}(k)$ if and only if $k \notin F$.)

We observe that free is absolute for non-empty transitive classes satisfying enough of **ZF** − **Powerset**.

Finally, given x, we can define a function $\mathrm{val}_x \colon \omega \times x^{<\omega} \to \{0, 1, 2\}$ by recursion on ω (here I interpret $x^{<\omega} = \{a : b \to x : b \text{ finite } \subset \omega\}$). Essentially, this is the model-theoretic satisfaction relation $\models$ adapted to our representation of ϕ by $\ulcorner\phi\urcorner$. In the big case distinction, I have put the logical symbol which is dealt with in square brackets before the $n+1$.

$$\mathrm{val}_x(0, a) = 0$$

$$\mathrm{val}_x(n+1, a) = \begin{cases} 0; & n+1 \notin F \\ 0; & n+1 \in F \wedge \mathrm{free}(n+1) \not\subseteq dom(a) \\ 0; & [=]\ n+1 \in F \wedge \mathrm{free}(n+1) \subseteq dom(a) \wedge \exists k, l \in \omega \left[n+1 = 3^{2^k} 5^{2^l} 7^1 \wedge a(k) \neq a(l)\right] \\ 1; & [=]\ n+1 \in F \wedge \mathrm{free}(n+1) \subseteq dom(a) \wedge \exists k, l \in \omega \left[n+1 = 3^{2^k} 5^{2^l} 7^1 \wedge a(k) = a(l)\right] \\ 0; & [\in]\ n+1 \in F \wedge \mathrm{free}(n+1) \subseteq dom(a) \wedge \exists k, l \in \omega \left[n+1 = 3^{2^k} 5^{2^l} 7^2 \wedge a(k) \notin a(l)\right] \\ 1; & [\in]\ n+1 \in F \wedge \mathrm{free}(n+1) \subseteq dom(a) \wedge \exists k, l \in \omega \left[n+1 = 3^{2^k} 5^{2^l} 7^2 \wedge a(k) \in a(l)\right] \\ 0; & [\neg]\ n+1 \in F \wedge \mathrm{free}(n+1) \subseteq dom(a) \wedge \exists k, l \in \omega \left[n+1 = 3^k 7^3 \wedge \mathrm{val}_x(k, a) = 1\right] \\ 1; & [\neg]\ n+1 \in F \wedge \mathrm{free}(n+1) \subseteq dom(a) \wedge \exists k, l \in \omega \left[n+1 = 3^k 7^3 \wedge \mathrm{val}_x(k, a) = 0\right] \\ 0; & [\wedge]\ n+1 \in F \wedge \mathrm{free}(n+1) \subseteq dom(a) \wedge \exists k, l \in \omega \big[n+1 = 3^k 5^l 7^4 \wedge \\ & \quad [\mathrm{val}_x(k, a) = 0 \vee \mathrm{val}_x(l, a) = 0]\big] \\ 1; & [\wedge]\ n+1 \in F \wedge \mathrm{free}(n+1) \subseteq dom(a) \wedge \exists k, l \in \omega \big[n+1 = 3^k 5^l 7^4 \wedge \\ & \quad [\mathrm{val}_x(k, a) = 1 \wedge \mathrm{val}_x(l, a) = 1]\big] \\ 0; & [\forall]\ n+1 \in F \wedge \mathrm{free}(n+1) \subseteq dom(a) \wedge \exists k, l \in \omega \big[n+1 = 3^k 5^l 7^5 \wedge \\ & \quad \exists \hat{a} \in x^{<\omega} \left[\hat{a}|_{\mathrm{free}(n+1)\setminus\{l\}} = a|_{\mathrm{free}(n+1)\setminus\{l\}} \wedge l \in dom(\hat{a}) \to \mathrm{val}_x(k, \hat{a}) = 0\right]\big] \\ 1; & [\forall]\ n+1 \in F \wedge \mathrm{free}(n+1) \subseteq dom(a) \wedge \exists k, l \in \omega \big[n+1 = 3^k 5^l 7^5 \wedge \\ & \quad \forall \hat{a} \in x^{<\omega} \left[\hat{a}|_{\mathrm{free}(n+1)\setminus\{l\}} = a|_{\mathrm{free}(n+1)\setminus\{l\}} \wedge l \in dom(\hat{a}) \to \mathrm{val}_x(k, \hat{a}) = 1\right]\big] \\ 0 & \text{otherwise - this case should not occur} \end{cases}$$

Note that because $x^{<\omega}$ is absolute (for transitive non-empty classes satisfying enough of **ZF** − **Powerset**), val_x is in fact absolute for these transitive non-empty classes.

For a formula $\phi(v_{k_1}, \ldots, v_{k_n})$ of LST with all free variables shown, and $a_1, \ldots, a_n \in x$ we define

$$(x, \in) \models \phi(a_1, \ldots, a_n) \equiv \mathrm{val}_x(\ulcorner\phi\urcorner, \{\langle k_i, a_i\rangle : i = 1, \ldots, n\}) = 1.$$

11.2 A few definable sets

We work in any non-empty transitive class A satisfying enough of **ZF**$-$**Powerset**.

Lemma 10. *Suppose x is a set.*

1. *$\emptyset \in Def(\emptyset)$ hence $Def(\emptyset) = \{\emptyset\}$.*
2. *$x \in Def(x)$.*
3. *For $y, z \in x$, $\{y, z\} \in Def(x)$. More generally, finite subsets of x are definable.*

Proof. We give the relevant formulae and parameters and leave it to the reader to verify that these define the appropriate sets:

1. $\phi(v_1) \equiv v_1 \neq v_1$ and $a = \emptyset$;
2. $\phi(v_1) \equiv v_1 = v_1$ and $a = \emptyset$
3. $\phi(v_1, v_2, v_3) \equiv v_1 = v_3 \vee v_2 = v_3$ and $a = \{\langle 1, y\rangle, \langle 2, z\rangle\}$; in general, we use $\phi(v_1, \ldots, v_n, v_{n+1}) \equiv v_1 = v_{n+1} \vee \cdots \vee v_n = v_{n+1}$ to obtain sets with (up to) n elements;

Remark 7. We will in fact spell out what really happens in the last of these cases (when we take $A = V$), just to convince you that you really don't want to do this (without a computer), but that it is not hard, just tedious. Note that since val is absolute for non-empty transitive classes satisfying enough of **ZF** $-$ **Powerset** we get that $(x, \in) \models \phi(a_1, \ldots, a_n)$ is absolute for these classes. Also $\phi^x \equiv \phi \equiv \phi^A$, so taking $A = V$ does not lose generality.

First we work out $\ulcorner v_1 = v_3 \vee v_2 = v_3\urcorner$. Note that for use, $\vee$ was only an abbreviation so $v_1 = v_3 \vee v_2 = v_3 \equiv \neg\,[\neg\,[v_1 = v_3] \wedge \neg\,[v_2 = v_3]]$.

With our concrete definition,

$$n_1 = \ulcorner v_1 = v_3\urcorner = 3^{2^1}5^{2^3}7$$

and

$$n_2 = \ulcorner v_2 = v_3\urcorner = 3^{2^2}5^{2^3}7.$$

Then

$$n_3 = \ulcorner\neg\,[v_1 = v_3]\urcorner = 3^{n_1}7^3$$

and

$$n_4 = \ulcorner\neg\,[v_2 = v_3]\urcorner = 3^{n_2}7^3.$$

Thus

$$n_5 = \ulcorner\neg\,[v_1 = v_3] \wedge \neg\,[v_2 = v_3]\urcorner = 3^{n_3}5^{n_4}7^4$$

and finally

$$n = \ulcorner \neg \left[\neg \left[v_1 = v_3 \right] \wedge \neg \left[v_2 = v_3 \right] \right] \urcorner = 3^{n_5} 7^3 = 3^{3^{3^{3^{2^1} 5^{2^3} 7} 7^3} 5^{3^{3^{2^2} 5^{2^3} 7} 7^3} 7^4} 7^3.$$

Clearly (well, by construction) $n \in F$ and $\text{free}(n) = \{1, 2, 3\}$.
We now need to check that

$$t \in \{y, z\} \leftrightarrow \text{val}\,(x, n, a \cup \{\langle 3, t \rangle\}) = 1.$$

So suppose that $t = y$. So, let us work out

$$\text{val}\,(x, n_1, a \cup \{\langle 3, t \rangle\}) = 1$$

since $n_1 \in F$, $\text{free}(3^{2^1} 5^{2^3} 7) = \{1, 3\} \subseteq \{1, 2, 3\} = dom\,(a \cup \{\langle 3, t \rangle\})$ and $k = 1, l3$ are such that $n_1 = 3^{2^k} 5^{2^l} 7$ and in fact $a(1) = y = t = a(3)$.

Thus

$$\text{val}\,(x, n_3, a \cup \{\langle 3, t \rangle\}) = 0$$

and so

$$\text{val}\,(x, n_5, a \cup \{\langle 3, t \rangle\}) = 0$$

giving

$$\text{val}\,(x, n, a \cup \{\langle 3, t \rangle\}) = 1$$

as required (we left out the detailed checks for these last three claims).

The case $t = z$ works similarly.

For the converse, if $t \neq y$ and $t \neq z$ then $\text{val}\,(x, n_1, a \cup \{\langle 3, t \rangle\}) = 0$ as well as $\text{val}\,(x, n_2, a \cup \{\langle 3, t \rangle\}) = 0$. Thus $\text{val}\,(x, n_3, a \cup \{\langle 3, t \rangle\}) = 1$ as well as $\text{val}\,(x, n_3, a \cup \{\langle 3, t \rangle\}) = 1$ giving $\text{val}\,(x, n_5, a \cup \{\langle 3, t \rangle\}) = 1$ so that $\text{val}\, x, n, a \cup \{\langle 3, t \rangle\} = 0$.

$\square$

11.3 The definable sets L

Definition 10. By recursion on On we define a class function $L : \text{On} \to V$ (strictly L^V since our definition happens relativized to V)

- $L_0 = \emptyset$;
- $\forall \alpha \in \text{On}\ L_{\alpha+1} = Def\,(L_\alpha)$;
- $\forall \gamma \in \text{Lim}\ L_\gamma = \bigcup_{\beta < \gamma} L_\beta$

We also write L for the class $\bigcup_{\alpha \in \text{On}} L_\alpha$

Lemma 11. $\mathbf{ZF}^-$ *proves: Forall* $\alpha \in \text{On}$*:*

1. $L_\alpha \subseteq V_\alpha$*;*
2. $L_\alpha \in L_{\alpha+1}$*;*

3. *L_α is transitive and $L_\alpha \subseteq L_{\alpha+1}$,*

4. *$L_\alpha \cap \mathrm{On} = \alpha$ and $\alpha \in L_{\alpha+1}$*

Proof. 1. By induction on α: base case and limit stages are trivial; for successor steps note that $Def\,(x) \subseteq \mathcal{P}\,(x)$.

2. See Lemma 10.

3. By induction on α: again, the base case and the limit stages are straightforward. So suppose $x \in L_{\alpha+1}$.

   If $t \in x \in L_{\alpha+1}$ then $x \subseteq L_\alpha$ so $t \in L_\alpha \subseteq L_{\alpha+1}$ by inductive assumption. Thus $L_{\alpha+1}$ is transitive.

   As $L_{\alpha+1}$ is transitive $x = \{t : t \in L_\alpha \wedge t \in x\} \in Def\,(L_{\alpha+1}) = L_{\alpha+2}$ giving $\subseteq$.

4. Again, by induction on α: The non-trivial bit of the base case has been covered in Lemma 10.

   For the successor step: assume $L_\alpha \cap \mathrm{On} = \alpha$ and $\alpha \in L_{\alpha+1}$. Let $\beta \in L_{\alpha+1} \cap \mathrm{On}$. If $\alpha \in \beta \subseteq L_\alpha$ then $\alpha \in L_\alpha \cap \mathrm{On} = \alpha$, a contradiction. So $\beta = \alpha$ or $\beta \in \alpha$ (as $\in$ totally orders On) and hence $\beta \in \alpha \cup \{\alpha\} = \alpha + 1$. Conversely, assume that $\beta \in \alpha + 1$. Either $\beta = \alpha$ and then by inductive assumption $\beta = \alpha \in L_{\alpha+1}$ or $\beta \in \alpha \in L_{\alpha+1}$ and by transitivity of $L_{\alpha+1}$ we obtain $\beta \in L_{\alpha+1}$. Also $\phi(v_1, v_2) \equiv v_1 \in v_2 \vee v_1 = v_2$ applied with $v_2 = \alpha$ shows that $\alpha + 1 \in L_{\alpha+2}$.

   For the limit: suppose $\gamma \in \mathrm{Lim}$ and the result is true for $\alpha < \gamma$. If $\beta \in L_\gamma \cap \mathrm{On}$ then $\beta \in L_\alpha$ for some $\alpha < \gamma$ and hence $\beta \in \gamma$. Conversely, if $\beta \in \gamma$ then $\beta < \gamma$ so $\beta \in L_{\beta+1} \subseteq L_\gamma$. But then $\phi(v_1) \equiv v_1$ is an ordinal shows that $\gamma \in Def\,(L_\gamma)$ as required.

□

11.4 ZF in L I

Theorem 10. *For each $\phi \in$ **ZF**,*

$$\boldsymbol{ZF} \vdash \phi^L$$

*i.e. $L \models$ **ZF**.*

Proof of ϕ^L for the 'easy' axioms ϕ. We check the axioms in turn.

Extensionality: L is transitive, so satisfies **Extensionality**^{L} (like V does).

Emptyset: $\emptyset^V \in L$ and $z = \emptyset$ is absolute, so $\emptyset^L = \emptyset^V$ witnesses **Emptyset**^{L}.

Pairing: Suppose $x, y \in L$ and find $\alpha \in \mathrm{On}$ such that $x, y \in L_{\alpha+1}$. By an earlier lemma $\{x, y\}^V \in Def\,(L_{\alpha+1}) = L_{\alpha+2} \subseteq L$ and by absoluteness of $z = \{x, y\}$, $\{x, y\}^L = \{x, y\}^V$ witnesses **Pairing**^{L} (for x, y).

Union: Suppose $x \in L$ and find $\alpha \in$ On such that $x \in L_{\alpha+1}$. Let $z = (\bigcup x)^V$. Note that if $t \in y \in x \in L_{\alpha+1}$ then by transitivity of $L_{\alpha+1}$ we have $t \in L_{\alpha+1}$ so that $z \subseteq L_{\alpha+1}$.

By definition of z,

$$t \in z \leftrightarrow [t \in L_{\alpha+1} \wedge \exists y \in x \ \ [t \in y]] .$$

The formula $\exists y \in x \ \ [t \in y]$ is Δ_0 so absolute and hence

$$t \in z \leftrightarrow \left[t \in L_{\alpha+1} \wedge [\exists y \in x \ \ [t \in y]]^{L_{\alpha+1}}\right] .$$

But then

$$t \in z \leftrightarrow [t \in L_{\alpha+1} \wedge (L_{\alpha+1}, \in) \models \exists y \in x \ \ [t \in y])] .$$

Thus we let $\phi(v_1, v_2) \equiv \exists v_3 \in v_1 [v_2 \in v_3]$ (and $a_1 = x$) to see that $z \in Def L_{\alpha+1} = L_{\alpha+2}$.

Finally, by absoluteness of $z = \bigcup x$, $z = (\bigcup x)^L$ witnesses $\textbf{Union}^L$ (for x).

Powerset: Suppose $x \in L$ and find $\alpha_x \in$ On such that $x \in L_{\alpha_x}$. Let $z = \mathcal{P}(x)^V \cap L$, so that

$$t \in z \leftrightarrow t \subseteq x \wedge t \in L.$$

For each $t \in z$, let $\alpha_t \in$ On be minimal such that $t \in L_{\alpha_t+1}$ and let $\alpha = \sup\{\alpha_t + 1 : t \in z\} \cup \{\alpha_x\} \in$ On so that $z \subseteq L_\alpha$ and $x \in L_\alpha$. Hence

$$t \in z \leftrightarrow t \subseteq x \wedge t \in L_\alpha \leftrightarrow [t \subseteq x]^{L_\alpha} \wedge t \in L_\alpha$$

by absoluteness of $t \subseteq x$. Thus

$$t \in z \leftrightarrow t \in L_\alpha \wedge (L_\alpha, \in) \models t \subseteq x$$

and hence $\phi(v_1, v_2) \equiv v_2 \subseteq v_1 \equiv \forall v_3 \in v_2 [v_3 \in v_1]$ (and $a_1 = x$) witnesses that $z \in Def(L_\alpha) = L_{\alpha+1} \subseteq L$.

We now need observe that the above discussion shows

$$[\forall t [t \in z \leftrightarrow t \subseteq x]]^L \equiv \forall t \in L \left[t \in z \leftrightarrow [t \subseteq x]^L\right]$$

so that indeed $z = \mathcal{P}(x)^L$ witnesses $\textbf{Powerset}^L$ (for x).

Foundation: If $x \in L$ then $x \in V$ so find $m \in V$ such that $[m \text{ is } \in\text{-minimal in } x]^V$. Then $m \in x \in L$ gives $m \in L$ and being $\in$-minimal in x is absolute ($\forall y \in x \ \ [y \notin m]$ is Δ_0), so $[m \text{ is } \in\text{-minimal in } x]^L$ as required.

Infinity: By the Lemma $\omega^V \in L_{\omega+1}$ and $\phi(z) \equiv \emptyset \in z \wedge \forall x \in z\ [x \cup \{x\} \in z]$ is absolute. Since $\phi(\omega^V)^V$ we obtain $\phi(\omega^V)^L$ as required (and we could note that $\omega^L = \omega^V$). □

Remark 8. It is worth noting that the above proof shows that

- Each L_α satsifies **Extensionality** and **Foundation** and if $\omega \in \alpha$ then L_α satisfies **Infinity**.
- For every limit ordinal γ, L_γ satisfies **Pairing** and **Union**.

Proof attempt at instances of ***Separation***^{L}. Suppose $\phi(v_1, \ldots, v_n, v_{n+1})$ is a formula of LST with all free variables displayed and that $a_1, \ldots, a_n, x \in L$. As before we find $\alpha \in L$ such that $a_1, \ldots, a_n, x \in L_\alpha$. By **Separation** in V applied with the formula ϕ^L, we obtain

$$z = \left\{t \in x : \phi(a_1, \ldots, a_n, t)^L\right\}$$

and since $z \subseteq x \subseteq L_\alpha$ we obtain $z \subseteq L_\alpha$.

So we may be tempted to show that $z \in Def\,(L_\alpha)$ with the formula $\phi(v_1, \ldots, v_n, v_{n+1})^L \wedge t \in x$. This encounters a problem: first the relativization L has to be relativized itself. Let's write $\psi_L(v_{n+2})$ for the formula defining L. If for example $\phi \equiv \forall v_{n+2} \psi$ then $\phi^L \equiv \forall v_{n+2}\ \left[\psi_L(v_{n+2}) \to \psi^L\right]$, so that

$$\left(\phi^L\right)^{L_\alpha} \equiv \forall v_{n+2}\ \left[v_{n+2} \in L_\alpha \wedge \psi_L(v_{n+2})^{L_\alpha} \wedge \left(\psi^L\right)^{L_\alpha}\right].$$

At this stage we are in no position to prove that ψ_L is absolute for L_α, V because L_α might not satisfy all of $\mathbf{ZF}^-$ we need (even if we assume that α is a limit ordinal). So, to avoid this, we try again, this time maybe with $\phi(v_1, \ldots, v_n, v_{n+1}) \wedge t \in x$ since we automatically relativize to $L_{\alpha+1}$ when checking $(L_{\alpha+1}, \in) \models \phi(a_1, \ldots, a_n, t)$. But here we also encounter a problem. To carry out this plan, we will need to show that

$$t \in z \leftrightarrow t \in x \wedge \phi(a_1, \ldots, a_n, t)^{L_\alpha}$$

and we only know that

$$t \in z \leftrightarrow t \in x \wedge \phi(a_1, \ldots, a_n, t)^L.$$

Thus we need to get our hands on some ordinal $\gamma \supseteq \alpha$ such that $\phi(v_1, \ldots, v_n, v_{n+1})$ is absolute for L_γ, L. Provided we have this γ, we then have

$$t \in z \leftrightarrow t \in x \wedge \phi(a_1, \ldots, a_n, t)^{L_\gamma}$$

so that $z \in Def\,(L_\gamma)$. This γ does exist as we will show in the next theorem, the Levy Reflection Principle.

Then by construction $[z = \{t \in x : \phi(a_1, \ldots, a_n, t)\}]^L$ and we are done. □

The Levy Reflection Principle

We aim to show the following meta-theorem. First the informal, wordy version:

Theorem 11. *For every formula ϕ of LST, if $A : \mathrm{On} \to V$ is a non-decreasing class function of non-empty transitive sets with $\forall \gamma \in \mathrm{Lim}\ A_\gamma = \bigcup_{\beta\in\gamma} A_\beta$ and $B = \bigcup_{\alpha\in\mathrm{On}} A_\alpha$ then there are arbitrarily large limit ordinals γ such that ϕ is absolute for A_γ, B.*

Or, more formally.

Theorem 12. *Suppose A is a formula of LST with one free variable and that $\phi(v_1, \ldots, v_n)$ is some formula of LST.*

We will assume that A is a class function on On *(see below) and then write $A_\alpha = A(\alpha)$ and $B = \bigcup_{\alpha\in\mathrm{On}} A_\alpha = \{x : \exists\alpha \in \mathrm{On}\ x \in A_\alpha\}$*

ZF* − *Powerset *proves:*

$$\begin{aligned}
A \textit{ is a class function on } \mathrm{On} \wedge \\
A_1 \neq \emptyset \wedge \\
\forall\alpha, \beta \in \mathrm{On}\ \ [\alpha \subseteq \beta \to A_\alpha \subseteq A_\beta] \wedge \\
\forall\alpha \in \mathrm{On}\ A_\alpha \textit{ is transitive } \ \wedge \\
\forall\gamma \in \mathrm{Lim}\ A_\gamma = \bigcup\{A_\beta : \beta \in \gamma\} \\
\to \\
\forall\alpha \in \mathrm{On}\ \exists\gamma \in \mathrm{Lim}\ \ [\alpha \subseteq \gamma \wedge \phi \textit{ is absolute for } A_\gamma, B]\,.
\end{aligned}$$

where of course 'ϕ is absolute for A_γ, B' means

$$\forall a_1, \ldots, a_n \in A_\gamma\ \ \left[\phi(a_1, \ldots, a_n)^{A_\gamma} \leftrightarrow \phi(a_1, \ldots, a_n)^B\right]$$

We first prove a lemma, the well known Tarski–Vaught criterion for absoluteness (from Model Theory). But first we need a definition in the meta-theory.

Definition 11. A list $\phi_1, \ldots, \phi_n$ of formulae of LST is subformula closed if and only if every subformula of each ϕ_k appears in the list.

Note that whenever $\phi_1, \ldots, \phi_n$ is a list of formulae of LST, we can add all the subformulae to obtain a subformula closed list $\phi_1, \ldots, \phi_n, \ldots, \phi_m$.

Lemma 12 (Tarksi–Vaught criterion)**.** *Suppose $\phi_1, \ldots, \phi_n$ is a subformula closed list of formulae of LST which do not contain the universal quantifier $\forall$ (but may contain $\exists$) and that $A \subseteq B$ are transitive, non-empty classes.*

Then each of $\phi_1, \ldots, \phi_n$ is absolute for A, B provided that for every formula $\phi_k(v_1, \ldots, v_m)$ of the form $\exists v_{m+1}\ \phi_j(v_1, \ldots, v_n, v_{m+1})$ we have

$$\forall a_1, \ldots, a_m \in A\ \ \left[\phi_k(a_1, \ldots, a_m)^B \to \exists a_{m+1} \in A\ \phi_j(a_1, \ldots, a_n, a_{m+1})^B\right].$$

Proof. By induction on the complexity of the (most complicated) formula. Atomic formulae are trivially absolute. The logical connectives $\wedge, \neg, \vee, \to, \leftrightarrow$ are trivial by induction. Thus assume that the most complicated formula is $\phi_n(v_1, \dots, v_m) \equiv \exists v_{m+1}\ \phi_j(v_1, \dots, v_m, v_{m+1})$ and that the statement is true for simpler formulae. In particular, ϕ_j is absolute for A, B.

So let $a_1, \dots, a_m \in A$. Now observe

$$\phi_n(a_1, \dots, a_m)^A \equiv \exists a_{m+1} \in A\ \phi_j(a_1, \dots, a_m, a_{m+1})^A.$$

Since ϕ_j is absolute

$$\left[\exists a_{m+1} \in A\ \phi_j(a_1, \dots, a_m, a_{m+1})^A\right] \leftrightarrow \left[\exists a_{m+1} \in A\ \phi_j(a_1, \dots, a_m, a_{m+1})^B\right].$$

Since $A \subseteq B$ (for the $\rightarrow$ direction) and by the assumption (for the $\leftarrow$-direction)

$$\left[\exists a_{m+1} \in A\ \phi_j(a_1, \dots, a_m, a_{m+1})^B\right] \leftrightarrow \phi_n(a_1, \dots, a_m)^B.$$

Hence $\phi_n(a_1, \dots, a_m)^A \leftrightarrow \phi_n(a_1, \dots, a_m)^B$ as required. □

The Tarski–Vaught criterion can be understood as a 'closure' criterion: if an existential formula ϕ is true in B then we can in fact find an element of A witnessing ϕ^B.

To prove the Levy Reflection principle, we emulate the proof of the Skolem Theorem, constructing the Skolem closure. We start with some A_α and 'add witnesses' to all existential subformulae in B that are true in B. Of course, adding these witnesses gives us new 'parameters' so that we have to consider more instantiations of formulae.

As a concrete example, consider (in the language of fields) the statement $\phi(x) \equiv \exists y\ y^2 = x$. Then ϕ is not absolute for $\mathbb{Q}, \mathbb{C}$ (because $2 \in \mathbb{Q}$ and $\exists y \in \mathbb{Q}\ y^2 = 2$ is false but $\exists y \in \mathbb{C}\ y^2 = 2$ is true). We start for example by adding all square roots of rationals and closing under the field operations to get a field F_1. Now $\sqrt{2} \in F_1$, but $\neg\exists y \in F_1\ y^2 = \sqrt{2}$. Thus we have to repeat: form F_2 by adding all square roots of elements of F_1 and close under the field operations. Again, there may (or may not) be (new) elements of F_2 which don't have square roots in F_2, so we keep adding them to form the F_n. Finally, we set $F = \bigcup_{n\in\omega} F_n$. Now F is a field (easy to check that the increasing union of fields is a field) and if $x \in F$ then $x \in F_n$ for some n and hence $\sqrt{x} \in F_{n+1} \subseteq F$ witnesses $\exists y \in F\ y^2 = x$.

Of course, we may have to do this with lots of formulae at the same time and we can't simply add elements, but will have to go up in the A_α-hierarchy. The principle of the proof is the same, though.

Proof of the Levy Reflection Principle. Let ϕ be a formula of LST. Replace all universal quantifiers $\forall x$ by logically equivalent $\neg\exists x\neg$ and let $\phi_1, \dots, \phi_n$ be a subformula closed list not mentioning $\forall$ and containing (the logical equivalent of) ϕ. Write i_k for the number of free variables of $\phi_k, k = 1, \dots, n$.

Let $\alpha \in \mathrm{On}$. By recursion on $m \in \omega$ we define $\alpha_m \in \mathrm{On}$, $f^k_m \colon A^{i_k}_{\alpha_m} \to \mathrm{On}$. Of course $\alpha_0 = \alpha + 1$ (so that $\alpha_0 \geq 0$).

Having defined α_m, we define f^k_m as follows. For $a \in A^{i_k}_{\alpha_m}$, writing $a_j = a(j)$, if $\phi_k(v_1, \dots, v_{i_k}) \equiv \exists t\ \psi(v_1, \dots, v_{i_k}, t)$ and $\phi_k(a_1, \dots, a_{i_k})^B$ then $f^k_m(a)$ is the least ordinal β such that $\exists t \in A_\beta\ \psi(a_1, \ldots, a_{i_k}, t)^B$ and $f^k_m(a) = 0$ otherwise. We need to argue that f^k_m is a function: if $\phi_k(a_1, \ldots, a_{i_k})^B$ then pick a witness $t \in B$ such that $\psi(a_1, \ldots, a_{i_k}, t)^B$ and $\hat{\beta} \in \mathrm{On}$ such that $t \in A_{\hat{\beta}}$ then $f^k_m(a) = \min\left\{\beta \in \hat{\beta} + 1 : \exists t \in A_\beta\ \psi(a_1, \ldots, a_{i_k})^B\right\}$.

We then set

$$\alpha_{m+1} = \sup\left\{\alpha_m + 1, \left\{\sup\left\{f^k_m(a) : a \in A^{i_k}\right\} : k = 1, \ldots, n\right\}\right\}.$$

Finally, we claim that $\gamma = \sup\{\alpha_{m+1} : m \in \omega\}$ is as required.

Firstly since $\alpha < \alpha + 1 = \alpha_0 < \alpha_1 < \ldots$ we have that $\gamma \in \mathrm{Lim}$ and $\alpha \subseteq \gamma$ and that each $\alpha_m \in \gamma$.

Next we check that A_γ, B satsify the Tarski–Vaught criterion. So let $\phi_l(v_1, \ldots, v_{i_k})$ be a formula of the form $\exists t\ \psi(v_1, \ldots, v_{i_k}, t)$. Let $a_1, \ldots, a_k \in A_\gamma$ and assume $\phi(a_1, \ldots, a_{i_k})^B$. Set $a = \{\langle j, a_j\rangle : j = \{1, \ldots, i_k\}\}$. For each $a_j, j = 1, \ldots, k$ note that $a_j \in A_\gamma = \bigcup_{\beta\in\gamma} A_\beta$ implies that there is $\beta \in \gamma$ such that $a_j \in A_\beta$. Since $\gamma = \sup_m \alpha_m$, we can find α_m such that $\beta \subseteq \alpha_m$ so that $a_j \in A_{\alpha_m}$. Thus (by taking the maximum of such witnesses), there is $m \in \omega$ such that all $a_j \in A_{\alpha_m}$ and so $a \in A^{i_k}_{\alpha_m}$. Since $\phi(a_1, \ldots, a_{i_k})^B$ by definition of $f^k_m(a)$ there is $t \in A_{f^k_m(a)}$ with $\psi(a_1, \ldots, a_{i_k}, t)^B$. Then $f^k_m(a) \leq \alpha_{m+1} \in \gamma$ so that in fact $t \in A_\gamma$. Thus we have shown $\exists t \in A_\gamma\ \psi(a_1, \ldots, a_{i_k}, t)^B$.

Hence by the Tarski–Vaught criterion, each ϕ_l is absolute for A_γ, B as claimed. $\square$

Remark 9. Note that in the definition of $f^k_m(a)$, we cannot argue that $f^k_m(a) = \min\left\{\beta \in \mathrm{On} : \exists t \in A_\beta\ \psi(a_1, \ldots, a_n, t)^B\right\}$ since the RHS is a class and we cannot use the well-order theorem of $\in$ on On for infinitely many classes.

In constructing the Skolem closure, we only would need to add a witness instead of going up the hierarchy. But we cannot choose (potentially) infinitely many witnesses because we do not assume **Choice** (yet).

Remark 10. Note that the proof works if we substitute On by a limit ordinal γ_0 that is closed under countable suprema, i.e. such that for all $\alpha\colon \omega \to \gamma_0; m \mapsto \alpha_m$ we have $\sup\{\alpha_m : m \in \omega\} \in \gamma_0$.

Remark 11. I have been extremely explicit in the proof. You can shorten it by quite a bit by leaving out the detailed checks, which are essentially easy.

Remark 12. Note that this proof is very much not uniform with respect to the formulae and as such is a true meta-theory proof. You cannot 'internalize' it (using codes for formulae etc) and expect to get the 'obvious' result.

ZF in L II

We now check the two axiom schemes, **Separation** and **Replacement** in L.

We write down a clean version of the proof of **Separation**.

Proof of instances of ***Separation***^{L}. Suppose $\phi(v_1, \ldots, v_n, v_{n+1})$ is a formula of LST (with all free variables shown) and $a_1, \ldots, a_n, x \in L$. Find $\alpha \in \mathrm{On}$ such that $a_1, \ldots, a_n, x \in L_\alpha$.

In V, apply separation with $v_{n+1} \in v_{n+2} \wedge \phi^L$ and $v_1 = a_1, \ldots, v_n = a_n, v_{n+2} = x$ to obtain $z \in V$ such that

$$\forall t \ \left[t \in z \leftrightarrow t \in x \wedge \phi(a_1, \ldots, a_n, t)^L\right]$$

(leaving out the trivial relativization to V).

Apply the Levy Reflection Principle to ϕ and the hierarchy L_α to find $\gamma \in \mathrm{On}$ such that $\alpha \subseteq \gamma$ and ϕ is absolute for L_γ, L. Thus

$$\forall t \ \left[t \in z \leftrightarrow t \in x \wedge \phi(a_1, \ldots, a_n, t)^{L_\gamma}\right].$$

But $\phi(a_1, \ldots, a_n, t)^L_\gamma \leftrightarrow (L_\gamma, \in) \models \phi(a_1, \ldots, a_n, t)$ so that $z \in \mathit{Def}\,(L_\gamma) = L_{\gamma+1} \subseteq L$.

Finally since L is transitive, $t \in x \in L \to t \in L$ and $t \in z \subseteq x \to t \in L$ so that we obtain

$$\forall t \in L \ \left[t \in z \leftrightarrow t \in x \wedge \phi(a_1, \ldots, a_n, t)^L\right]$$

which of course is equivalent to

$$[z = \{t \in x : \phi(a_1, \ldots, a_n, t)\}]^L$$

as required. □

For **Replacement** we proceed similarly.

Proof of instances of ***Replacement***^{L}. Suppose $\phi(v_1, \ldots, v_n, v_{n+1}, v_{n+2})$ is a formula of LST (with all free variables shown) and $a_1, \ldots, a_n, d \in L$. Assume

$$[\forall x \in d \ \exists! y \ \phi(a_1, \ldots, a_n, x, y)]^L \equiv \forall x \in d \ \exists! y \in L \ \phi(a_1, \ldots, a_n, x, y)^L$$

(since L is transitive $x \in d$ is equivalent to $x \in d \wedge x \in L$).

Let $\psi(v_1, \ldots, v_n, v_{n+1}, v_{n+2}) \equiv v_{n+2} \in L \wedge \psi(v_1, \ldots, v_n, v_{n+1}, v_{n+2})^L$.

We check that ψ codes a function on d in V, i.e.

$$\forall x \in d \ \exists! y \ \psi(a_1, \ldots, a_n, x, y) \equiv \forall x \in d \ \exists! y \left[y \in L \wedge \phi(a_1, \ldots, a_n, x, y)^L\right].$$

So assume $x \in d$. By assumption, we can find $y \in L$ such that $\phi(a_1, \ldots, a_n, x, y)^L$ so that $y \in \wedge \phi(a_1, \ldots, a_n, x, y)^L$, showing existence. Now if $y, y' \in V$ such that $\psi(a_1, \ldots, a_n, x, y)$. Then $y, y' \in L$ and $\phi(a_1, \ldots, a_n, x, y)^L$ by definition of ψ so that by assumption $[y = y']^L \equiv y = y'$.

Thus we can apply **Replacement** in V to obtain $z \in V$ such that

$$\forall y \ [y \in z \leftrightarrow \exists x \in d \ \psi(a_1, \ldots, a_n, x, y)] \equiv \forall y \ \left[y \in z \leftrightarrow \exists x \in d \ \left[y \in L \wedge \phi(a_1, \ldots, a_n, x, y)^L\right]\right].$$

Note that therefore $z \subseteq L$. For each $y \in z$, let $\alpha_y \in \mathrm{On}$ be minimal such that $y \in L_{\alpha_y}$ (see remark) and let $\alpha \in \mathrm{On}$ be such that $a_1, \ldots, a_n, d \in L_\alpha$. Set $\alpha = \sup \{\alpha\} \cup \{\alpha_y : y \in z\} \in \mathrm{On}$ so that $z \subseteq L_\alpha$ and $a_1, \ldots, a_n, d \in L_\alpha$.

Apply Levy's Reflection Principle, to obtain $\gamma \in \text{Lim}$, $\alpha \subseteq \gamma$ such that ϕ is absolute for L_γ, L. Then

$$y \in z \leftrightarrow y \in L_\gamma \wedge \exists x \in d\ \phi(a_1, \ldots, a_n, x, y)^{L_\gamma}$$

so that $z \in Def(L_\gamma) = L_{\gamma+1}$.

By construction

$$[z = \{y : \exists x \in d\ \phi(a_1, \ldots, a_n, x, y)\}]^L$$

as required. □

Remark 13. It is worth spelling out what exactly goes on in 'let $\alpha_y \in \text{On}$ be minimal ...'. Formally, we consider the class

$$F = \{\langle y, \alpha \rangle : y \in z \wedge \alpha \in \text{On} \wedge y \in L_\alpha \wedge [\forall \beta \in \text{On}\ y \in L_\beta \to \alpha \subseteq \beta]\}$$

and the formula $\theta(v_1, v_2) \equiv \langle v_1, v_2 \rangle \in F$.

We show that this satisfies the condition for **Replacement** (in V) with $d = z$. So let $y \in z \subseteq L$ and pick $\hat{\alpha} \in \text{On}$ such that $y \in L_{\hat{\alpha}}$. Now note that (by **Separation** in V) and well-foundedness of $\in$ on On (the 'set'-version), $\alpha = \min\{\beta \in \hat{\alpha} \cup \{\hat{\alpha}\} : y \in L_\beta\}$ is well-defined (since the set on the RHS contains $\hat{\alpha}$ and is thus non-empty). Since $\langle y, \alpha \rangle \in F$ we obtain existence. Uniqueness on the other hand is clear from the definition and the fact that $\in$ is a total order on On.

The same proof shows that F is a class function on z (and because z is a set, it is in fact a function!). So by **Replacement** we can find $z \in V$ such that

$$z = \{F(y) : y \in z\}$$

and note that $F(y) = \alpha_y$ (as they were called in the proof).

Remark 14. We don't need to explicitly evoke Levy's Reflection Principle in the proof of **Replacement**^{L}. Having obtained α such that $z \subseteq L_\alpha$, we could simply apply **Separation** in L to get

$$\{y \in L_\alpha : y \in z\}$$

(where $y \in z$ has to be replaced by an appropriate formula) defining z in V. Of course, we still use the Levy Reflection Principle implicitly in the proof of the appropriate instance of **Separation**^{L}.

12 Choice

We use a 'very strong' form of choice (of course, they are all equivalent under **ZF** - but if your 'base theory' is slightly weaker then this form of choice typically implies all the other commonly used ones):

Definition 12.

$$\textbf{Choice} \equiv \forall x\ \exists r\ r \text{ is a well-order on } x$$

We will in fact prove an even stronger version about L, generally called 'Global Choice':

Definition 13. We say that a non-empty transitive class A satisfies 'Global Choice' if and only if we can write down an explicit formula $\phi(x, y)$ with two free variables such that ϕ is a well-order on A, i.e.

$$\begin{aligned}\textbf{ZF} \vdash & \forall x \in A\ \neg\phi(x, x)\ \wedge \\ & \forall x, y, z \in A\ [\phi(x, y) \wedge \phi(y, z) \rightarrow \phi(x, z)]\ \wedge \\ & \forall x, y \in A\ [\phi(x, y) \vee \phi(y, x) \vee x = y]\ \wedge \\ & \forall x \in A\ [x \neq \emptyset \rightarrow [\exists m \in x\ \forall y \in x\ \neg\phi(y, m)]]\end{aligned}$$

Lemma 13. ***ZF*** *proves that* ***Global Choice*** *implies* ***Choice****.*

Proof. Suppose $\phi(a, b)$ is as stated in **Global Choice** and fix x. By **Separation**, we let

$$r = \{\langle a, b\rangle \in x \times x : \phi(a, b)\}$$

and note that r 'works'. □

Remark 15. When applying this to L, we are of course implicitly using the Reflection Principle (which we used to show **Separation**) to show that $r \in L$.

Assuming that $<$ is a well-order on x, we proceed to write down a well-order $\hat{<}$ on $x^{<\omega} = \{f : n \rightarrow x : n \in \omega\}$:

Definition 14. Suppose $<$ is a relation on x. We define

$$\begin{aligned}f \hat{<} g \equiv & f, g \in x^{<\omega} \wedge \\ & [dom(f) \subsetneq dom(g) \vee \\ & [dom(f) = dom(g) \wedge \exists m \in dom(f)\ [f(m) < g(m) \wedge \forall k \in m\ f(k) = g(k)]]]\end{aligned}$$

Lemma 14. *If $<$ is a well-order on x, then $\hat{<}$ is a well-order on $x^{<\omega}$.*

We use this to lift well-orders from x to $Def(x)$. Essentially, we use $\hat{<}$ to well-order $\omega \times x^{<\omega}$ and then use the smallest witness $(n, a) \in \omega \times x^{<\omega}$ which 'codes' $z \in Def(x)$ to be the 'size' of z.

Definition 15. Suppose $<$ is a relation on x. We define

$$\begin{aligned}u \tilde{<} v \equiv & u, v \in Def(x) \wedge \\ & \exists n_u \in F \exists a_u \in x^{<\omega} \\ & [u = \{t \in x : \mathrm{val}(x, n_u, a_u) = 1\} \wedge \\ & \forall n_v \in F \forall a_v \in x^{<\omega}\ [v = \{t \in x : \mathrm{val}(x, n_v, a_v) = 1\} \rightarrow [n_u < n_v \vee [n_u = n_v \wedge a_u \hat{<} a_v]]]]\end{aligned}$$

Lemma 15. *If $<$ is a well-order on x then $\tilde{<}$ is a well-order on $Def(x)$.*

Theorem 13. ***ZF*** *proves that L satisfies* ***Global Choice****.*

Proof. By Recursion on On we define the class function $<_\alpha$ on On by

$$<_\emptyset = \emptyset,$$

$$<_{\alpha+1} = <_\alpha \cup \{\langle x,y\rangle : x \in L_\alpha \wedge y \in L_{\alpha+1} \setminus L_\alpha\} \cup \{\langle x,y\rangle : x,y \in L_{\alpha+1} \setminus L_\alpha \wedge x \tilde{<}_\alpha y\}$$

and of course if $\gamma \in \mathrm{Lim}$ then

$$<_\gamma = \bigcup \{<_\beta : \beta \in \gamma\}$$

It is easy to verify inductively that each $<_\alpha$ is a well-order on L_α extending every $<_\beta$ for $\beta \in \alpha$.

Thus we can define

$$\phi(x,y) \equiv x,y \in L \wedge \forall \alpha \in \mathrm{On}\ [x,y \in L_\alpha \to x <_\alpha y]$$

and again easily verify that this is a global well-order.

□

Corollary 2. *If* ***ZFC*** *is inconsistent, then so is* ***ZF****.*

13 $V = L$

To proceed we will prove a strong statement that L satisfies, namely that '$V = L$'. Let us first formally define what this means.

Recall that on the one hand L and V were class functions on On and on the other hand classes $\{x : \exists\alpha \in \mathrm{On}\ x \in L_\alpha\}$ and $\{x : \exists\alpha \in \mathrm{On}\ x \in V_\alpha\}$. The statement $V = L$ then simply means that these two classes are equal, i.e.

$$\forall x\, [\exists\alpha \in \mathrm{On}\ x \in L_\alpha \leftrightarrow \exists\alpha \in \mathrm{On}\ x \in V_\alpha]\,.$$

Since we assume **Foundation** (which was shown equivalent to $\forall x\ x \in V$) this is in fact equivalent to

$$\forall x\ \exists\alpha \in \mathrm{On}\ x \in L_\alpha.$$

For a non-empty class A that satisfies enough **ZF** − **Powerset** to define a class function L^A on On^A, we can relativize this to obtain

$$[V = L]^A \equiv \forall x \in A\ \exists\alpha \in \mathrm{On}^A\ x \in L^A_\alpha.$$

(Note that in fact we can relativize to any non-empty class A, but then L might not in fact be a class function and we would have to write out carefully what we mean by $x \in L_\alpha$.)

That L satisfies $V = L$, i.e. that

$$\mathbf{ZF} \vdash [V = L]^L$$

is remarkable. To appreciate this, assume for the moment that $V \neq L$. Then V^L (V defined in L) will in fact be L and thus different from 'the real' V. Thus it could be that $L^L \subsetneq L^V$ (or in fact $L^L \supsetneq L^V$).

We thus need to delve into the definition of L to prove

Theorem 14. *The class function Def is absolute for non-empty transitive classes satisfying enough of* $\boldsymbol{ZF-Powerset}$.

Hence the class function L is absolute for non-empty transitive classes A satisfying enough of $\boldsymbol{ZF-Powerset}$ *in the sense that* $\forall\alpha\in\mathrm{On}^A\ L^A_\alpha=L_\alpha$.

The proof is essentially straightforward: we verify that val and hence Def are absolute and then absoluteness will 'trickle up' (through the Recursion Theorem) to L.

But for absoluteness of val we need a Lemma: recall the quantifier case of the definition of val:

$$\mathrm{val}(x,\ulcorner\forall v_k\phi\urcorner,a)=1\leftrightarrow\forall\hat{a}\in x^{<\omega}\dots$$

What is written in $\dots$ is absolute (can be expressed by a Δ_0-formula), but we will also need that $x^{<\omega}$ is absolute and exists as a set.

Let's write

$$\phi(f,x)\equiv\exists n\in\mathrm{On}\,[n\text{ is finite}\wedge f\text{ is a function from }n\text{ to }x]$$

and that

$$x^{<\omega}=\{f:\phi(f,x)\}\,.$$

Lemma 16. *Suppose $A\subseteq B$ are non-empty transitive classes satisfying enough of* $\boldsymbol{ZF-Powerset}$.

$$\boldsymbol{ZF-Powerset}\vdash\forall x\in A\left[\left(x^{<\omega}\right)^A=\left(x^{<\omega}\right)^B\right].$$

Proof. Recall that

$$f\in x^{<\omega}\equiv\phi(f,x)\equiv f\text{ is a function on }\omega\wedge\exists n\in\omega\ dom(f)\subseteq n.$$

ϕ is absolute for transitive classes satisfying enough of **ZF − Powerset** since it can be expressed with a Δ_0-formula.

Hence $(x^{<\omega})^A\subseteq(x^{<\omega})^B$.

So assume that the inclusion is strict. Let $n\in\omega$ be minimal such that

$$\exists f\in\left(x^{<\omega}\right)^B\setminus\left(x^{<\omega}\right)^A\wedge dom(f)\subseteq n.$$

Because the formula ϕ above is absolute, this means that $f\in B\setminus A$.

If $n=\emptyset$ then $f=\emptyset\in A$, a contradiction.

So suppose $n=m+1$ and note that by minimality of n, $m\in dom(f)$. Again by minimality of n, $f|_m\in A$. By transitivity of A and $f(m)\in x\in A$ and the fact that A satisfies **Pairing** and **Union**, we get $m\in A$ and $f=f|_m\cup\{\langle m,f(m)\rangle\}\in A$, a contradiction. ☐

Lemma 17. *Suppose A is a non-empty transitive class satisfying enough of* $\boldsymbol{ZF-Powerset}$.

$$\boldsymbol{ZF-Powerset}\vdash\forall x\in A\ x^{<\omega}\in A.$$

Proof. All the notions used in this proof are absolute, and we will use this silently. Also, we will not state which axioms we need exactly, but take care not to use **Powerset**.

Note that $x^{\emptyset} = \emptyset \in A$. Also

$$x^1 = \{\{\langle 0, t\rangle\} : t \in x\} \in A.$$

Next, if $a, b \in A$ then

$$a \times b = \bigcup \{\{\langle r, t\rangle : r \in a\} : t \in b\} \in A.$$

Assume that $x^n \in A$. Then

$$x^{n+1} = \{f \cup \{\langle n, t\rangle\} : \langle f, t\rangle \in x^n \times x\} \in A.$$

Thus by induction on $n \in \omega$, $\forall n \in \omega\ x^n \in A$.

Finally

$$x^{<\omega} = \bigcup \{x^n : n \in \omega\} \in A.$$

$\square$

Proof of Theorem. Suppose A is a non-empty transitive class satisfying enough of **ZF** $-$ **Powerset**. All 'absolute' below refer to 'absolute for $A, \{x : x = x\}$'.

By absoluteness of ordinal addition and multiplication and existence and absoluteness of $x^{<\omega}$, the sets T and F are absolute, as are free and val.

Hence by absoluteness of ω and again absoluteness of x^{ω},

$$Def(x) = \left\{ z : \exists n \in F\ \exists a \in x^{<\omega} \left[\begin{array}{l} dom(a) = \mathrm{free}(n) \setminus \{0\} \wedge 0 \in \mathrm{free}(n) \wedge \\ \forall t \in x\ \ [t \in z \leftrightarrow \mathrm{val}(x, n, a \cup \{\langle 0, t\rangle\}) = 1)] \end{array} \right] \right\}$$

is absolute.

Finally, by the absoluteness of class functions defined by Recursion (on On) (from absolute class functions F, here $F = Def$), L is absolute. $\square$

Theorem 15.

$$\textbf{\textit{ZF}} - \textbf{\textit{Powerset}} \vdash [V = L]^L.$$

Proof. The statement $[V = L]^L$ means

$$\forall x \in L\ \exists \alpha \in \mathrm{On}^L\ x \in L_\alpha^L.$$

Since $\mathrm{On}^L \leftrightarrow \mathrm{On}$ (being an ordinal is absolute for L, V) and $L_\alpha^L = L_\alpha$ (by absoluteness of Def and induction on α), $[V = L]^L$ is equivalent to

$$\forall x \in L\ \exists \alpha \in \mathrm{On}\ x \in L_\alpha$$

which is true by definition of L. $\square$

Corollary 3. *If **ZFC**+ **V=L** is inconsistent, then so is **ZF**.*

14 Mostowski Collapse

Definition 16. Suppose that A is a set (in fact the proof also works for classes) and R is a well-founded, set-like relation on A. Writing $pred(x) = \{y \in A : yRx\}$ we define by generalized recursion $mos(x) = \{mos(y) : y \in pred(x)\}$. We write $mos[A]$ for the image of A under mos.

Lemma 18. *If R is well-founded and set-like on a set (or class) A, then $mos[A] = \{mos(x) : x \in A\}$ is transitive.*

Proof. If $r \in t \in mos[A]$ then find $x \in A$ with $mos(x) = t$. Then $r \in t = mos(x)$ so there is $y \in A$ such that yRx and $mos(y) = r$. Hence $r \in mos[A]$. $\square$

Lemma 19. *If R is well-founded, set-like and extensional ($\forall x, y \in A\ pred(x) = pred(y) \to x = y$) on A then mos is injective on A and in fact an isomorphism between (A, R) and $(mos[A], \in)$.*

Proof. Suppose mos is not injective. Then the set (class)

$$\{x \in A : \exists b \in A\ \ [b \neq x \wedge mos(b) = mos(x)]\}$$

is non-empty so has an R-minimal element a. Fix $b \in A$ with $mos(b) = mos(a)$ and $b \neq a$. If there is $c \in A$ such that cRb then $mos(c) \in mos(b) = mos(a)$ so there is $d \in A$ such that dRa and $mos(c) = mos(d)$. But dRa so by R-minimality of a, $c = d$ giving cRa. Similarly, if there is $d \in A$ such that dRa then $mos(d) \in mos(a) = mos(b)$ so there is $c \in A$ such that cRb and $mos(c) = mos(d)$. Again, R-minimality of a gives $d = c$. Thus $pred(b) = pred(a)$ and since R is extensional $b = a$ follows, a contradiction.

Clearly, mos is thus a bijection between A and $mos[A]$ and by construction $\forall a, b \in A\ \ [aRb \leftrightarrow mos(a) \in mos(b)]$. $\square$

Lemma 20. *If $R = \in$ (which from **ZF** is well-founded, set-like and extensional) and $T \subseteq A$ is transitive, then $mos|_T = id|_T$.*

Proof. If not, then set (class) $\{t \in T : mos(t) \neq t\}$ is non-empty, so has an $\in$-minimal element m. Now $mos(m) \neq m$ means that

- either there is $d \in mos(m) \setminus m$. Find $t \in A$ such that tRm (which means $t \in m$ as $R = \in$) and $mos(t) = d$. Then $t \in m \in T$ so $t \in T$ and thus by R-minimality of m, we must have $d = mos(t) = t \in m$, a contradiction;
- or there is $t \in m \setminus mos(m)$. Again $t \in m \in T$ gives $t \in T$ and tRm so that $t = mos(t) \in mos(m)$, a contradiction.

$\square$

15 Cardinals

We now assume **ZFC** but indicate the use of **Choice** and **Powerset** when they are necessary for our proofs.

You should have seen most of this material in your first Set Theory course. Specific attention should be paid to Lemma 24 and its proof which requires neither **Powerset** nor **Choice**.

Definition 17. For $x, y \in V$ we write $x \preceq y$ if and only if there is an injection $f : x \to y$.

We write $x \approx y$ if and only if there is a bijection $f : x \to y$.

Lemma 21.

1. *$\preceq$ is a reflexive, transitive relation on V (i.e. a pre-order);*
2. *$\approx$ is a symmetric, reflexive, transitive relation on V (i.e. an equivalence relation)*

Theorem 16 (Schröder-Bernstein)**.** *For every $x, y \in V$, $x \preceq y \wedge y \preceq x \to x \approx y$.*

Lemma 22 (requires **Choice**)**.** *For $x, y \in V$: if $f : x \to y$ is a surjection then $y \preceq x$.*

Proof. Well-order x and define $g : y \to x$ by $g(t) =$ the least element of $f^{-1}[t]$. □

Theorem 17 (requires **Choice**)**.** *Every set is bijective with an ordinal. Hence $\preceq$ is total (i.e. $\forall x, y\ [x \preceq y \vee y \preceq x]$.*

Proof. Suppose x is a set. Well-order x by $<$ and apply the Mostowski Collapse (where the relation is the well-order on x - note that this is well-founded, set-like and extensional) to obtain a t such that t is $\in$-well-ordered and transitive, so that t is an ordinal and $\mathrm{mos}\colon (x, <) \to t$ is a bijection. □

Definition 18 (requires **Chioice**)**.**

$$|x| = \text{the least } \alpha \in \mathrm{On} \text{ such that } \alpha \approx x$$

An ordinal α is a cardinal if and only if $\forall \beta \in \alpha\ \beta \not\approx \alpha$.

We write Card for the class of cardinals.

Lemma 23.

1. *$\forall x\ |x|$ is a cardinal;*
2. *$\forall \alpha \in \mathrm{On}\ |\alpha| \subseteq \alpha$;*
3. *$\forall \kappa \in \mathrm{Card}\ |\kappa| = \kappa$;*
4. *every infinite cardinal is a limit ordinal;*

Proof.

1. By minimality of $|x|$.
2. By the definition of $|\alpha|$ and the fact that $\mathrm{id}\colon \alpha \to \alpha$ is a bijection.

3. By the previous results $|\kappa| \subseteq \kappa$ and $|\kappa| \in \kappa$ being an immediate contradiction to the definition of cardinal.

4. If $\omega \subseteq \alpha \in \mathrm{On}$ then
$$\beta \mapsto \begin{cases} \beta+1 & \beta \in \omega \\ \beta & \beta \in \alpha \setminus \omega \\ 0 & \beta = \alpha \end{cases}$$
is a bijection $\alpha + 1 \to \alpha$ so that $\alpha + 1$ cannot be a cardinal.

□

Lemma 24. *For all* $\alpha \in \mathrm{On}$ *with* $\alpha \geq \omega$, $|\alpha| = |\alpha \times \alpha|$.

Proof. Clearly $t \mapsto \langle t, t \rangle$ is an injection from α into $\alpha \times \alpha$, so $|\alpha| \leq |\alpha \times \alpha|$

For the other direction:

We well-order $\mathrm{On} \times \mathrm{On}$ by $(t, s) < (t', s')$ if and only if $\max\{t, s\} < \max\{t', s'\}$ or $\max\{t, s\} = \max\{t', s'\} \wedge [t < t' \vee [t = t' \wedge s < s']]$.

Now, induct on α. For $\alpha = \omega$, write down an explicit injection (e.g. $(n, m) \mapsto 2^n 3^m$).

So assume this is true below some ordinal α: If α is not a cardinal, then $\alpha \approx \beta$ for $\beta < \alpha$ and then clearly
$$\alpha \approx \beta \approx \beta \times \beta \approx \alpha \times \alpha.$$

So assume that α is a cardinal: Note that $(\alpha \times \alpha, <)$ is isomorphic to an ordinal δ (by using the Mostowski Collapse). If $\alpha \in \delta$, then let $\langle \beta_1, \beta_2 \rangle \in \alpha \times \alpha$ so that $\alpha = mos(\langle \beta_1, \beta_2 \rangle)$. Let $\beta = \max\{\beta_1 + 1, \beta_2 + 1\}$. As α is a limit ordinal, $\beta \in \alpha$ so that $|\beta| = |\beta \times \beta|$. But then $\langle \beta_1, \beta_2 \rangle \in \beta \times \beta$ so that $\alpha \in mos[\beta \times \beta]$. But $|\beta \times \beta| = |\beta|$ so that $mos[\beta \times \beta] \leq |\beta|$ giving $\alpha \leq |\beta| \preceq \beta$, a contradiction to α being a cardinal. Hence $\delta \leq \alpha$ and thus $|\alpha \times \alpha| \leq |\alpha| = \alpha$ as required. □

In Part B 'Set Theory'you may have seen a different proof which uses **Choice** (in the form of Zorn's Lemma) and **Powerset** to show this for all infinite sets x. Of course with the well-ordering principle (our form of **Choice**) our proof yields the same result: let $f : x \to \alpha$ be a bijection and $g : \alpha \times \alpha \to \alpha$ be the bijection from what we've just proven. Then $f^{-1} \circ g \circ (f \times f) : x \times x \to x$ is the required bijection.

Our proof has two advantages: it defines an explicit, absolute injection $\alpha \times \alpha \to \alpha$ (and in fact an explicit absolute bijection $\mathbf{On} \times \mathbf{On} \to \mathbf{On}$ by the class version of the Mostowski collapse (absolute follows because Recursion preserves absoluteness and $<$ is absolute). Also we avoid the use of **Powerset**.

Definition 19. For cardinals α, β, define $\alpha \oplus \beta = |\alpha \times \{0\} \cup \beta \times \{1\}|$ and $\alpha \otimes \beta = |\alpha \times \beta|$.

Note that $\oplus$ and $\otimes$ do not require **Choice** as the lexicographic order is an explicit well-order.

Lemma 25. *1.* $\oplus, \otimes$ *are weakly order preserving in both arguments;*

2. $\oplus$ *is associative, commutative and for infinite* α, β, $\alpha \oplus \beta = \max\{\alpha, \beta\}$.

3. $\otimes$ *is associative and commutative and for infinite* α, β, $\alpha \otimes \beta = \max\{\alpha, \beta\}$.

4. $\otimes$ *distributes over* $\oplus$;

5. $\oplus$ *and* $\otimes$ *coincide with the usual (ordinal) addition and multiplication on* ω.

Definition 20 (requires **Choice** and **Powerset**). For cardinals α, β, define $\alpha^\beta = \left|\alpha^\beta\right|$.

Note that on the right hand side α^β is the set of all functions from β to α.

Lemma 26 (requires **Choice** and **Powerset**). *Using* $+$ *and* $.$ *instead of* $\oplus$ *and* $\otimes$ *(i.e. cardinal arithmetic). For cardinals* κ, λ, θ:

1. $\kappa^{\lambda.\theta} = \left[\kappa^\lambda\right]^\theta$;

2. $\kappa^{\lambda+\theta} = \kappa^\lambda.\kappa^\theta$;

3. $2^\kappa = |\mathcal{P}(\kappa)|$;

4. *if* $2 \leq \kappa \leq \lambda$, $\omega \leq \lambda$ *then* $\kappa^\lambda = 2^\lambda$

Definition 21 (requires **Choice** and **Powerset**). Suppose I is a non-empty set and $f : I \to \mathrm{Card}; i \mapsto \kappa_i$ a functions. We define

$$\sum_{i \in I} \kappa_i = \left|\bigcup_{i \in I} \kappa_i \times \{i\}\right|$$

and

$$\prod_{i \in I} \kappa_i = \left|\left\{g : I \to \bigcup_{i \in I} \kappa_i : \forall i \in I\ g(i) \in \kappa_i\right\}\right|.$$

Lemma 27 (requires **Choice** and **Powerset**). *Under hypothesis as in the definition:*

1. *if* $I = \{0, 1\}$ *then* $\sum_{i \in I} \kappa_i = \kappa_0 \oplus \kappa_1$ *and* $\prod_{i \in I} \kappa_i = \kappa_0 \otimes \kappa_1$;

2. *if* κ *is any cardinal then* $\kappa \otimes \sum_{i \in I} \kappa_i = \sum_{i \in I} \kappa \otimes \kappa_i$;

3. *if* $f : I \to \mathrm{Card}$ *is constant with value* κ *(i.e.* $\forall i \in I\ \kappa_i = \kappa$*) and* $I \in \mathrm{Card}$ *then* $\prod_{i \in I} \kappa_i = \kappa^I$;

Theorem 18 (König's Inequality - requires **Choice** and **Powerset**). *Suppose* I *is a non-empty set and* $f, g : I \to \mathrm{Card}$. *Write* $\kappa_i = f(i)$ *and* $\lambda_i = g(i)$.

If $\forall i \in I\ \kappa_i < \lambda_i$ *then* $\sum_i \kappa_i < \prod_i \lambda_i$.

Proof. Suppose that $h : \sum_i \kappa_i \to \prod_i \lambda_i$ is a function. We will show that h is not a surjection.

For $i \in I$, let $\pi_i : \prod_i \lambda_i \to \lambda_i; x \mapsto x_i$ be the projection onto the i^{th} coordinate.

For each $i \in I$, note that $|\pi_i \circ h[\kappa_i \times \{i\}]| \leq \kappa_i < \lambda_i$ so that $\pi_i \circ h$ cannot be onto. Let $c_i \in \lambda_i \setminus \pi_i \circ h[\kappa_i \times \{i\}]$ be least (in the well-order of λ_i).

Now let $c = \{\langle i, c_i \rangle : i \in I\} \in \prod_i \lambda_i$. If $\alpha \in \sum_i \kappa_i$ then $\pi_i \circ h(\alpha) \neq c_i$ so that $h(\alpha) \neq c$. Thus h is not surjective. $\square$

Definition 22. For any ordinal α, define $\mathrm{cf}(\alpha)$ to be the least ordinal β such that there is an unbounded function $f : \beta \to \alpha$ (i.e. f satisfies $\forall \gamma \in \alpha \exists \delta \in \beta\ \gamma \leq f(\delta)$).

α is regular if and only if $\mathrm{cf}(\alpha) = \alpha$. α is singular if it is not regular.

Note that the identity from α to itself is unbounded, so $\mathrm{cf}(\alpha)$ is well-defined.

Lemma 28. *For all* $\alpha \in \mathrm{On}$*:*

1. $\mathrm{cf}(\alpha) \leq \alpha$*;*
2. $\mathrm{cf}(\alpha) \in \mathrm{Card}$*;*
3. $\mathrm{cf}(\alpha + 1) = 1$ *and for limit ordinals* α*,* $\mathrm{cf}(\alpha) \geq \omega$*;*
4. *if* $f : \mathrm{cf}(\alpha) \to \alpha$ *is unbounded then there is a weakly increasing* $g : \mathrm{cf}(\alpha) \to \alpha$ *that is unbounded;*
5. $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$*, so* $\mathrm{cf}(\alpha)$ *is regular;*

Proof.

1. $\alpha \to \alpha; \beta \mapsto \beta$ is unbounded;
2. Suppose $\beta < \mathrm{cf}(\alpha)$ and assume $\beta \approx \mathrm{cf}(\alpha)$. Let h be a witnessing bijection and $f : \mathrm{cf}(\alpha) \to \alpha$ unbounded. Then $f \circ h : \beta \to \alpha$ is unbounded, contradicting minimality of $\mathrm{cf}(\alpha)$.
3. $\alpha+1$ is non-empty, so $\mathrm{cf}(\alpha+1) \neq 0$; but $0 \mapsto \alpha$ defines an unbounded map from 1 into $\alpha + 1$. If on the other hand $f : n + 1 \to \alpha$ is unbounded then either $f(n)$ is the maximal element of $f[n+1]$ or $f|_n$ is also unbounded. In the former case $\alpha = f(n) + 1$, in the other $\mathrm{cf}(\alpha) < n + 1$. So by induction on n, $\mathrm{cf}(\alpha) \neq n$ for limit ordinals α.
4. This is clear for successor ordinals α, so assume that α is a limit and hence that $\mathrm{cf}(\alpha)$ is a limit ordinal. Define $g : \mathrm{cf}(\alpha) \to \mathrm{On}$ by $g(\beta) = \sup_{\delta<\beta} f(\delta) = \bigcup_{\delta<\beta} f(\delta)$. Clearly g is weakly increasing. To see that g maps into α, use minimality of $\mathrm{cf}(\alpha)$: if $g(\beta) \geq \alpha$ for some $\beta \in \mathrm{cf}(\alpha)$ then $f|_\beta$ is unbounded in in α: if $\eta \in \alpha \subseteq g(\beta)$ then there is $\delta < \beta$ with $\eta \in f(\delta)$ a contradiction. Finally if $\eta \in \alpha$ then find $\beta \in \mathrm{cf}(\alpha)$ with $\eta \leq f(\beta)$ and note that as $\mathrm{cf}(\alpha)$ is a limit ordinal, $\eta \leq f(\beta) \leq g(\beta + 1)$ makes sense.

5. clear if α is a successor from above. So assume α is a limit ordinal and let $h : \mathrm{cf}(\mathrm{cf}(\alpha)) \to \mathrm{cf}(\alpha)$ and $f : \mathrm{cf}(\alpha) \to \alpha$ be unbounded. Without loss of generality, h, f are weakly increasing. Define $g : \mathrm{cf}(\mathrm{cf}(\alpha)) \to \alpha$ by $g(\beta) = f(\gamma)$ where $\gamma \in \mathrm{cf}(\alpha)$ is least such that $h(\beta) \leq \gamma$.

   To see that g is unbounded, let $\eta \in \alpha$, find $\delta \in \mathrm{cf}(\alpha)$ such that $\eta \leq f(\delta)$ find $\beta \in \mathrm{cf}(\mathrm{cf}(\alpha))$ such that $f(\delta) \leq h(\beta)$ and check that $g(\beta) \geq f(h(\beta)) \geq f(\delta) \geq \eta$ as required.

□

Theorem 19 (König's Inequality). *For every infinite cardinal κ, $\kappa < \mathrm{cf}(2^\kappa)$.*

Proof. Let $\lambda = 2^\kappa$ and $\theta = \mathrm{cf}(2^\kappa)$ and assume $\theta \leq \kappa$. Letting $f : \theta \to \lambda$ be unbounded and weakly increasing, and writing $\kappa_\alpha = f(\alpha)$ and noting that $\kappa_\alpha \in \lambda$ so that $|\kappa_\alpha| < \lambda$, we have

$$\lambda = \bigcup_{\alpha\in\theta} \kappa_\alpha \leq \sum_{\alpha\in\theta} |\kappa_\alpha| < \prod_{\alpha\in\theta} \lambda = \lambda^\theta = 2^{\kappa\otimes\theta} \leq 2^{\kappa\otimes\kappa} = 2^\kappa = \lambda$$

a contradiction. □

16 GCH

We will now show that $ZFC + VL \vdash GCH$ where GCH is the formula

$$\forall\lambda \in Card\ 2^\lambda = \lambda^+.$$

We will state and prove a couple of theorems first which are essentially refinements of what we have done before.

Theorem 20 (ZFC).

$$\forall\alpha \in \mathrm{On}\ \ [\alpha \geq \omega \to |L_\alpha| = |\alpha|]$$

Proof. By induction on α. First by induction on $n \in \omega$, we show that $|L_n| = |V_n|$ is finite so that $|V_\omega| = |L_\omega| = \omega$ establishing the base case.

Now, if $|L_\alpha| = |\alpha| \geq \aleph_0$ then observe that $|\omega \times A_{L_\alpha} \times \{L_\alpha\}| = \aleph_0 \otimes |\alpha| \otimes 1 = |\alpha|$ and that the map $(n, \sigma, L_\alpha) \mapsto \{t \in L_\alpha : val(n, \sigma \cup \{\langle 0, t\rangle\}, L_\alpha) = T\}$ is a surjection from $\omega \times A_{L_\alpha} \times \{L_\alpha\}$ onto $L_{\alpha+1}$ so that (using **Choice**) $|L_\alpha| \leq |\alpha| = |\alpha + 1|$. On the other hand $\alpha + 1 \subseteq L_{\alpha+1}$ so that $|\alpha + 1| \leq |L_{\alpha+1}|$.

The limit case is trivial: if $\gamma \in \mathrm{Lim}$, $|\gamma| = \left|\bigcup_{alpha<\gamma} \alpha\right| = \left|\bigcup_{\alpha<\gamma} L_\alpha\right| \leq |L_\gamma|$. □

Theorem 21 (ZFC). *If M is a transitive set which satisfies (enough of)* ***ZF − Powerset*** *$+ V = L$ then $M = L_\gamma$ for $\gamma = M \cap \mathrm{On} = \min \mathrm{On} \setminus M$ and γ is a limit ordinal.*

Proof. Firstly note that transitivity of M shows that $\min \mathrm{On} \setminus M = M \cap \mathrm{On}$. Next, if $\alpha \in M$ then $\alpha + 1 = \alpha \cup \{\alpha\} \in M$ by **Pairing** and **Union** and the fact that $\alpha + 1$ is absolute. Hence γ must be a limit.

Now, suppose $x \in M$. Since $M \models V = L$ this means $\exists \delta \in M \cap \mathrm{On}^M\ x \in L_\delta^M$. Both On and L_δ are absolute for M, V, so we have in fact $\exists \delta \in \gamma\ x \in L_\delta$. Fixing some such δ we see that $x \in L_\delta \subseteq L_\gamma$.

Finally, assume $x \in L_\gamma$. As γ is a limit ordinal, there is $\delta \in \gamma$ with $x \in L_\delta$. But then $\delta \in M$ and $L_\delta = L_\delta^M \subseteq M$ so that $x \in M$ as required. $\square$

Theorem 22 (ZFC)**.** *For every regular uncountable cardinal κ, $L_\kappa \models$ **ZF** $-$ **Powerset** $+ V = L$ (or enough thereof if you are nervous).*

Proof. We have seen earlier that for example $x, y \in L_\alpha \to \{x, y\} \in L_{\alpha+1}$, $x \in L_\alpha \to \bigcup x \in L_\alpha$, $\omega \in L_{\omega+1}$ etc so that our proofs that L satisfies each of **Union**, **Pairing**, **Infinity**, **Foundation**, **Extensionality** go through. We need to check **Separation** and **Replacement**.

For **Separation**: first modify the theorem and proof of Levy's Reflection principle as follows: replace On by κ (regular uncountable cardinal) and add the condition $\forall \beta \in \kappa\, |A_\beta| < \kappa$. Then each $F_i(\vec{a}) < \kappa$ and there are $< \kappa$ many $\vec{a} \in A_\alpha^{n_i}$ so that $G_i(\alpha) < \kappa$ by regularity. Since there are only finitely many i, $K(\alpha) = \max G_i(\alpha) < \kappa$. But then the sequence α_n consists of countably (so $< \kappa$) many ordinals $< \kappa$ so by regularity $\gamma = \sup \alpha_n < \kappa$. Now apply the proof that $L \models$ **Separation** to L_κ, noting that $|L_\alpha| = |\alpha| < \kappa$ for $\alpha < \kappa$ as κ is a cardinal.

For **Replacement**: we employ a similar strategy, essentially checking that the set we produce still belongs to L_κ. If $\phi(x, y, \vec{v})$ is the formula, $\vec{a} \in L_\kappa^n$ and L_κ believes that $\forall x \forall y, y'\ [\phi(x, y, \vec{a}) \wedge \phi(x, y', \vec{a}) \to y = y']$ and $d \in L_\kappa$ then firstly $d \in L_\alpha$ for some $\alpha < \kappa$ and hence $|d| \leq |\alpha + 1| < \kappa$. So, writing α_x for the least ordinal such that $y_x \in L_{\alpha_x}$ (where y_x is the unique element of L_λ such that $\phi(x, y_x, \vec{a})$) we see that $\alpha_x < \kappa$. Regularity of κ shows that $\alpha = \sup \{\alpha_x : x \in d\} < \kappa$ so that the $z = \left\{y_x : x \in d \wedge \phi(x, y_x, \vec{a})^{L_\kappa}\right\} \subseteq L_\alpha$ so that $z \in L_{\alpha+1} \subseteq L_\lambda$.

For $V = L$, we essentially follow the proof that $L \models V = L$, i.e. use absoluteness of the class function L (for L_κ, V - we now know that L_κ satisfies enough of **ZF** $-$ **Powerset**): recall that $V = L \equiv \forall x\ exists \alpha \in \mathbf{On}\ x \in L_\alpha$, that $\mathbf{On}^{L_\kappa} = \kappa$ and that for $\alpha \in \kappa$ we have $(L_\alpha)^{L_\kappa} = L_\alpha$. Thus (after relativizing and using these facts) $(V = L)^{L_\kappa} \leftrightarrow \forall x \in L_\kappa\ \exists \alpha \in \kappa\ x \in L_\alpha$ which follows from the definition of $L_\kappa = \bigcup_{\alpha \in \kappa} L_\alpha$ since κ is a limit ordinal (infinite cardinals are limit ordinals). $\square$

Theorem 23 (Downward Löwenheim Skolem Theorem)**.** *If A is an infinite subset of a set B then there is C such that $A \subseteq C \subseteq B$, $|A| = |C|$ and for every formula ϕ of LST, ϕ is absolute for C, B (i.e. C is an elementary submodel of B, often written $C \preceq B$).*

Proof. See a book on model theory. For a rough idea (see remarks after the proof), follow the proof of the Reflection Principle: Enumerate the formulas of LST by $\phi_i, i \in \omega$. Instead of having that F_i choose ordinals α, let them choose (using **Choice**) witnessing elements b (i.e. well order B; if $B \models \phi_i(b_1, \ldots, b_{n_i}) = \exists x \psi(x, b_1, \ldots, b_{n_i})$ we let $F_i(b_1, \ldots, b_{n_i})$ be the least element b of B such that $\psi(b, b_1, \ldots, b_n)$). Now, starting with $A_0 = A$, let $A_{n+1} = A_n \cup \{F_i(a_1, \ldots, a_{n_i}) : i \in \omega \wedge a_1, \ldots, a_{n_i} \in A_n\}$. Note that $|A_{n+1}| = |A_n|$ and that $C = \bigcup_{n \in \omega} A_n$ is as required. □

Of course we cannot enumerate the 'real'formulas inside our theory (and specifically not with the ω of our theory), so we would need to internalize them, i.e. we enumerate the Gödel numbers (in our theory) of formulas and then things like $B \models \phi(b_1, \ldots, b_n)$ is replaced by $val(\ulcorner \phi \urcorner, \sigma, B)$ for a suitable σ. We need to convince ourselves once more that this works as intended.

Alternatively, we note that for our purposes later, we only ever consider a specific (though not explicitly described) finite set of formulas, so we could just use these.

Recall that $H_\kappa = \{x : |TC(x)| < \kappa\}$ where $TC(x)$ is the smalles transitive set t with $x \subseteq t$.

Theorem 24. ***ZFC*** *$+ V = L$ proves for infinite cardinals κ, $H_\kappa = L_\kappa$ and hence GCH.*

Proof. The critical case is to show $L_{\kappa^+} = H_{\kappa^+}$ (since we have already seen that $L_\omega = H_\omega$ and 'limit cardinals' work by unions):

First $L_\kappa \subseteq H_\kappa$: If $x \in L_\kappa$ then $x \in L_\alpha$ for $\alpha < \kappa$ so that $x \subseteq L_\alpha$ (transitivity) and thus $|TC(x)| \leq |L_\alpha| = |\alpha| < \kappa$.

For $H_{\kappa^+} \subseteq L_{\kappa^+}$: Fix $x \in H_{\kappa^+}$ and let $t = TC(\{x\})$. Note that $t = \{x\} \cup TC(x)$ so that $|t| \leq \kappa$. Using $V = L$, find a regular, uncountable λ such that $t \in L_\lambda$ (find an infinite $\alpha \in \mathrm{On}$ such that $t \in L_\alpha$ and then note that $\lambda = |\alpha|^+$ is regular and uncountable). Noting that L_λ satsifies (enough of) **ZF**$-$**Powerset**$+ V = L$ apply the Downward Löwenheim Skolem Theorem to obtain s such that $t \subseteq s \subseteq L_\lambda$, $|s| = |t|$ and s satsifies (enough of) **ZF** $-$ **Powerset** $+ V = L$. Then consider the Mostowski Collapse $mos[s]$ of s. On the one hand $t \subseteq s$ is transitive so $mos|_t = id|_t$ and in particular $x = mos(x) \in mos[s]$. On the other hand $mos[s]$ satisfies the same $\in$-sentences as s (since mos is an $\in$-isomorphism) so that $mos[s]$ satisfies (enough of) **ZF** $-$ **Powerset** $+ V = L$ and hence $mos[s] = L_\gamma$ for $\gamma = s \cap \mathrm{On}$ and $x \in L_\gamma$. But then

$$|\gamma| = |L_\gamma| = |mos[s]| \leq |s| = |t| \leq \kappa$$

so that $\gamma < \kappa^+$ and $x \in L_{\kappa^+}$ as required.

Finally, to see that this implies GCH, consider any infinite cardinal λ. If $A \in \mathcal{P}(\lambda)$ (the powerset being taken in L of course) then $TC(A) \subseteq A \cup \lambda$ (as the RHS is transitive) so that $|TC(A)| \leq \lambda$. Hence $A \in H_{\lambda^+}$. Thus $\mathcal{P}(\lambda) \subseteq H_{\lambda^+} = L_{\lambda^+}$ and thus $|\mathcal{P}(\lambda)| \leq |L_{\lambda^+}| \leq \lambda^+$. On the other hand by Cantor's Theorem $|\mathcal{P}(\lambda)| \geq \lambda^+$ so that $|\mathcal{P}(\lambda)| = \lambda^+$ as required.

□