

Lectures on classical algebraic geometry (C3.4 Algebraic Geometry).^{*†}

Damian RÖSSLER[‡]

Oxford, Michaelmas Term

PLEASE LET ME KNOW OF ANY MISTAKES OR TYPOS THAT YOU FIND IN
THESE NOTES

Contents

1	Introduction	3
2	Hilbert's Nullstellensatz and algebraic sets	4
3	Regular maps between algebraic sets	6
4	Varieties	10
5	Open and closed subvarieties	15
6	Projective space	16
7	Projective varieties	18
8	Dimension	21
9	Rational maps	24
10	Products	28
11	Intersections in affine and projective space	35

^{*}I am so very grateful to Yutong Dai, who carefully read the text and provided me with a long list of typos.

[†]Last modified on 2025/05/03 at 16:07:11

[‡]Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford OX2 6GG, United Kingdom

12 Separatedness and completeness	37
13 Smoothness	42
14 Blowing up	48
15 Appendix. Proof of Theorem 8.7.	50
Exercise sheet 1. Chapters 1-4.	54
Exercise sheet 2. Chapters 1-8.	57
Exercise sheet 3. Chapters 1-12.	64
Exercise sheet 4. All lectures.	71

1 Introduction

Classical algebraic geometry is the study of the sets of simultaneous solutions of collections of polynomial equations in several variables with coefficients in an algebraically closed field. Such sets are called algebraic varieties. So eg the set of simultaneous solutions of the equations $x^2 + y^2 - 1 = 0, xy = 0$ in \mathbb{C}^2 is an algebraic variety.

Because they are so easy to define, algebraic varieties appear in almost every area of mathematics. They play a crucial role in number theory, in topology, in differential geometry and complex geometry (ie the theory of complex manifolds). When the base field is \mathbb{C} , an algebraic variety defines a complex manifold provided it has "no kinks" (we shall give a precise definition later).

A basic reference for classical algebraic geometry is chap. I of D. Mumford's book *The Red Book of Varieties and Schemes* (Springer Lecture Notes in Mathematics 1358). Another reference is chap. I of R. Hartshorne's book *Algebraic Geometry* (Springer). One might also consult the book by M. Reid *Undergraduate algebraic geometry* (London Mathematical Society Student Texts **12**, Cambridge University Press 1988). An updated free online version of M. Reid's lectures can be found under

<https://homepages.warwick.ac.uk/staff/Miles.Reid/MA4A5/UAG.pdf>

The natural generalisation of classical algebraic geometry is the theory of schemes, which will be taught in Hilary Term. In Grothendieck's theory of schemes, the base field can be replaced by any commutative ring but the absence of Hilbert's Nullstellensatz, which is at the root of the material presented here, means that different techniques have to be used.

There are three important tools, which will not be presented in this course:

- The theory of sheaves
- Cohomological techniques
- The technique of base change

These tools are very powerful but there will not be enough time to present them in these lectures. Also, the best framework for them is the theory of schemes (although they could also be used in the restricted setting of this text).

There is also a tool from Commutative Algebra, which will not be used here but which is very useful in Algebraic Geometry: the tensor product of modules over a ring. Tensor products are ubiquitous in the theory of schemes.

The prerequisites for this course are the part A course *Rings and Modules* and the part B course *Commutative Algebra*. It is assumed that the reader is familiar with the terminology used in the notes of the commutative algebra course. We shall often quote results proven in that course, referring to it as "CA". I have put the CA notes on the web page of the present course for easy reference.

Throughout the course, we shall work over a fixed algebraically closed field k . As in the CA course, a ring will be a commutative ring with unit, unless stated otherwise. The reader may assume that for any $n \geq 1$, the ring of polynomials $k[x_1, \dots, x_n]$ is a UFD (Unique Factorisation Domain). It can also be assumed that the localisation $k[x_1, \dots, x_n]_S$ is a UFD for any multiplicative set $S \subseteq k[x_1, \dots, x_n]$.

2 Hilbert's Nullstellensatz and algebraic sets

Let $n \geq 0$ and let $R_n := k[x_1, \dots, x_n]$. Let $\Sigma \subseteq R_n$. The *algebraic set* associated with Σ is

$$Z(\Sigma) = \text{zero set of } \Sigma := \{(t_1, \dots, t_n) \in k^n \mid \forall P \in \Sigma : P(t_1, \dots, t_n) = 0\}$$

Note the following simple fact. Let ΣR_n be the ideal generated by Σ in R_n . Then we have $Z(\Sigma) = Z(\Sigma R_n)$.

We now recall two basic results in commutative algebra.

Theorem 2.1 (Hilbert's basis theorem; see Th. 7.6 in CA). *The ring $k[x_1, \dots, x_n]$ is noetherian.*

Recall that a noetherian ring is a ring all of whose ideals are finitely generated. In particular, by the remark above any algebraic set in k^n is the zero set of a finite number of polynomials.

Theorem 2.2 (Hilbert's strong Nullstellensatz; see Cor. 9.5 in CA). *For any ideal $I \subseteq R_n$ we have*

$$\mathfrak{r}(I) = \{P \in R_n \mid \forall (t_1, \dots, t_n) \in Z(I) : P(t_1, \dots, t_n) = 0\}$$

Recall that $\mathfrak{r}(I)$ is the radical (or nilradical) of I , ie the intersection of all the prime ideals of R_n containing I . One can show that $\mathfrak{r}(I)$ consists of all the elements Q of R_n , such that $Q^l \in I$ for some $l = l(Q)$ (see Prop. 3.2 in CA). Recall also that a radical ideal is an ideal which coincides with its own radical.

If $A \subseteq k^n$ is subset, we shall write

$$\mathcal{I}(A) := \{P \in R_n \mid \forall (t_1, \dots, t_n) \in A : P(t_1, \dots, t_n) = 0\}.$$

The set $\mathcal{I}(A)$ is clearly an ideal in R_n . Note that in terms of the operator $\mathcal{I}(\cdot)$, the strong HNS implies that $\mathcal{I}(Z(I)) = \mathfrak{r}(I)$ for any ideal I of R_n .

We may now prove the basic

Proposition 2.3. *Let $V \subseteq k^n$ be an algebraic set and let $I \subseteq R_n$ be an ideal. Then the identities $Z(I) = Z(\mathfrak{r}(I))$, $\mathcal{I}(Z(I)) = \mathfrak{r}(I)$ and $Z(\mathcal{I}(V)) = V$ hold.*

In particular, the two maps

$$\{\text{algebraic sets in } k^n\} \xrightleftharpoons[\mathcal{I}]{Z} \{\text{radical ideals in } R_n\}$$

are inverse to each other. Note that in this correspondence, we have $V_1 \subseteq V_2$ iff $Z(V_1) \supseteq Z(V_2)$ for any two algebraic sets V_1 and V_2 (why?).

Proof. (of Proposition 2.3) The identity $Z(I) = Z(\mathfrak{r}(I))$ follows from the definitions and the identity $\mathcal{I}(Z(I)) = \mathfrak{r}(I)$ was noted just before the statement of the proposition. We thus only have to prove that $Z(\mathcal{I}(V)) = V$. To see this, note that by definition we have $V \subseteq Z(\mathcal{I}(V))$. On the other hand, by definition $V = Z(J)$ for some ideal J in $k[x_1, \dots, x_n]$. By construction, we have $J \subseteq \mathcal{I}(V)$, so $Z(J) = V \supseteq Z(\mathcal{I}(V))$. Hence $V = Z(\mathcal{I}(V))$. \square

We also note the following identities, whose proof is left as an exercise for the reader:

$$(1) \mathcal{I}(V_1 \cup V_2) = \mathcal{I}(V_1) \cap \mathcal{I}(V_2)$$

- (2) $\mathcal{I}(\cap_i V_i) = \mathfrak{r}(\sum_i \mathcal{I}(V_i))$
- (3) $Z(I_1 \cap I_2) = Z(I_1) \cup Z(I_2)$
- (4) $Z(\sum_i I_i) = \cap_i Z(I_i)$

(where the V_i are algebraic sets, the I_i are ideals and the symbol \sum refers to the sum of ideals).

In view of the properties (4) and (3) above, the algebraic sets in k^n can be viewed as the closed sets of a topology on k^n , called the *Zariski topology*. If $V \subseteq k^n$ is an algebraic set, we endow V with the topology induced by the Zariski topology of k^n . This topology is called the Zariski topology of V .

We can refine the correspondence above as follows.

Say that an algebraic set $V \subseteq k^n$ is *reducible* if $V = V_1 \cup V_2$, where $V_1, V_2 \subseteq k^n$ are non empty algebraic sets, $V_1 \not\subseteq V_2$ and $V_2 \not\subseteq V_1$. An algebraic set $V \subseteq k^n$ is said to be *irreducible* if it is not reducible. One verifies from the definition that an algebraic set is irreducible iff all its non empty open subsets are dense.

For the following two lemmata, we shall need the following result from CA:

Theorem 2.4. *Let R be a noetherian commutative ring and let $I \subseteq R$ be a radical ideal. Then there is a unique finite set of prime ideals $\{\mathfrak{p}_l\}$ such that $I = \cap_l \mathfrak{p}_l$ and such that for all indices l we have $\mathfrak{p}_l \not\supseteq \cap_{j \neq l} \mathfrak{p}_j$. Furthermore, the \mathfrak{p}_l are the prime ideals of R , which are minimal for the inclusion relation among the prime ideals containing I .*

Proof. This follows from the Lasker-Noether theorem (see Prop. 7.8 in CA) and the remark after Th. 6.7 in CA. \square

Lemma 2.5. *Let $V \subseteq k^n$ be an algebraic subset. Then V is irreducible iff $\mathcal{I}(V)$ is a prime ideal.*

Proof. " \Leftarrow ": Suppose that V is reducible. Then $V = V_1 \cup V_2$, where V_1 and V_2 are two algebraic subsets not contained in each other (and in particular not empty). By property (1) above, we have $\mathcal{I}(V) = \mathcal{I}(V_1) \cap \mathcal{I}(V_2)$, where $\mathcal{I}(V_1)$ and $\mathcal{I}(V_2)$ are two ideals not contained in each other. In particular, there is $a_1 \in \mathcal{I}(V_1)$ such that $a_1 \notin \mathcal{I}(V_2)$ and $a_2 \in \mathcal{I}(V_2)$ such that $a_2 \notin \mathcal{I}(V_1)$. In particular $a_1, a_2 \notin \mathcal{I}(V)$. On the other hand $a_1 a_2 \in \mathcal{I}(V)$ so that $\mathcal{I}(V)$ is not prime.

" \Rightarrow ": Suppose that $\mathcal{I}(V)$ is not prime. Let $\{\mathfrak{p}_l\}_{l \in \Lambda}$ be the set of prime ideals in R , which are minimal among the prime ideals containing $\mathcal{I}(V)$. By Theorem 2.4 we know that Λ is finite and that $\mathcal{I}(V) = \cap_l \mathfrak{p}_l$. Hence $\#\Lambda > 1$ since $\mathcal{I}(V)$ is not prime. Let l_1 be any element of Λ . By Theorem 2.4 again (or Prop. 6.1 (ii) in CA and the minimality of the \mathfrak{p}_l) we have $\mathfrak{p}_{l_1} \not\supseteq \cap_{l \neq l_1} \mathfrak{p}_l$. On the other hand, we also have $\mathfrak{p}_{l_1} \not\subseteq \cap_{l \neq l_1} \mathfrak{p}_l$ by minimality. Hence $Z(\mathfrak{p}_{l_1}) \not\subseteq Z(\cap_{l \neq l_1} \mathfrak{p}_l)$ and $Z(\mathfrak{p}_{l_1}) \not\supseteq Z(\cap_{l \neq l_1} \mathfrak{p}_l)$. Finally, we have $Z(\mathcal{I}(V)) = V = Z(\mathfrak{p}_{l_1}) \cup Z(\cap_{l \neq l_1} \mathfrak{p}_l)$ by (3) above and Proposition 2.3 so that V is reducible. \square

Lemma 2.6. *Let $V \subseteq k^n$ be an algebraic set. Then there is a unique finite collection $\{V_l\}_{l \in \Lambda}$ of irreducible algebraic subsets of k^n such that*

- (1) $V = \cup_l V_l$;
- (2) $\forall l : V_l \not\subseteq \cup_{j \neq l} V_j$.

Furthermore, the V_l are the irreducible algebraic sets in k^n , which are maximal among the irreducible algebraic sets contained in V .

Proof. In view of the remark after Prop. 2.3, the properties (1)...(4) above and Lemma 2.5, this is equivalent to Theorem 2.4 for $R = R_n$. \square

Example. The algebraic set defined by the equation $x_1x_2 = 0$ in k^2 has the irreducible components $x_1 = 0$ and $x_2 = 0$. Indeed, the two components are clearly not contained in each other and $x_i = 0$ defines an irreducible algebraic set because $k[x_1, x_2]/(x_i) \simeq k[x]$ and thus the ideal (x_i) is prime in $k[x_1, x_2]$ (use Lemma 2.5).

Proposition 2.7. Let $V \subseteq k^n$ be an algebraic set defined by a radical ideal I . Let $\bar{v} = \langle v_1, \dots, v_n \rangle \in V$ and let \mathfrak{m} be a maximal ideal of R_n . Suppose that $\mathfrak{m} \supseteq I$. Then

- (1) $\mathcal{I}(\{\bar{v}\}) \supseteq I$ and $\mathcal{I}(\{\bar{v}\})$ is a maximal ideal of R_n ;
- (2) $Z(\mathfrak{m})$ consists of one point $\bar{u} = \langle u_1, \dots, u_n \rangle \in V$;
- (3) $\mathfrak{m} = (x_1 - u_1, \dots, x_n - u_n)$ where \bar{u} is as in (2).

Proof. Unravel the definitions and apply Proposition 2.3. \square

The last proposition in particular provides a correspondence between the points of V and the maximal ideals of R_n containing $\mathcal{I}(V)$, or equivalently with the maximal ideals of $R_n/\mathcal{I}(V)$. In other words, if we write for any ring R

$$\text{Spm}(R) := \{\text{maximal ideals of } R\}$$

then there is a natural bijection $\text{Spm}(R_n/\mathcal{I}(V)) \simeq V$.

Lemma 2.8. Let $V \subseteq k^n$ be an algebraic set. Under the bijection $\text{Spm}(R_n/\mathcal{I}(V)) \simeq V$, the closed subsets of V correspond to the subsets of $\text{Spm}(R_n/\mathcal{I}(V))$ of the form

$$Z(S) := \{\mathfrak{m} \in \text{Spm}(R_n/\mathcal{I}(V)) \mid \mathfrak{m} \supseteq S\}$$

where $S \subseteq R_n/\mathcal{I}(V)$. The closed subsets of V are in one to one correspondence with the radical ideals of $R_n/\mathcal{I}(V)$ via $Z(\cdot)$.

Proof. Left to the reader. Unroll the definitions. \square

Note that the set $\{\mathfrak{m} \in \text{Spm}(R_n/\mathcal{I}(V)) \mid \mathfrak{m} \supseteq S\}$ corresponds in V to the set $Z(S') \cap V$ for any lifting S' of S to R_n . So the notation $Z(S)$ will not lead to any confusion. Also, if $C \subseteq V$ is a closed subset, then we have $C = Z(\mathcal{I}(C) \pmod{\mathcal{I}(V)}) = Z(\mathcal{I}(C)) \cap V$. So we will sometimes use the shorthand $\mathcal{I}(C)$ for $\mathcal{I}(C) \pmod{\mathcal{I}(V)} \subseteq R_n/\mathcal{I}(V)$ if C is a closed subset of V . With this notation, the properties (1), ..., (4) listed above are also valid for the correspondence described in Lemma 2.8.

3 Regular maps between algebraic sets

Let $n, t \geq 0$. A map $\phi : k^n \rightarrow k^t$ is said to be *polynomial* if there are elements $P_1(x_1, \dots, x_n), \dots, P_t(x_1, \dots, x_n) \in R_n = k[x_1, \dots, x_n]$, such that

$$\phi(a_1, \dots, a_n) = \langle P_1(a_1, \dots, a_n), \dots, P_t(a_1, \dots, a_n) \rangle$$

for all $\langle a_1, \dots, a_n \rangle \in k^n$.

Note that the polynomials P_i define a map of k -algebras $\phi^* : R_t \rightarrow R_n$ by the formula

$$\phi^*(Q(y_1, \dots, y_t)) := Q(P_1(x_1, \dots, x_n), \dots, P_t(x_1, \dots, x_n))$$

and on the other hand, if we are given a map of k -algebras $\Phi : k[y_1, \dots, y_t] = R_t \rightarrow R_n = k[x_1, \dots, x_n]$, then we can define polynomials $T_1(x_1, \dots, x_n), \dots, T_t(x_1, \dots, x_n) \in R_n$ by the formula

$$T_i(x_1, \dots, x_n) := \Phi(y_i)$$

and these two processes are obviously inverse to each other. So to give polynomials P_i as above is equivalent to giving a map of k -algebras $R_t \rightarrow R_n$. Note that from definitions we see that the composition of two polynomial maps is a polynomial map.

If $\Phi : R_t \rightarrow R_n$ is a map of k -algebras, we shall write $\text{Spm}(\Phi) : k^n \rightarrow k^t$ for the corresponding polynomial map (defined as above from the polynomials arising from Φ).

Lemma 3.1. *The map*

$$\text{Spm} : \{\text{maps of } k\text{-algebras } R_t \rightarrow R_n\} \rightarrow \{\text{polynomial maps } k^n \rightarrow k^t\}$$

is bijective.

Proof. The surjectivity of Spm is a tautology so we only have to prove injectivity. Let $\Phi_1, \Phi_2 : R_t \rightarrow R_n$ be two maps of k -algebras. Suppose that $\text{Spm}(\Phi_1) = \text{Spm}(\Phi_2)$. We have to prove that $\Phi_1 = \Phi_2$. Suppose that Φ_1 (resp. Φ_2) is defined by polynomials $P_{11}(x_1, \dots, x_n), \dots, P_{1t}(x_1, \dots, x_n)$ (resp. $P_{21}(x_1, \dots, x_n), \dots, P_{2t}(x_1, \dots, x_n)$). Let $i \in \{1, \dots, t\}$. If $\text{Spm}(\Phi_1) = \text{Spm}(\Phi_2)$ then the polynomial $P_{1i} - P_{2i}$ vanishes for all the values of its variables. This implies that $P_{1i} = P_{2i}$ (why?). Since i was arbitrary, we conclude that $\Phi_1 = \Phi_2$. \square

In view of the lemma, for any polynomial map $\phi : k^n \rightarrow k^t$, there is a unique map of k -algebras $\phi^* : R_t \rightarrow R_n$ such that $\text{Spm}(\phi^*) = \phi$. Note that the operation $(\cdot)^*$ (resp. $\text{Spm}(\cdot)$) is compatible with composition of polynomial maps (resp. composition of maps of k -algebras). This follows from the definitions.

Let now $V \subseteq k^n$ and $W \subseteq k^t$ be algebraic sets in k^n and k^t , respectively. A map $\psi : V \rightarrow W$ is said to be *regular* if there is a polynomial map $\phi : k^n \rightarrow k^t$ such that $\phi(V) \subseteq W$ and such that $\psi(v) = \phi(v)$ for all $v \in V$. Note that if ψ is given, there might be several different ϕ inducing ψ (what is an obvious example of this phenomenon?). Note also that a regular map is continuous for the Zariski topology (why?). Also, a composition of regular maps is regular (unroll the definitions).

We shall attempt to generalise Lemma 3.1 to algebraic sets.

For this, we make the following definition.

Definition 3.2. *Let $V \subseteq k^n$ be an algebraic set. The coordinate ring $\mathcal{C}(V)$ of V is the ring*

$$\mathcal{C}(V) := R_n / \mathcal{I}(V).$$

Note that since $\mathcal{I}(V)$ is a radical ideal (see above - this also follows directly from the definitions), the ring $\mathcal{C}(V)$ is a reduced ring, ie the only nilpotent element of $\mathcal{C}(V)$ is the zero element. We also recall that any finitely generated algebra over a field is a Jacobson ring (see Cor. 9.4 in CA). In particular, $\mathcal{C}(V)$ is a Jacobson ring. Recall that a Jacobson ring R is a ring such that for any ideal $I \subseteq R$, we have

$$\bigcap_{\mathfrak{m} \in \text{Spm}(R), \mathfrak{m} \supseteq I} \mathfrak{m} = \bigcap_{\mathfrak{p} \in \text{Spec}(R), \mathfrak{p} \supseteq I} \mathfrak{p} =: \mathfrak{r}(I)$$

where $\text{Spec}(R)$ is the set of prime ideals of R (see section 4 in CA).

Let again $V \subseteq k^n$ and $W \subseteq k^t$ be algebraic sets in k^n and k^t , respectively. Let $\psi : V \rightarrow W$ be a regular map and let $\phi : k^n \rightarrow k^t$ be a polynomial map inducing ψ , as above. Suppose that $\phi = \text{Spm}(\Phi)$ for the map of k -algebras $\Phi : R_t \rightarrow R_n$.

Lemma 3.3. *We have $\Phi(\mathcal{I}(W)) \subseteq \mathcal{I}(V)$.*

Proof. I am grateful to one of the members of the audience for pointing out a simplified proof of this lemma. Suppose Φ is given by elements $P_1(x_1, \dots, x_n), \dots, P_t(x_1, \dots, x_n) \in R_n = k[x_1, \dots, x_n]$, as above. By assumption, for all $\bar{v} \in V$, we have

$$\langle P_1(\bar{v}), \dots, P_t(\bar{v}) \rangle \in W$$

and so for any $Q(y_1, \dots, y_t) \in \mathcal{I}(W)$ and any $\bar{v} \in V$, we have $Q(P_1(\bar{v}), \dots, P_t(\bar{v})) = 0$. In other words,

$$\Phi(Q) = Q(P_1(x_1, \dots, x_n), \dots, P_t(x_1, \dots, x_n)) \in \mathcal{I}(V)$$

as required. \square

From the lemma, we see that Φ induces a map of k -algebras $\Phi_{V,W} : \mathcal{C}(W) \rightarrow \mathcal{C}(V)$.

The next lemma is needed in the next proposition.

Lemma 3.4. *If $\bar{v} := \langle v_1, \dots, v_n \rangle \in V$ then the maximal ideal of $\mathcal{C}(W)$ corresponding to $\psi(\bar{v})$ is the ideal*

$$\Phi_{V,W}^{-1}((x_1 - v_1, \dots, x_n - v_n) \pmod{\mathcal{I}(V)}) = \Phi^{-1}((x_1 - v_1, \dots, x_n - v_n) \pmod{\mathcal{I}(W)}).$$

In particular, $\Phi_{V,W}^{-1}$ sends maximal ideals to maximal ideals and $\Phi_{V,W}$ entirely determines $\psi : V \rightarrow W$.

Proof. Note first that $\Phi^{-1}((x_1 - v_1, \dots, x_n - v_n))$ is maximal in R_t because there is by construction an injection of k -algebras

$$R_t / \Phi^{-1}((x_1 - v_1, \dots, x_n - v_n)) \hookrightarrow R_n / (x_1 - v_1, \dots, x_n - v_n) \simeq k$$

so that $R_t / \Phi^{-1}((x_1 - v_1, \dots, x_n - v_n)) \simeq k$ (isomorphism of k -algebras). On the other hand, any maximal ideal in $R_t = k[y_1, \dots, y_t]$ is likewise of the form $(y_1 - u_1, \dots, y_t - u_t)$ by Proposition 2.7. So in order to determine the ideal $\Phi^{-1}((x_1 - v_1, \dots, x_n - v_n))$ we only need to find $u_1, \dots, u_t \in k$ such that

$$\Phi(y_i - u_i) \in (x_1 - v_1, \dots, x_n - v_n). \quad (1)$$

By the correspondence between algebraic sets and radical ideals, condition (1) is equivalent to the condition that the polynomial $\Phi(y_i - u_i)$ vanishes on $\langle v_1, \dots, v_n \rangle$. We compute

$$\Phi(y_i - u_i)(\langle v_1, \dots, v_n \rangle) = \Phi(y_i)(\langle v_1, \dots, v_n \rangle) - u_i = \phi_i(\langle v_1, \dots, v_n \rangle) - u_i$$

where ϕ_i is the projection of the map $\phi : k^n \rightarrow k^t$ to the i -th coordinate. We thus see that $\Phi(y_i - u_i)$ vanishes on $\langle v_1, \dots, v_n \rangle$ for all $i \in \{1, \dots, t\}$ iff $\phi(\langle v_1, \dots, v_n \rangle) = \langle u_1, \dots, u_t \rangle$. Hence

$$\Phi^{-1}((x_1 - v_1, \dots, x_n - v_n)) = (y_1 - \phi_1(\bar{v}), \dots, y_t - \phi_t(\bar{v})).$$

In particular, the maximal ideal of $\mathcal{C}(W)$ corresponding to $\psi(\bar{v})$ is the ideal $\Phi_{V,W}^{-1}((x_1 - v_1, \dots, x_n - v_n) \pmod{\mathcal{I}(V)})$. \square

We now have the

Proposition 3.5. *The map $\Phi_{V,W} : \mathcal{C}(W) \rightarrow \mathcal{C}(V)$ depends only on ψ .*

Proof. Suppose that ψ is also induced by another polynomial map $\phi' : k^n \rightarrow k^t$, associated with a map of k -algebras $\Phi' : R_t \rightarrow R_n$. Let $\Phi'_{V,W} : \mathcal{C}(W) \rightarrow \mathcal{C}(V)$ be the map of k -algebras induced by ϕ' via Lemma 3.3. Let $r \in \mathcal{C}(W)$ and let $\mathfrak{m} \in \text{Spm}(V)$. By the above remark and the assumptions, we have $(\Phi')_{V,W}^{-1}(\mathfrak{m}) = \Phi_{V,W}^{-1}(\mathfrak{m}) \in \text{Spm}(\mathcal{C}(W))$. Let $\mathfrak{n} := (\Phi')_{V,W}^{-1}(\mathfrak{m}) = \Phi_{V,W}^{-1}(\mathfrak{m})$. We have commutative diagrams

$$\begin{array}{ccc} \mathcal{C}(W) & \xrightarrow{\Phi_{V,W}} & \mathcal{C}(V) \\ \downarrow & & \downarrow \\ \mathcal{C}(W)/\mathfrak{n} & \longrightarrow & \mathcal{C}(V)/\mathfrak{m} \\ \simeq \uparrow & & \simeq \uparrow \\ k & \xrightarrow{=} & k \end{array}$$

and also

$$\begin{array}{ccc} \mathcal{C}(W) & \xrightarrow{\Phi'_{V,W}} & \mathcal{C}(V) \\ \downarrow & & \downarrow \\ \mathcal{C}(W)/\mathfrak{n} & \longrightarrow & \mathcal{C}(V)/\mathfrak{m} \\ \simeq \uparrow & & \simeq \uparrow \\ k & \xrightarrow{=} & k \end{array}$$

In particular, we see that $\Phi_{V,W}(r) \pmod{\mathfrak{m}} = \Phi'_{V,W}(r) \pmod{\mathfrak{m}}$. Since \mathfrak{m} was an arbitrary maximal ideal of $\mathcal{C}(V)$, we conclude that $\Phi_{V,W}(r) - \Phi'_{V,W}(r)$ lies in the Jacobson radical of $\mathcal{C}(V)$. Since $\mathcal{C}(V)$ is a Jacobson ring and is reduced, we thus see that $\Phi_{V,W}(r) = \Phi'_{V,W}(r)$. Since $r \in \mathcal{C}(W)$ was arbitrary, we conclude that $\Phi_{V,W} = \Phi'_{V,W}$. \square

From the last lemma, we see that we may write $\Phi_{V,W} =: \psi^*$.

Lemma 3.6. *Let $\Lambda : \mathcal{C}(W) \rightarrow \mathcal{C}(V)$ be a map of k -algebras. Then there is a regular map $\lambda : V \rightarrow W$ such that $\lambda^* = \Lambda$.*

Proof. Let $\Lambda_0 : R_t \rightarrow R_n$ be a map of k -algebras such that the diagram

$$\begin{array}{ccc} R_t = k[y_1, \dots, y_t] & \xrightarrow{\Lambda_0} & R_n = k[x_1, \dots, x_n] \\ \downarrow & & \downarrow \\ \mathcal{C}(W) & \xrightarrow{\Lambda} & \mathcal{C}(V) \end{array}$$

commutes. We may obtain such a map by choosing representatives in R_n of $\Lambda(y_i \pmod{\mathcal{I}(W)})$ for each $i \in \{1, \dots, t\}$. By construction, we then have $\Lambda_0(\mathcal{I}(W)) \subseteq \mathcal{I}(V)$. Applying Lemma 3.4, Lemma 3.1 and Proposition 2.7, we conclude that Λ_0 arises from a polynomial map $\text{Spm}(\Lambda_0) : k^n \rightarrow k^t$ such that $\text{Spm}(\Lambda_0)(V) \subseteq W$. By construction, we have $(\text{Spm}(\Lambda_0)|_V)^* = \Lambda$ so we may choose $\lambda = \text{Spm}(\Lambda_0)|_V$. \square

From the last lemma, Lemma 3.4 and Proposition 3.5, we see that given a map of k -algebras $\Lambda : \mathcal{C}(W) \rightarrow \mathcal{C}(V)$, there is a unique regular map $\text{Spm}(\Lambda) : V \rightarrow W$ such that $\text{Spm}(\Lambda)^* = \Lambda$ (note that this generalises the operator $\text{Spm}(\cdot)$ defined before Lemma 3.1). On the other hand, by Proposition 3.5, Lemma 3.4 and the previous lemma, given a regular map $\lambda : V \rightarrow W$, the map of k -algebras $\lambda^* : \mathcal{C}(W) \rightarrow \mathcal{C}(V)$ is the unique one such that $\text{Spm}(\lambda^*) = \lambda$.

We conclude that there is a bijection from the set of regular maps $V \rightarrow W$ to the set of maps of k -algebras $\mathcal{C}(W) \rightarrow \mathcal{C}(V)$, which sends $\lambda : V \rightarrow W$ to λ^* and whose inverse is given by $\text{Spm}(\cdot)$.

Finally note that any finitely generated reduced k -algebra is isomorphic as a k -algebra to the coordinate ring of some algebraic set (why?).

All this leads to an intrinsic characterisation of algebraic sets and regular maps between them. We may view algebraic sets as a category whose objects are pairs (V, n) ($n \geq 0$), where V is the zero set in k^n of a set of k -polynomials in n variables, and where the arrows from (V, n) to (W, t) are the maps from V to W which are restrictions of polynomial maps from k^n to k^t .

The following theorem summarises the previous discussion.

Theorem 3.7. *The category of algebraic sets is antiequivalent to the category of finitely generated reduced k -algebras.*

Note that in this antiequivalence, a finitely reduced k -algebra R is not naturally associated with an algebraic set. However if $V_1 \subseteq k^{n_1}$ and $V_2 \subseteq k^{n_2}$ are two algebraic sets such that $\mathcal{C}(V_1) \simeq \mathcal{C}(V_2)$, then the algebraic sets $V_1 \subseteq k^{n_1}$ and $V_2 \subseteq k^{n_2}$ are isomorphic, and any such isomorphism corresponds to precisely one isomorphism of k -algebras $\mathcal{C}(V_1) \simeq \mathcal{C}(V_2)$. Also, if $V \subseteq k^n$ is an algebraic set, there is a canonical identification between the set V and the set $\text{Spm}(\mathcal{C}(V))$. Finally, we see from Lemma 2.8 that the topology induced on $\text{Spm}(\mathcal{C}(V))$ by this identification is determined by $\mathcal{C}(V)$ only.

4 Varieties

Let $V \subseteq k^n$ be an algebraic set. Note that from Theorem 3.7, there is a natural identification between the regular maps from V to k (where k is viewed as an algebraic set) and the elements of $\mathcal{C}(V)$. Indeed the elements of $\mathcal{C}(V)$ are in one-to-one correspondence with the morphisms of k -algebras $k[x] \rightarrow \mathcal{C}(V)$ and in turn these morphisms correspond to regular maps $V \rightarrow k$. More concretely, let $f \in \mathcal{C}(V) = R_n/\mathcal{I}(V)$ and let \tilde{f} be an arbitrary lifting of f to $R_n = k[x_1, \dots, x_n]$. The regular function $V \rightarrow k$ corresponding to f is then the restriction of the map $k^n \rightarrow k$ given by the polynomial \tilde{f} .

We would also like to make sense of regular maps from open subsets of V to k .

We first note the

Lemma 4.1. *Any open set in V is a union of open subsets of the form $V \setminus Z(f)$, for $f \in \mathcal{C}(V)$.*

Proof. Left to the reader. Unroll the definitions. \square

Definition 4.2. *Let $U \subseteq V$ be an open subset. A function $u : U \rightarrow k$ is said to be regular if for any regular map of algebraic sets $\tau : T \rightarrow V$ such that $\tau(T) \subseteq U$, the function $\tau \circ u$ is regular on T (ie corresponds to an element of $\mathcal{C}(T)$).*

To show that this definition is useable, we shall need the following

Lemma 4.3. *Suppose that the regular map $h : V' \rightarrow V$ makes $\mathcal{C}(V')$ isomorphic to $\mathcal{C}(V)[f^{-1}]$ as a $\mathcal{C}(V)$ -algebra for some $f \in \mathcal{C}(V)$. Then*

(1) *h is injective and h is a homeomorphism onto $V \setminus Z(f)$;*

(2) if $g : V'' \rightarrow V$ is a regular map such that $g(V'') \subseteq V \setminus Z(f)$, then there is a unique regular map $g' : V'' \rightarrow V'$ such that $g = h \cdot g'$.

Proof. (1) The injectivity follows from the fact that for any maximal ideal \mathfrak{m} of $\mathcal{C}(V)[f^{-1}]$, \mathfrak{m} is generated by the image of $\mathfrak{m} \cap \mathcal{C}(V) = (h^*)^{-1}(\mathfrak{m})$ in $\mathcal{C}(V)[f^{-1}]$ (recall that $(h^*)^{-1}(\mathfrak{m})$ is maximal by Lemma 3.4 - one could also appeal to Cor. 10.4 in CA). See Lemma 5.6 in CA for this.

Next, we show that $h(V') \subseteq V \setminus Z(f)$. In terms of maximal ideals, this translates to the statement that $f \notin (h^*)^{-1}(\mathfrak{m}) = \mathfrak{m} \cap \mathcal{C}(V)$ for all the maximal ideals of $\mathcal{C}(V)[f^{-1}]$. By the general properties of localisations (see Lemma 5.6 in CA), $\mathfrak{m} \cap \mathcal{C}(V)$ does not meet the multiplicative set generated by f , so in particular $f \notin \mathfrak{m} \cap \mathcal{C}(V)$. This shows that $h(V') \subseteq V \setminus Z(f)$.

We now show that $h|_{V'} : V' \rightarrow V \setminus Z(f)$ is surjective. For this, note that if \mathfrak{n} is a maximal ideal of $\mathcal{C}(V)$ such that $f \notin \mathfrak{n}$ then there is a unique prime ideal \mathfrak{n}_0 of $\mathcal{C}(V)[f^{-1}]$ such that $\mathfrak{n}_0 \cap \mathcal{C}(V) = (h^*)^{-1}(\mathfrak{n}_0) = \mathfrak{n}$ (again use Lemma 5.6 in CA). The ideal \mathfrak{n}_0 is also maximal. To see this, let $\mathfrak{n}'_0 \supseteq \mathfrak{n}_0$ be a maximal ideal. Since we have $(h^*)^{-1}(\mathfrak{n}'_0) \supseteq \mathfrak{n}$, we thus have $(h^*)^{-1}(\mathfrak{n}'_0) = \mathfrak{n}$. Since as before \mathfrak{n}_0 (resp. \mathfrak{n}'_0) is generated by the image of \mathfrak{n} (resp. $(h^*)^{-1}(\mathfrak{n}'_0)$) in $\mathcal{C}(V)[f^{-1}]$, we see that $\mathfrak{n}_0 = \mathfrak{n}'_0$, ie \mathfrak{n}_0 is maximal.

To show that h is a homeomorphism onto $V \setminus Z(f)$, it is sufficient to show that image of any closed subset of V' maps to a closed subset of $V \setminus Z(f)$. In terms of ideals, this translates to the statement that for any ideal J of $\mathcal{C}(V)[f^{-1}]$, there exists an ideal $I = I(J)$ of $\mathcal{C}(V)$ such that $\mathfrak{m} \supseteq J$ iff $\mathfrak{m} \cap \mathcal{C}(V) \supseteq I$ for all $\mathfrak{m} \in \text{Spm}(\mathcal{C}(V)[f^{-1}])$. Letting J be an ideal of $\mathcal{C}(V)[f^{-1}]$, define $I := J \cap \mathcal{C}(V)$. If $\mathfrak{m} \supseteq J$ we clearly have $\mathfrak{m} \cap \mathcal{C}(V) \supseteq I$. On the other hand, if $\mathfrak{m} \cap \mathcal{C}(V) \supseteq I$, we have $\mathfrak{m} = (\mathfrak{m} \cap \mathcal{C}(V)) \cdot \mathcal{C}(V)[f^{-1}] \supseteq I \cdot \mathcal{C}(V)[f^{-1}] = J$ (again use Lemma 5.6 in CA, in particular (ii) in the proof of Lemma 5.6). So for any ideal J of $\mathcal{C}(V)[f^{-1}]$, we may choose $I = I(J) = J \cap \mathcal{C}(V)$.

(2) We first translate this into a statement of commutative algebra. We are given a map of k -algebras $g^* : \mathcal{C}(V) \rightarrow \mathcal{C}(V'')$ such that for any maximal ideal \mathfrak{m} of $\mathcal{C}(V'')$, we have $f \notin (g^*)^{-1}(\mathfrak{m})$. We would like to show that there is a map of $\mathcal{C}(V)$ -algebras from $\mathcal{C}(V)[f^{-1}]$ to $\mathcal{C}(V'')$. In view of the universal property of localisation (see Lemma-Definition 5.1 in CA), it is sufficient for this to show that $g^*(f)$ is a unit in $\mathcal{C}(V'')$. Now suppose for contradiction that $g^*(f)$ is not a unit in $\mathcal{C}(V'')$. Then $g^*(f)$ is contained in a maximal ideal \mathfrak{m} of $\mathcal{C}(V'')$. Hence $f \in (g^*)^{-1}(\mathfrak{m})$, which is a contradiction. \square

We give a description of regular functions in terms of the ambient space in the next corollary.

Corollary 4.4. *Let $f \in \mathcal{C}(V)$. The regular functions on $V \setminus Z(f)$ are the restrictions of the functions $k^n \rightarrow k$ which are of the form $\frac{P(x_1, \dots, x_n)}{(F(x_1, \dots, x_n))^l}$ ($l \geq 0$), where $P(x_1, \dots, x_n) \in R_n$ and $F(x_1, \dots, x_n) \in R_n$ is any lifting of f to R_n .*

Proof. Note first that $\mathcal{C}(V)[f^{-1}] \simeq \mathcal{C}(V)[t]/(tf - 1)$ as a $\mathcal{C}(V)$ -algebra (see Lemma 5.3 in CA). Hence $\mathcal{C}(V)[f^{-1}]$ corresponds to the algebraic set Z in k^{n+1} given by the ideal generated by the sets $\mathcal{I}(V)$ and $tF(x_1, \dots, x_n) - 1$ in $k[x_1, \dots, x_n, t]$. The polynomial map $\phi : k^{n+1} \rightarrow k^n$ inducing the map of k -algebras $\mathcal{C}(V) \rightarrow \mathcal{C}(V)[t]/(tf - 1)$ is simply given by the formula $\phi(\langle v_1, \dots, v_n, z \rangle) = \langle v_1, \dots, v_n \rangle$. The inverse of the map $Z \xrightarrow{\phi|_Z} V \setminus Z(f)$ is given by the formula $\langle v_1, \dots, v_n \rangle \mapsto \langle v_1, \dots, v_n, F(v_1, \dots, v_n)^{-1} \rangle$ (it must be this map since the map $Z \xrightarrow{\phi|_Z} V \setminus Z(f)$ is bijective and $(v_1, \dots, v_n, F(v_1, \dots, v_n)^{-1}) \in Z$ by construction). Hence a regular map on $V \setminus Z(f)$ is given by the evaluation of a polynomial in the variables x_1, \dots, x_n, t on the vector $\langle v_1, \dots, v_n, F(v_1, \dots, v_n)^{-1} \rangle$ (for $\langle v_1, \dots, v_n \rangle \in V \setminus Z(f)$). This is the conclusion of the corollary. \square

Note that the last lemma implies that the regular functions on $V \setminus Z(f)$ are all quotients of restrictions of

regular functions on V by powers of f . Also, Lemma 4.3 implies that if $h \in \mathcal{C}(V)$ is a regular function on V and $h|_{V \setminus Z(f)} = 0$, then $f \cdot h = 0$ (by the definition of localisation and the fact that $\mathcal{C}(V)$ is reduced).

Proposition 4.5. *Let U be an open subset of the algebraic set $V \subseteq k^n$. A function $a : U \rightarrow k$ is regular iff for any point $\bar{u} \in U$, there is a polynomial $F \in R_n$, such that $F(\bar{u}) \neq 0$ and a polynomial $P \in R_n$ such that a coincides with P/F in a neighbourhood of \bar{u} .*

This implies in particular that if a function $a : U \rightarrow k$ is regular and nowhere vanishing, then $1/a : U \rightarrow k$ is also a regular function. In other words, the units in the ring of regular functions $U \rightarrow k$ are the nowhere vanishing regular functions.

Proof. (of Proposition 4.5). We first show the following.

Let $W \subseteq k^t$ be an algebraic set. Let $f_1, \dots, f_l \in \mathcal{C}(W)$ and suppose that $(f_1, \dots, f_l) = \mathcal{C}(W)$. Let $h : W \rightarrow k$ be a function (not assumed regular) and suppose that for each $i \in \{1, \dots, l\}$ there is an integer $n_i \geq 0$ and an element $c_i \in \mathcal{C}(W)$ such that $h|_{W \setminus Z(f_i)} = c_i/f_i^{n_i}$. We claim that the function h is then regular on W (ie arises from an element of $\mathcal{C}(W)$), or in other words by restriction to W of a polynomial map $k^t \rightarrow k$.

To prove this, note first that we may assume that all the n_i are equal to some $m \geq 1$. Indeed, if we let $m := 1 + \sup_i n_i$ then we may write $h|_{W \setminus Z(f_i)} = c_i f_i^{m-n_i}/f_i^m$ for all i . Now notice that for all $i, j \in \{1, \dots, l\}$ we have

$$h|_{W \setminus Z(f_i f_j)} = c_i/f_i^m = c_j/f_j^m$$

so that $(f_i f_j)^m (c_i/f_i^m - c_j/f_j^m) = f_j^m c_i - c_j f_i^m = 0$ on $W \setminus Z(f_i f_j)$. We deduce from the remark preceding the proposition that

$$(f_i f_j) f_j^m c_i = (f_i f_j) c_j f_i^m$$

on V . Now let $b_i \in \mathcal{C}(W)$ be functions such that

$$\sum_i b_i f_i^{2m} = 1$$

(note that we also have $(f_1^{2m}, \dots, f_l^{2m}) = \mathcal{C}(W)$ - prove this or see Lemma 12.2 in CA). Let

$$\tilde{h} := \sum_i b_i f_i^m c_i.$$

We compute

$$\tilde{h} f_j^{2m} = \sum_i b_i f_i^m f_j^{2m} c_i = \sum_i b_i (f_i f_j)^m f_j^m c_i = \sum_i b_i (f_i f_j)^m f_i^m c_j = \left(\sum_i b_i f_i^{2m} \right) f_j^m c_j = f_j^m c_j$$

so that $\tilde{h}|_{W \setminus Z(f_j)} = c_j/f_j^m$. Hence $\tilde{h} = h$. This completes the proof of the claim.

Coming back to the proposition, note that the " \Rightarrow " direction of the equivalence stated in the proposition is clear from Lemma 4.1 and Corollary 4.4. Thus we only have to prove the " \Leftarrow " direction of the equivalence. Since the topology of U is quasi-compact (this will be proven in exercise sheet 2, question 2.5 (4) - you can also prove this directly), we may reword this implication as follows. Let $g_1, \dots, g_l \in \mathcal{C}(V)$ and suppose that $U = \cup_i (V \setminus Z(g_i))$. Let $V' \subseteq k^{n'}$ be an algebraic set and let $H : V' \rightarrow V$ be a regular map such that $H(V') \subseteq U$. Suppose that for all $i \in \{1, \dots, l\}$ we have $a|_{V \setminus Z(g_i)} = d_i/g_i$ for some $n_i \geq 0$ and some $d_i \in \mathcal{C}(V)$. The " \Leftarrow " direction of the equivalence of the proposition is then the statement that $a \circ H = H^*(a)$ is a regular function on V' . So we only have to prove this last statement under the just stated assumptions.

Note first that by construction, for all $i \in \{1, \dots, l\}$ we have

$$H^*(a)|_{V' \setminus Z(H^*(g_i))} = H^*(d_i)/H^*(g_i).$$

Also, since $H(V') \subseteq U$, we have $(H^*(g_1), \dots, H^*(g_l)) = \mathcal{C}(V')$. Hence we may apply the preceding claim to $W = V'$, $f_i = H^*(g_i)$ and $h = H^*(a)$ to conclude that $H^*(a)$ is regular on V' . \square

Note that in view of the previous proposition, the following property holds trivially: if $U' \subseteq U$ is an inclusion of open subsets of V , then the restriction to U' of a regular function on U is also regular. We encapsulate this property in the following

Definition 4.6. *Let T be a topological space. A sheaf of functions \mathcal{O}_T on T with values in k is an assignment, which associates with each open subset O of T a sub k -algebra $\mathcal{O}_T(O)$ of $\text{Maps}(O, k)$, with the following property: for any open covering $\{O_i\}$ of an open subset O , a function $f : O \rightarrow k$ lies in $\mathcal{O}_T(O)$ iff $f|_{O_i} \in \mathcal{O}_T(O_i)$ for all i .*

Here $\text{Maps}(O, k)$ is the set of functions from O to k , with its natural k -algebra structure (given by pointwise multiplication and addition).

Note that if O is an open subset of a topological space endowed with a sheaf of k -valued functions, O inherits a sheaf of k -valued functions from T .

Proposition 4.5 implies that for any algebraic set $V \subseteq k^n$, the regular functions on Zariski open subsets of V define a sheaf of functions \mathcal{O}_V with values in k on V .

There is a natural notion of mapping between topological spaces endowed with sheaves of k -valued functions:

Definition 4.7. *Let (T, \mathcal{O}_T) and $(T', \mathcal{O}_{T'})$ be two topological spaces endowed with sheaves of functions with values in k . A morphism (sometimes loosely called a map) from (T, \mathcal{O}_T) to $(T', \mathcal{O}_{T'})$ is a continuous map $a : T \rightarrow T'$ such that for any open subset $U' \subseteq T'$ and any element $f \in \mathcal{O}_{T'}(U')$, the function $f \circ a|_{a^{-1}(U')}$ on $a^{-1}(U')$ lies in $\mathcal{O}_T(a^{-1}(U'))$.*

We will also need the following definition.

Let T be a topological space endowed with a sheaf of functions \mathcal{O}_T with values in k . Let $t \in T$. Let $\widehat{\mathcal{O}}_{T,t} := \cup_{O \text{ open}, t \in O} \mathcal{O}_T(O)$ (where all the $\mathcal{O}_T(O)$ are considered to be disjoint from each other). Define an equivalence relation on $\widehat{\mathcal{O}}_{T,t}$ by declaring two functions in $\widehat{\mathcal{O}}_{T,t}$ equivalent if they coincide in some open neighbourhood of t . The set of equivalence classes in $\widehat{\mathcal{O}}_{T,t}$ has a natural k -algebra structure and we denote it by $\mathcal{O}_{T,t}$. The k -algebra $\mathcal{O}_{T,t}$ is called *the local ring at t* . Note that by definition, for any open neighbourhood O of t , there is a natural map of k -algebras $\mathcal{O}_T(O) \rightarrow \mathcal{O}_{T,t}$. Also, there is a natural map of k -algebras $\mathcal{O}_{T,t} \rightarrow k$, which is given by evaluation at t .

If we are given a morphism from (T, \mathcal{O}_T) to $(T', \mathcal{O}_{T'})$ as in the last definition, the pull-back of functions gives a map of k -algebras $\mathcal{O}_{T',a(t)} \rightarrow \mathcal{O}_{T,t}$ for any $t \in T$.

From the very definition of regularity, we see that any regular map from an algebraic set to another induces a morphism between the associated topological spaces with sheaves of k -valued functions.

We are now ready to define a general variety.

Definition 4.8. *Let T be a topological space endowed with a sheaf of functions with values in k . We say that T is a variety if there is a finite open covering $\{U_i\}$ of T , such that U_i with its induced sheaf of k -valued*

functions is isomorphic to an algebraic set endowed with its sheaf of regular functions. A morphism of varieties is a morphism of the corresponding topological spaces with sheaves of k -valued functions.

Lemma 4.9. *Let $V \subseteq k^n$ be an algebraic set and let (V, \mathcal{O}_V) be the associated topological space with sheaf of k -valued functions. Let $\bar{v} \in V$. Then the natural map of k -algebras $\mathcal{C}(V) = \mathcal{O}_V(V) \rightarrow \mathcal{O}_{V, \bar{v}}$ extends (necessarily uniquely) to an isomorphism of k -algebras $\mathcal{C}(V)_{\bar{v}} \simeq \mathcal{O}_{V, \bar{v}}$.*

Here we identified \bar{v} with the corresponding maximal ideal $\mathcal{I}(\{\bar{v}\})$ when writing $\mathcal{C}(V)_{\bar{v}}$ (so that $\mathcal{C}(V)_{\bar{v}}$ is the localisation of $\mathcal{C}(V)$ at the multiplicative set $\mathcal{C}(V) \setminus \mathcal{I}(\{\bar{v}\})$).

Proof. We first show that the map $\mathcal{C}(V) \rightarrow \mathcal{O}_{V, \bar{v}}$ extends to a map of k -algebras $\mathcal{C}(V)_{\bar{v}} \rightarrow \mathcal{O}_{V, \bar{v}}$.

To show this, we have to show that a regular function $f \in \mathcal{C}(V)$, which does not vanish at \bar{v} , maps to a unit in $\mathcal{O}_{V, \bar{v}}$. Now by definition a unit in $\mathcal{O}_{V, \bar{v}}$ is represented by a regular function in a neighbourhood of \bar{v} , which vanishes nowhere in that neighbourhood (see the remark before Definition 4.6). Now since f does not vanish at \bar{v} , it is nowhere vanishing in the set $V \setminus Z(f)$, which is a neighbourhood of \bar{v} . So the image of f in $\mathcal{O}_{V, \bar{v}}$ is a unit.

So we have a unique extension of the map $\mathcal{C}(V) \rightarrow \mathcal{O}_{V, \bar{v}}$ to a map of k -algebras $\mathcal{C}(V)_{\bar{v}} \rightarrow \mathcal{O}_{V, \bar{v}}$. We still have to show that this last map is injective and surjective.

We first show injectivity. Let $f/s \in \mathcal{C}(V)_{\bar{v}}$ (where $s \in \mathcal{C}(V) \setminus \mathcal{I}(\{\bar{v}\})$). Suppose that the image of f/s in $\mathcal{O}_{V, \bar{v}}$ vanishes. By definition, this means that the function f vanishes in a neighbourhood of \bar{v} . In particular, there exists an $h \in \mathcal{C}(V)$ such that f vanishes in $V \setminus Z(h)$, where h does not vanish at \bar{v} (use Lemma 4.1). In other words, the image of f in $\mathcal{C}(V)[h^{-1}]$ vanishes (use Lemma 4.4 and the commentary thereafter). Since $h \notin \mathcal{I}(\{\bar{v}\})$, the natural map $\mathcal{C}(V) \rightarrow \mathcal{C}(V)_{\bar{v}}$ factors through $\mathcal{C}(V)[h^{-1}]$ and hence the image of f in $\mathcal{C}(V)_{\bar{v}}$ also vanishes. This settles injectivity.

Now for surjectivity. By Lemma 4.1, an element $\tilde{e} \in \mathcal{O}_{V, \bar{v}}$ is represented by a regular function on $V \setminus Z(h)$, for some h which does not vanish at \bar{v} . Such a function corresponds to an element of $\mathcal{C}(V)[h^{-1}]$ and again since the natural map $\mathcal{C}(V) \rightarrow \mathcal{C}(V)_{\bar{v}}$ factors through $\mathcal{C}(V)[h^{-1}]$, we see that \tilde{e} lies in the image of $\mathcal{C}(V)_{\bar{v}}$. Since $\tilde{e} \in \mathcal{O}_{V, \bar{v}}$ was arbitrary, the natural map $\mathcal{C}(V)_{\bar{v}} \rightarrow \mathcal{O}_{V, \bar{v}}$ is surjective. \square

Note the following consequences of the last lemma. With the terminology of the lemma, the ring $\mathcal{O}_{V, \bar{v}}$ is local. Also, note that the natural evaluation map $\mathcal{O}_{V, \bar{v}} \rightarrow k$ is surjective, because all constant functions are regular on V . Hence the kernel of the map $\mathcal{O}_{V, \bar{v}} \rightarrow k$ is maximal. Hence this kernel coincides with the unique maximal ideal of $\mathcal{O}_{V, \bar{v}}$.

For Definition 4.8 to be coherent, we need to check that we can recover an algebraic set from its associated topological space with sheaf of k -valued functions:

Lemma 4.10. *Let $V \subseteq k^n$ and $W \subseteq k^t$ be two algebraic sets. Let (V, \mathcal{O}_V) and (W, \mathcal{O}_W) be the associated topological spaces with sheaves of k -valued functions. Let g be a morphism from (V, \mathcal{O}_V) to (W, \mathcal{O}_W) . Then g is induced by a regular map $\psi : V \rightarrow W$.*

Proof. By definition, the morphism g provides a map of k -algebras $\mathcal{C}(W) \rightarrow \mathcal{C}(V)$. Furthermore, for any $\bar{v} \in V$, we have a commutative diagram of k -algebras

$$\begin{array}{ccc}
\mathcal{C}(W) & \xrightarrow{g^*} & \mathcal{C}(V) \\
\downarrow & & \downarrow \\
\mathcal{O}_{W,g(\bar{v})} & \xrightarrow{g^*} & \mathcal{O}_{V,\bar{v}}
\end{array}$$

From the remark after Lemma 4.9, the ring $\mathcal{O}_{V,\bar{v}}$ is a local ring and its maximal ideal consists of the elements represented by the regular functions h defined in a neighbourhood of \bar{v} such that $h(\bar{v}) = 0$. A similar statement is true for $\mathcal{O}_{W,g(\bar{v})}$ and $g(\bar{v})$ in place of \bar{v} . In particular, the map $g^* : \mathcal{O}_{W,g(\bar{v})} \rightarrow \mathcal{O}_{V,\bar{v}}$ sends the maximal ideal of $\mathcal{O}_{W,g(\bar{v})}$ into the maximal ideal of $\mathcal{O}_{V,\bar{v}}$. Since the involved rings are local, this implies that the inverse image by g^* of the maximal ideal of $\mathcal{O}_{V,\bar{v}}$ is the maximal ideal of $\mathcal{O}_{W,g(\bar{v})}$. Using standard properties of localisations and Lemma 4.9, we conclude that the inverse image of $\mathcal{I}(\{\bar{v}\}) \subseteq \mathcal{C}(V)$ by $g^* : \mathcal{C}(V) \rightarrow \mathcal{C}(W)$ is $\mathcal{I}(\{g(\bar{v})\})$. In particular, $g(\bar{v}) = \text{Spm}(g^*)(\bar{v})$ (use Lemma 3.4). Hence g is induced by the map of k -algebras $g^* : \mathcal{C}(W) \rightarrow \mathcal{C}(V)$ and hence by a regular map $V \rightarrow W$ (by Theorem 3.7). \square

In categorical terms, this implies that the category of algebraic sets embeds in the category of topological spaces with sheaves of k -valued functions by a fully faithful functor. We shall from now on call *affine variety* a variety isomorphic to a variety associated with an algebraic set.

We shall often abbreviate "topological space with sheaf of k -valued functions" as "Topskf" from now on.

5 Open and closed subvarieties

Proposition 5.1. *Let (V, \mathcal{O}_V) be a variety. Let $U \subseteq V$ be an open subset and let \mathcal{O}_U be the sheaf of k -valued functions induced by \mathcal{O}_V . Then (U, \mathcal{O}_U) is a variety and the inclusion map is a morphism of varieties.*

Proof. Let $\{V_i\}$ be an open covering of V such that each V_i is isomorphic as a Topskf to an affine variety (where V_i is endowed with the sheaf of k -valued functions induced by V). Then $\{V_i \cap U\}$ is an open covering of U . Since $V_i \cap U$ is open in V_i , there is for each i a subset $E_i \subseteq \mathcal{C}(V_i)$ such that $\cup_{e \in E_i} (V_i \setminus Z(e)) = V_i \cap U$ (use Lemma 4.1). Hence we only have to show that the open subset $V_i \setminus Z(e)$ of V_i is isomorphic as a Topskf to an affine variety. But this follows from Lemma 4.3. \square

An open subset of a variety is called an *open subvariety* if it is endowed with the structure of Topskf described in the last Proposition.

Let (V, \mathcal{O}_V) be a variety. Let $Z \subseteq V$ be a closed subset. Endow Z with the topology induced by V . For any open subset O of Z , define a function $f : O \rightarrow k$ to be regular if there is a collection of open subsets $\{U_i\}$ of V and regular functions $g_i : U_i \rightarrow k$ such that

- $(\cup_i U_i) \cap Z = O$;
- $g_i|_{O \cap U_i} = f|_{O \cap U_i}$.

In words, $f : O \rightarrow k$ is regular iff in the neighbourhood of every point of O the function f is the restriction of a regular function on some open subset of V . This endows Z with a structure of topological space with k -valued functions. We shall write \mathcal{O}_Z for the corresponding sheaf of k -valued functions. The sheaf of k -valued functions \mathcal{O}_Z on Z is said to be *induced* by \mathcal{O}_V .

Proposition 5.2. *The topological space Z with sheaf of k -valued functions \mathcal{O}_Z is a variety. The inclusion map $Z \rightarrow V$ is a morphism of varieties.*

Proof. The inclusion map $Z \rightarrow V$ provides us with a morphism $(Z, \mathcal{O}_Z) \rightarrow (V, \mathcal{O}_V)$ of Topskf by construction. Hence we only have to show that (Z, \mathcal{O}_Z) is a variety (see Definition 4.8). Let $\{V_i\}$ be a covering of V by open subsets such that (V_i, \mathcal{O}_{V_i}) is isomorphic as a Topskf to an affine variety. By definition, it is sufficient to show that for each i , the Topskf $Z \cap V_i$ is isomorphic to an affine variety. Hence we may assume that V is affine to begin with. Hence we are reduced to the situation where $V \subseteq k^n$ is an algebraic set and $Z \subseteq k^n$ is another algebraic set such that $Z \subseteq V$. Endow Z with the sheaf of functions \mathcal{O}_Z induced by \mathcal{O}_V . We would like to show that (Z, \mathcal{O}_Z) is isomorphic to an affine variety as a Topskf. Now note that by Proposition 4.5 the sheaf \mathcal{O}_Z is precisely the sheaf of regular functions on Z viewed as an algebraic subset of k^n . So (Z, \mathcal{O}_Z) is isomorphic to an affine variety as a Topskf. \square

An closed subset of a variety V is called a *closed subvariety* if it is endowed with the structure of Topskf induced by V .

Lemma 5.3. *Let (W, \mathcal{O}_W) and (V, \mathcal{O}_V) be two varieties. Let Z (resp. O) be a closed subset (resp. open subset) of V . Endow Z (resp. O) with its structure of closed (resp. open) subvariety. Let $\lambda : W \rightarrow V$ be a morphism of Topskf such that $\lambda(W) \subseteq Z$ (resp. $\lambda(W) \subseteq O$). Then the induced map $W \rightarrow Z$ (resp. $W \rightarrow O$) is a morphism of Topskf.*

Proof. Left to the reader. Unroll the definitions. \square

We also record a consequence of the proof of Proposition 5.2:

Lemma 5.4. *Let $V \subseteq W \subseteq k^n$, where V and W are algebraic sets in k^n . Let $(V, \mathcal{O}_V) \rightarrow (W, \mathcal{O}_W)$ be the corresponding morphism of topological spaces with sheaves of k -valued functions. Then \mathcal{O}_V is induced by \mathcal{O}_W .*

6 Projective space

Projective varieties arise when one tries to find an algebraic counterpart of the topological notion of compactness. We will revisit this later when we consider complete varieties.

Let $n \geq 0$. A *line through the origin* of k^{n+1} is by definition the vector subspace $[\bar{v}]$ of k^{n+1} generated by a vector $\bar{v} \in k^{n+1} \setminus \{0\}$. We define *projective space of dimension n* to be the set $\mathbb{P}^n(k)$ of lines through the origin of k^{n+1} . If $\bar{v} = \langle v_0, \dots, v_n \rangle \in k^{n+1} \setminus \{0\}$, we shall write $[v_0, \dots, v_n]$ for $[\langle v_0, \dots, v_n \rangle]$.

We shall endow $\mathbb{P}^n(k)$ with a variety structure. For $i \in \{0, \dots, n\}$, define

$$U_i = \{[v_0, \dots, v_n] \in \mathbb{P}^n(k) \mid v_i \neq 0\}.$$

In the following, we shall write the symbol \sim over a term that is to be omitted. The map $u_i : k^n \rightarrow U_i$ such that

$$u_i(\langle v_0, \dots, \check{v}_i, \dots, v_n \rangle) := [v_0, \dots, v_{i-1}, 1, v_{i+1}, \dots, v_n]$$

is clearly a bijection and we have

$$u_i^{-1}([v_0, \dots, v_n]) = \langle \frac{v_0}{v_i}, \dots, \frac{\check{v}_i}{v_i}, \dots, \frac{v_n}{v_i} \rangle.$$

if $[v_0, \dots, v_n] \in U_i$.

If $j < i$ and $v_j \neq 0$, we compute

$$(u_j^{-1} \circ u_i)(\langle v_0, \dots, \check{v}_i, \dots, v_n \rangle) = u_j^{-1}([v_0, \dots, v_{i-1}, 1, v_{i+1}, \dots, v_n]) = \langle \frac{v_0}{v_j}, \dots, \frac{\check{v}_j}{v_j}, \dots, \frac{1}{v_j}, \frac{v_{i+1}}{v_j}, \dots, \frac{v_n}{v_j} \rangle$$

and if $j > i$ and $v_j \neq 0$, we have similarly

$$(u_j^{-1} \circ u_i)(\langle v_0, \dots, \check{v}_i, \dots, v_n \rangle) = \langle \frac{v_0}{v_j}, \dots, \frac{v_{i-1}}{v_j}, \frac{1}{v_j}, \dots, \frac{\check{v}_j}{v_j}, \dots, \frac{v_n}{v_j} \rangle$$

Hence, if $i \neq j$, the map $u_j^{-1} \circ u_i$ gives a map from the open subset of k^n

$$\mathcal{U}_{ij} := \{ \langle v_0, \dots, \check{v}_i, \dots, v_n \rangle \in k^n \mid v_j \neq 0 \}$$

into the open subset of k^n

$$\mathcal{U}_{ji} := \{ \langle v_0, \dots, \check{v}_j, \dots, v_n \rangle \in k^n \mid v_i \neq 0 \}$$

and $u_i(\mathcal{U}_{ij}) = U_i \cap U_j = u_j(\mathcal{U}_{ji})$. Let $u_{ij} := u_j^{-1} \circ u_i : \mathcal{U}_{ij} \rightarrow \mathcal{U}_{ji}$.

Note that if one sees \mathcal{U}_{ij} as an open subvariety of k^n , then \mathcal{U}_{ij} is an affine variety associated with the coordinate ring

$$k[x_0, \dots, \check{x}_i, \dots, x_n][x_j^{-1}] \simeq k[x_0, \dots, \check{x}_i, \dots, x_n][t]/(tx_j - 1)$$

and similarly, \mathcal{U}_{ji} is an affine variety associated with the coordinate ring

$$k[y_0, \dots, \check{y}_j, \dots, y_n][y_i^{-1}] \simeq k[y_0, \dots, \check{y}_j, \dots, y_n][z]/(zy_i - 1)$$

One checks from the definitions that u_{ij} arises from the polynomial map which sends z to x_j and y_l to $x_l \cdot t$ if $l \neq i$ and to t if $l = i$. Hence u_{ij} defines a morphism of varieties from \mathcal{U}_{ij} to \mathcal{U}_{ji} . One checks from the formula just given that u_{ij} and u_{ji} are inverse to each other, so u_{ij} is an isomorphism of varieties.

Now we define a topology on $\mathbb{P}^n(k)$ by declaring a subset $O \subseteq \mathbb{P}^n(k)$ to be open iff $u_i^{-1}(O)$ is open in k^n for all $i \in \{0, \dots, n\}$ (why does this define a topology?). Furthermore, if $O \subseteq \mathbb{P}^n(k)$ is open, we define a k -valued function $f : O \rightarrow k$ to be *regular* iff $f \circ u_i|_{u_i^{-1}(O)}$ is a regular function on $u_i^{-1}(O)$ for all i . Since (k^n, \mathcal{O}_{k^n}) is a Topskf, we see that with this definition, $\mathbb{P}^n(k)$ becomes a Topskf (why? - unroll the definitions). We shall write $\mathcal{O}_{\mathbb{P}^n(k)}$ for the just defined sheaf of k -valued functions on $\mathbb{P}^n(k)$.

Proposition 6.1. *The sets U_i are open in $\mathbb{P}^n(k)$ for all $i \in \{0, \dots, n\}$ and the maps $u_i : k^n \rightarrow \mathbb{P}^n(k)$ restrict to isomorphisms of Topskf between k^n and (U_i, \mathcal{O}_{U_i}) , where \mathcal{O}_{U_i} is the sheaf of k -valued functions induced on U_i by $\mathcal{O}_{\mathbb{P}^n(k)}$. In particular, the Topskf $(\mathbb{P}^n(k), \mathcal{O}_{\mathbb{P}^n(k)})$ is a variety.*

The U_i are called the *standard coordinate charts* of $\mathbb{P}^n(k)$. We shall sometimes write U_i^n for U_i to emphasise the dependence on n .

Proof. To show that U_i is open, we have to show that $u_j^{-1}(U_i)$ is open in k^n for all j . We have shown above that $u_j^{-1}(U_i) = \mathcal{U}_{ji}$ is open, so U_i is open.

Next, we have to show that the map u_i is a homeomorphism onto its image. The map u_i is continuous and injective by definition, so we only have to show that u_i is an open map. So let $O \subseteq k^n$ be an open set. We have to show that $u_i(O)$ is open, or in other words that $u_j^{-1}(u_i(O))$ is open for all j . Now we have

$$u_j^{-1}(u_i(O)) = u_j^{-1}(u_i(O) \cap (U_i \cap U_j)) = u_j^{-1}(u_i(O \cap \mathcal{U}_{ij})) = u_{ij}(O \cap \mathcal{U}_{ij})$$

and $u_{ij}(O \cap \mathcal{U}_{ij})$ is open in \mathcal{U}_{ji} since $u_{ij} : \mathcal{U}_{ij} \rightarrow \mathcal{U}_{ji}$ is a homeomorphism by the above. On the other hand \mathcal{U}_{ji} is open in U_j , so $u_{ij}(O \cap \mathcal{U}_{ij})$ is also open in U_j . So u_i is a homeomorphism onto its image.

Finally, we have to show that if $O \subseteq U_i$ is an open set then $f : O \rightarrow k$ is a regular function iff $f \circ u_i|_{u_i^{-1}(O)}$ is a regular function on the open subset $u_i^{-1}(O)$ of k^n . By definition, if $f : O \rightarrow k$ is a regular function, then $f \circ u_i|_{u_i^{-1}(O)}$ is regular. So suppose that $f \circ u_i|_{u_i^{-1}(O)}$ is regular. We have to show that for all j the function $f \circ u_j|_{u_j^{-1}(O)}$ is regular on $u_j^{-1}(O)$ viewed as an open subset of k^n . Now we have by definition

$$u_j|_{\mathcal{U}_{ji}} = u_i|_{\mathcal{U}_{ij}} \circ u_{ji}$$

and since $u_j^{-1}(O) \subseteq \mathcal{U}_{ji}$ we thus have

$$u_j|_{u_j^{-1}(O)} = u_i|_{\mathcal{U}_{ij} \cap u_i^{-1}(O)} \circ u_{ji}|_{u_j^{-1}(O)}$$

where $u_{ji}|_{u_j^{-1}(O)}$ is viewed as a map from $u_j^{-1}(O)$ to $\mathcal{U}_{ij} \cap u_i^{-1}(O)$. The map $u_{ji}|_{u_j^{-1}(O)}$ is a morphism of Topskf since u_{ji} is a morphism of Topskf by the above. Also, the function $f \circ u_i|_{\mathcal{U}_{ij} \cap u_i^{-1}(O)}$ is a regular function by the definition of a sheaf of k -valued functions. Hence

$$f \circ u_j|_{u_j^{-1}(O)} = (f \circ u_i|_{\mathcal{U}_{ij} \cap u_i^{-1}(O)}) \circ u_{ji}|_{u_j^{-1}(O)}$$

is a regular function. This completes the proof. \square

Example. The space $\mathbb{P}^1(k)$ only has two coordinate charts, the charts U_0 and U_1 . By inspection, we see that $\mathbb{P}^1(k) \setminus U_i$ consists of only one point. So one can see $\mathbb{P}^1(k)$ as the "compactification" of k obtained by adding a "point at ∞ " to k . If $k = \mathbb{C}$, the space $\mathbb{P}^1(k)$ can be naturally identified (as a set) with the Riemann sphere of complex analysis.

7 Projective varieties

What are the closed subsets of projective space? To answer this question, we shall need the following definitions.

A polynomial $P(x_0, \dots, x_n) \in k[x_0, \dots, x_n]$ is said to be *homogenous* if it is a sum of monomials of the same degree. Any polynomial $P(x_0, \dots, x_n)$ has a canonical decomposition

$$P = \sum_{i=0}^{\deg(P)} P_{[i]}$$

where $P_{[i]}$ is the sum of the monomials of degree i appearing in P (so that in particular $P_{[i]}$ is homogenous).

Example. The polynomials $x_0, x_0^2 + x_0x_1$ are homogenous but $x_0^2 + x_1$ is not.

We have a decomposition of $k[x_0, \dots, x_n]$ as an internal direct sum

$$k[x_0, \dots, x_n] = \bigoplus_{l \geq 0} k[x_0, \dots, x_n]_{[l]}$$

where $k[x_0, \dots, x_n]_{[l]}$ is the k -vector space of homogenous polynomials of degree l . In particular, we have $k[x_0, \dots, x_n]_{[0]} = k$. This decomposition into a direct sum makes $k[x_0, \dots, x_n]$ into a graded ring in the sense of section 11.2 of CA.

Example. We have $(x_0^2 + x_1)_{[2]} = x_0^2$, $(x_0^2 + x_1)_{[1]} = x_1$, $(x_0^2 + x_1)_{[0]} = 0$.

Note the following elementary fact. If $P(x_0, \dots, x_n) \in k[x_0, \dots, x_n]$ is homogenous then $P(s \cdot x_0, \dots, s \cdot x_n) = s^{\deg(P)} P(x_0, \dots, x_n)$ for all $s \in k$.

We thus see that if $P(x_0, \dots, x_n) \in k[x_0, \dots, x_n]$ is a homogenous polynomial and $\bar{v} \in k^{n+1}$ is non zero, we have $P(\bar{v}) = 0$ iff $P(s \cdot \bar{v}) = 0$ for all $s \in k^*$. This gives rise to the following definition. Let $S \subseteq k[x_0, \dots, x_n]$ be a set of homogenous polynomials. We define

$$Z(S) := \{[\bar{v}] \in \mathbb{P}^n(k) \mid \bar{v} \in k^{n+1} \setminus \{0\}, \forall P \in S : P(\bar{v}) = 0\}.$$

A *projective algebraic set* in $\mathbb{P}^n(k)$ is a subset of the form $Z(S)$, where $S \subseteq k[x_0, \dots, x_n]$ is a set of homogenous polynomials.

For convenience, we shall extend the operator $Z(\cdot)$ to non homogenous polynomials. For any set $S \subseteq k[x_0, \dots, x_n]$ (not necessarily consisting of homogenous polynomials), we set

$$Z(S) := \{[\bar{v}] \mid \bar{v} \in k^{n+1} \setminus \{0\}, P_{[i]}(\bar{v}) = 0 \forall i \geq 0\}.$$

Just as in the affine case, we have $Z(S) = Z(S \cdot k[x_0, \dots, x_n])$ (why?). Hence the projective algebraic sets in $\mathbb{P}^n(k)$ are the sets of the type $Z(I)$, where $I \subseteq k[x_0, \dots, x_n]$ is an ideal generated by homogenous elements.

We shall say that an ideal of $k[x_0, \dots, x_n]$ is *homogenous* if it is generated by homogenous elements.

Lemma 7.1. *Let $I \subseteq k[x_0, \dots, x_n]$ be an ideal. Then I is homogenous iff for all $P \in I$ and all $i \geq 0$, we have $P_{[i]} \in I$. If I is homogenous then its radical $\mathfrak{r}(I)$ is also homogenous.*

In other words, a homogenous ideal is a graded ideal in $k[x_0, \dots, x_n]$ (ie a graded $k[x_0, \dots, x_n]$ -submodule of $k[x_0, \dots, x_n]$).

Proof. See exercises. \square

Proposition 7.2. *Projective algebraic sets are closed in $\mathbb{P}^n(k)$. Furthermore, if $C \subseteq \mathbb{P}^n(k)$ is a closed subset and J is the ideal generated by the homogenous polynomials which vanish on C , then $Z(J) = C$. In particular, the closed subsets of $\mathbb{P}^n(k)$ are precisely the projective algebraic sets.*

Proof. Let $S := \{P_i\}$ be a set of homogenous polynomials in $k[x_0, \dots, x_n]$. By construction, we have

$$u_i^{-1}(Z(S)) = Z(\{P_i(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)\})$$

so that $u_i^{-1}(Z(S))$ is closed in k^n . By Proposition 6.1, the set $Z(S) \cap U_i$ is thus closed in U_i (for the induced topology). Since the U_i cover $\mathbb{P}^n(k)$, we thus see that $Z(S)$ is closed in $\mathbb{P}^n(k)$.

As to the second assertion, we clearly have $Z(J) \supseteq C$. So we need to prove that $Z(J) \subseteq C$. In other words, we have to prove that if $[\bar{v}] \notin C$, then there is a homogenous polynomial $H \in J$, such that $H([\bar{v}]) \neq 0$. Now let $j \in \{0, \dots, n\}$ and suppose that $[\bar{v}] \in U_j$. We then have $[\bar{v}] \notin C \cap U_j$. Since $u_j^{-1}(C)$ is the zero set of an ideal in $k[x_0, \dots, \tilde{x}_j, \dots, x_n]$, there is a polynomial $P(x_0, \dots, \tilde{x}_j, \dots, x_n) \in k[x_0, \dots, \tilde{x}_j, \dots, x_n]$ such that $P(u_j^{-1}([\bar{v}])) \neq 0$ and such that $P \in \mathcal{I}(u_j^{-1}(C))$. Let

$$\beta_j(P) := x_j^{\deg(P)} P\left(\frac{x_0}{x_j}, \dots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \dots, \frac{x_n}{x_j}\right).$$

This is a homogenous polynomial (the "homogenisation" of P with respect of the variable x_j) such that

$$(\beta_j(P))(x_0, \dots, x_{j-1}, 1, x_j, \dots, x_n) = P_j.$$

In particular we have $Z(\beta_j(P)) \supseteq C \cap U_j$ and $(\beta_j(P))([\bar{v}]) = P(u_j^{-1}([\bar{v}])) \neq 0$. Now let $Q_j = x_j \beta_j(P)$. Then Q_j is still homogenous and we have $Q_j([\bar{v}]) \neq 0$ and $Z(Q_j) \supseteq C$ (because x_j vanishes on $\mathbb{P}^n(k) \setminus U_j$). Hence we may set $H = Q_j$. This completes the proof. \square

If $A \subseteq \mathbb{P}^n(k)$ is a subset, we shall write $\mathcal{I}(A) \subseteq k[x_0, \dots, x_n]$ for the ideal generated by the homogenous polynomials vanishing on A . This notation clashes with the notation in the affine case but the context should make it clear which definition of $\mathcal{I}(\cdot)$ we use.

Now we have the analogue of Proposition 2.3:

Proposition 7.3. *Let $C \subseteq \mathbb{P}^n(k)$ be a closed subset and let $J \subseteq k[x_0, \dots, x_n]$ be a homogenous radical ideal. Suppose that $Z(J) \neq \emptyset$. Then $\mathcal{I}(C)$ is a (by definition homogenous) radical ideal and we have $Z(\mathcal{I}(C)) = C$ and $\mathcal{I}(Z(J)) = J$.*

Proof. We first show that $\mathcal{I}(C)$ is a radical ideal. To see this, let $H \subseteq \mathfrak{r}(\mathcal{I}(C))$ be the subset of $\mathfrak{r}(\mathcal{I}(C))$ consisting of the homogenous elements of $\mathfrak{r}(\mathcal{I}(C))$. By the definition of the nilradical of an ideal, all the elements of H vanish on C . On the other hand, $\mathfrak{r}(\mathcal{I}(C))$ is a homogenous ideal by Lemma 7.1 and so H generates $\mathfrak{r}(\mathcal{I}(C))$. Hence $\mathfrak{r}(\mathcal{I}(C)) \subseteq \mathcal{I}(C)$. Hence $\mathfrak{r}(\mathcal{I}(C)) = \mathcal{I}(C)$.

The equality $Z(\mathcal{I}(C)) = C$ is contained in Proposition 7.2. For the second equality, note first that the inclusion $J \subseteq \mathcal{I}(Z(J))$ follows from the definitions. We thus only have to prove that $J \supseteq \mathcal{I}(Z(J))$. So let Q be a non zero homogenous polynomial vanishing on $Z(J)$. We need to show that $Q \in J$. Note that $\deg(Q) > 0$. Indeed, if $\deg(Q) = 0$ then Q is a non zero constant polynomial and then $Z(Q) = \emptyset$, which implies that $Z(J) = \emptyset$. This is not possible by assumption. Also, note that J does not contain any constant polynomial, for otherwise $Z(J) = \emptyset$. Now consider the map

$$q : k^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n(k)$$

given by the formula $q(\bar{v}) := [\bar{v}]$. Note that $q^{-1}(Z(J))$ is by construction the set of zeroes of J in $k^{n+1} \setminus \{0\}$. Hence the set of zeroes of J in k^n is the set $q^{-1}(Z(J)) \cup \{0\}$ (since every non constant homogenous polynomial vanishes at the 0 vector). Now Q also vanishes on $q^{-1}(Z(J)) \cup \{0\}$ and so by the strong Nullstellensatz we have $Q \in \mathfrak{r}(J) = J$. \square

Lemma 7.4. *Let $J \subseteq k[x_0, \dots, x_n]$ be a homogenous radical ideal. Then the subset $Z(J)$ of $\mathbb{P}^n(k)$ is empty iff $J = k[x_0, \dots, x_n]$ or $J = k[x_0, \dots, x_n]_+$.*

Here $k[x_0, \dots, x_n]_+$ is the homogenous ideal of $k[x_0, \dots, x_n]$ generated by all the non constant homogenous polynomials.

Proof. We first prove the \Leftarrow direction of the equivalence. So let $\bar{v} = \langle v_1, \dots, v_n \rangle \in k^{n+1} \setminus \{0\}$. Suppose that $v_{i_0} \neq 0$ for some $i_0 \in \{0, \dots, n\}$. The homogenous polynomial $x_{i_0} \in k[x_0, \dots, x_n]_+$ does not vanish at $[\bar{v}]$. Since $\bar{v} \in k^{n+1} \setminus \{0\}$ was arbitrary, we see that $Z(J)$ is empty if $J = k[x_0, \dots, x_n]_+$ or $J = k[x_0, \dots, x_n]$.

We now prove the \Rightarrow direction. So suppose that $Z(J) = \emptyset$. To avoid notational confusion, write $Z_{\text{aff}}(I)$ for the set of common zeroes in k^{n+1} of the elements of a (not necessarily homogenous) ideal $I \subseteq k[x_0, \dots, x_n]$. By using the map $q : k^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n(k)$ described in the proof of Proposition 7.3, we see that

$$Z_{\text{aff}}(J) \cap (k^{n+1} \setminus \{0\}) = \emptyset.$$

Now suppose first that J does not contain any non zero constant polynomials. Then $0 \in Z_{\text{aff}}(J)$ (because J is generated by non constant homogenous polynomials) so that $Z_{\text{aff}}(J) = \{0\}$. Using the correspondence described after Proposition 2.3, we conclude that J is the radical ideal of $k[x_0, \dots, x_n]$ associated with the point 0, which is $k[x_0, \dots, x_n]_+$. If J contains a non zero constant polynomial then $J = k[x_0, \dots, x_n]$ (because J contains a unit). So we conclude that if $Z(J) = \emptyset$ then either $J = k[x_0, \dots, x_n]_+$ or $J = k[x_0, \dots, x_n]$. \square

We shall call the ideal $k[x_0, \dots, x_n]_+$ the *irrelevant* ideal of $k[x_0, \dots, x_n]$.

We conclude from Lemma 7.4 and Proposition 7.3 that there is a correspondence

$$\{\text{closed sets in } \mathbb{P}^n(k)\} \xrightleftharpoons[Z]{\mathcal{I}} \{\text{non irrelevant homogenous radical ideals in } R_n\}$$

where the maps $Z(\cdot)$ and $\mathcal{I}(\cdot)$ are inverse to each other.

A *projective variety* is a variety isomorphic (as a variety) to a closed subvariety of $\mathbb{P}^n(k)$ (for some $n \geq 0$).

A *quasi-projective variety* is a variety isomorphic to an open subvariety of a projective variety.

8 Dimension

Let T be a topological space. Then T is said to be *noetherian* if for any descending sequence

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \dots$$

of closed subsets of T , there is an $i_0 \geq 0$ such that $C_{i_0} = C_{i_0+1} = \dots$. In this situation, we say that the sequence stabilises at i_0 . Note that any subset of a noetherian topological space is also noetherian (in the induced topology) (why?). Finally, note that a noetherian topological space is quasi-compact (ie any covering of the space has a finite subcovering). See exercises.

The topological space T is said to be *irreducible* if T is not empty and any open subset of T is dense in T .

Example. The Zariski topology on k^n is noetherian. Indeed any descending sequence

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \dots$$

of closed subsets of k^n corresponds uniquely to a sequence

$$\mathcal{I}(C_1) \subseteq \mathcal{I}(C_2) \subseteq \mathcal{I}(C_3) \subseteq \dots$$

(see the first section) and such a sequence stabilises for some index because $k[x_1, \dots, x_n]$ is a noetherian ring (by Hilbert's basis theorem). Consequently, the topology of any algebraic set is noetherian. A closed subspace Z of k^n is irreducible iff Z is irreducible as an algebraic set (why?).

Lemma 8.1. *Let T be a non empty noetherian topological space. Then there is a unique finite collection $\{T_i\}$ of irreducible closed subsets of T such that*

$$(1) T = \cup_i T_i$$

$$(2) T_i \not\subseteq \cup_{j \neq i} T_j \text{ for all } i.$$

Note that a consequence of the lemma is that the T_i are the irreducible closed subsets of T which are maximal for the relation of inclusion among all the irreducible closed subsets contained in T (why?).

Proof. See exercises. \square

The closed subsets T_i described in Lemma 8.1 are called the *irreducible components* of T . If T is an algebraic set, the decomposition of T into irreducible components coincides with the decomposition given by Lemma 2.6 (why?).

Lemma 8.2. *A variety is noetherian.*

Proof. Let V be a variety. Let

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \dots$$

be a descending sequence of closed subsets of V . Let $\{U_i\}$ be a finite covering of V by open affine subvarieties. Since the U_i are noetherian (as topological spaces) by the remark above and since there are only finitely many U_i , there is an integer $l \geq 1$ such that $C_l \cap U_i = C_{l+1} \cap U_i = \dots$ for all i . Since the U_i cover V , this implies that $C_l = C_{l+1} = \dots$ \square

Now consider again a non empty topological space T . The *dimension* $\dim(T)$ of T is

$$\dim(T) := \sup\{t \mid \text{there are irreducible closed subsets } C_0, \dots, C_t \subseteq T \text{ such that } C_0 \subsetneq C_1 \subsetneq \dots \subsetneq C_t\}.$$

Note that $\dim(T)$ might be infinite. Dimension is not defined for the empty topological space (note that some authors define the dimension of the empty topological space to be -1).

Lemma 8.3. *Let $V \subseteq k^n$ be an algebraic set. Then $\dim(V) = \dim(\mathcal{C}(V))$.*

Here $\dim(\mathcal{C}(V))$ is the dimension of $\mathcal{C}(V)$ as a ring (see Def. 11.1 in CA). Recall that by definition we have

$$\dim(R) := \sup\{n \mid \exists \mathfrak{p}_0, \dots, \mathfrak{p}_n \in \text{Spec}(R) : \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \dots \supsetneq \mathfrak{p}_n\}$$

for any ring R .

Proof. We have already seen that irreducible closed subsets of V correspond to prime ideals of $\mathcal{C}(V)$ (see Lemma 2.5). Hence the definition of $\dim(\mathcal{C}(V))$ corresponds with the definition of $\dim(V)$ under the correspondence between radical ideals of $\mathcal{C}(V)$ and closed subsets of V described at the beginning of section one. \square

Theorem 8.4. (1) *The dimension of k^n is n .*

(2) *The dimension of $\mathbb{P}^n(k)$ is n .*

Proof. (1) We saw in CA that $\dim(k[x_1, \dots, x_n]) = n$ (see Cor. 11.27 in CA). Hence $\dim(k^n) = n$ by Lemma 8.3. (2) Apply question 2.7 to the open covering of $\mathbb{P}^n(k)$ by its standard coordinate charts and use (1). \square

Definition 8.5. *Let T be a topological space. Let $C \subseteq T$ be a closed irreducible subspace. The codimension, or height of C is*

$$\text{cod}(C, T) = \text{ht}(C, T) := \sup\{t \mid \text{there are irreducible closed subsets } C_1, \dots, C_t \subseteq T \text{ such that } C \subsetneq C_1 \subsetneq \dots \subsetneq C_t\}$$

We shall sometimes write $\text{cod}(C)$ and $\text{ht}(C)$ instead of $\text{cod}(C, T)$ and $\text{ht}(C, T)$, respectively, when the ambient topological space T is clear from the context.

Note that from the definitions, we have

$$\dim(T) = \sup_{C \text{ closed irreducible subset of } T} \text{ht}(C, T).$$

Suppose that $C, V \subseteq k^n$ are algebraic sets in k^n and that $C \subseteq V$. Suppose that C is irreducible. Then the height of C in V is the height of the prime ideal $\mathcal{I}(C) \pmod{\mathcal{I}(V)}$ of $\mathcal{C}(V)$ (in the sense of section 11 of CA). The proof is similar to the proof of Lemma 8.3 (we leave the details to the reader).

Proposition 8.6. *Let V be a variety. Let $C \subseteq V$ be an irreducible closed subset. Then $\dim(V)$ and $\text{cod}(C, V)$ are finite.*

Proof. See question 2.7 (4). \square

Finally, we also have the following difficult result of commutative algebra, which justifies the use of the word "codimension".

Theorem 8.7. *Let R be a finitely generated k -algebra. Suppose that R is an integral domain. Let $\mathfrak{p} \subseteq R$ be a prime ideal. Then we have*

$$\text{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim(R)$$

The proof of this theorem is given in the Appendix. The proof is in several steps and is structured as an exercise with model solution. We suggest the reader go through the steps by themselves first without looking at the model solution. Note that the proof of Theorem 8.7 is not examinable.

Corollary 8.8. *Let V be an irreducible variety. Let $C \subseteq V$ be an irreducible closed subset. Then*

$$\text{cod}(C, V) + \dim(C) = \dim(V)$$

Note first that from the definitions, we have

$$\text{cod}(C, V) + \dim(C) \leq \dim(V)$$

(why?). So we only have to prove that $\text{cod}(C, V) + \dim(C) \geq \dim(V)$.

Proof. Let $\{V_i\}$ be a finite open covering of V . We suppose that each V_i is isomorphic to an affine variety when viewed as an open subvariety of V . Note that since V is irreducible, each V_i is irreducible as well (why?). We use question 2.7 and we obtain

$$\sup_{i, C \cap V_i \neq \emptyset} \text{cod}(C \cap V_i, V_i) = \text{cod}(C, V)$$

and

$$\sup_{i, C \cap V_i \neq \emptyset} \dim(C \cap V_i) = \dim(C)$$

Let R_i be the coordinate ring associated with V_i (so that $V_i \simeq \text{Spm}(R_i)$ as a set). Then R_i is integral by question 2.5 (3). Hence we may apply Theorem 8.7 and we compute

$$\text{cod}(C \cap V_i, V_i) + \dim(C \cap V_i) = \dim(V_i)$$

if $C \cap V_i \neq \emptyset$. Applying $\sup_{i, C \cap V_i \neq \emptyset}(\cdot)$ to both sides of this equality and using question 2.7 again, we see that there is an index i_0 such that $C \cap V_{i_0} \neq \emptyset$ and such that

$$\text{cod}(C \cap V_{i_0}, V_{i_0}) + \dim(C \cap V_{i_0}) = \dim(V)$$

and hence

$$\text{cod}(C, V) + \dim(C) \geq \text{cod}(C \cap V_{i_0}, V_{i_0}) + \dim(C \cap V_{i_0}) = \dim(V).$$

which is what we wanted to prove. \square

Here is another fundamental result from the CA course, which is relevant to the theory of dimension.

Theorem 8.9. *Let $n \geq 0$ and let $V, W \subseteq k^n$ be algebraic sets. Suppose that $V \subseteq W$. Suppose that $I \subseteq k[x_1, \dots, x_n]$ is such that $Z(I) = V$.*

Let $l \geq 1$ and suppose that the ideal $I \pmod{\mathcal{I}(W)} \subseteq \mathcal{C}(W)$ is generated by l elements.

Then every irreducible component of V has codimension $\leq l$ in W .

Furthermore, if C is an irreducible component of V then there is an ideal $J \subseteq \mathcal{I}(C) \subseteq \mathcal{C}(W)$ which is generated by $\text{cod}(C, W)$ elements and such that C is an irreducible component of $Z(J) \subseteq W$ (in other words $\mathcal{I}(C)$ is a prime ideal, which is minimal among the prime ideals which contain J).

See Cor. 11.15 and Cor. 11.17 in CA for the proof. This is a consequence of Krull's principal ideal theorem.

9 Rational maps

Let V, W be varieties. Consider the set $H = H_{V,W}$ whose elements are morphisms $f : U \rightarrow W$, where U is a non empty open subvariety of V . Let $\sim = \sim_{V,W}$ be the relation on H , such that $f : U \rightarrow W$ and $g : O \rightarrow W$ are related by \sim iff there is a open subvariety UO of $U \cap O$, which is dense in V and which is such that $f|_{UO} = g|_{UO}$. The relation \sim is easily seen to be an equivalence relation. We shall write $\text{Rat}(V, W)$ for the set of equivalence classes of H under the relation \sim . We call elements of $\text{Rat}(V, W)$ *rational maps* from V to W . Beware that rational maps are not actual maps but equivalence classes of maps. We will use the notation $f : V \dashrightarrow W$ to denote a rational map f from V to W .

Remark 9.1. If V is irreducible and W is quasi-projective, the condition $f|_{UO} = g|_{UO}$ is equivalent to the stronger condition $f|_{U \cap O} = g|_{U \cap O}$. This follows from the fact that W is separated if it is quasi-projective. The term *separated* will be defined in section 12. See also the solution of Question 3.4.

Suppose now until further notice that V is irreducible.

Note the following. Let $f : U \rightarrow W$ be a representative of a rational map from V to W . If f is dominant, then any other representative of the same rational map is dominant as well. Indeed, let $g : O \rightarrow W$ be another representative of the rational map defined by f . Then $f|_{UO} = g|_{UO}$ for some non empty open subset $UO \subseteq U \cap O$ (which is automatically dense in V). Suppose for contradiction that g is not dominant. Then

$W \setminus g(O)$ contains a non empty open subset W_1 . Since $f : U \rightarrow W$ is dominant, we know that $f^{-1}(W_1) \neq \emptyset$. Thus, since V is irreducible, we have $f^{-1}(W_1) \cap UO = g^{-1}(W_1) \cap UO \neq \emptyset$. In particular $g^{-1}(W \setminus g(O)) \neq \emptyset$, which is a contradiction. So g is also dominant.

We thus see from the discussion in the last paragraph that it makes sense to speak of a *dominant rational map* from V to W : it is a rational map all of whose representative are dominant (or equivalently, it is a rational map with one dominant representative). We shall write $\text{Rat}_{\text{dom}}(V, W)$ for the set of dominant rational maps from V to W .

Remark 9.2. One can show that the image of a dominant morphism between two irreducible varieties contains an open set of the target. A special case of this statement is proven in Question 4.6.

We shall write $\kappa(V)$ as a shorthand for $\text{Rat}(V, k)$. If $f : U \rightarrow k$ and $g : O \rightarrow k$ are two elements of $H_{V,k}$, one may define a new element $f + g : U \cap O \rightarrow k$ of $H_{V,k}$ by declaring that $(f + g)(u) = f(u) + g(u)$ for all $u \in U \cap O$ (note that $U \cap O$ is not empty because V is irreducible). Note that $f + g : U \cap O \rightarrow k$ is a morphism of varieties (use Proposition 4.5). Similarly, one may define an element $fg = f \cdot g : U \cap O \rightarrow k$ by declaring that $(f \cdot g)(u) = f(u) \cdot g(u)$ for all $u \in U \cap O$. Again, $f \cdot g : U \cap O \rightarrow k$ is a morphism (same reasoning as before). Finally, if $f : U \rightarrow k$ does not vanish on all of U , then we may define $f^{-1} : U \setminus Z(f) \rightarrow k$ by the formula $f^{-1}(u) = 1/f(u)$. Here again, $f^{-1} : U \setminus Z(f) \rightarrow k$ is a morphism (reason as before and use the remark after Proposition 4.5). It is again easily verified that these operations are compatible with $\sim_{V,k}$ and we thus obtain a structure of field on $\kappa(V)$. This field is called the *function field* of V . There is an obvious injection $k \hookrightarrow \kappa(V)$ which makes $\kappa(V)$ into a k -algebra. Note finally that for any $v \in V$, there is a natural injection $\mathcal{O}_{V,v} \hookrightarrow \kappa(V)$, which sends any representative of an equivalence class in $\mathcal{O}_{V,v}$ to its equivalence class in $\kappa(V)$. So $\kappa(V)$ naturally contains the local rings at all the points of V .

Now suppose that we are given a dominant morphism of irreducible varieties $a : V \rightarrow W$. Then we may define a map $H_{W,k} \rightarrow H_{V,k}$ by the recipe

$$(f : O \rightarrow k) \mapsto (f \circ a|_{f^{-1}(O)} : f^{-1}(O) \rightarrow k)$$

where O is a non empty open subvariety of W and $f : O \rightarrow k$ is an element of $H_{W,k}$. This definition makes sense because $f^{-1}(O) \neq \emptyset$ as f is dominant. One checks (we skip the details) that this map is compatible with the relations $\sim_{W,k}$ and $\sim_{V,k}$ and also with the operations $+$, $(\cdot)^{-1}$ and \cdot . One thus obtains a map of rings

$$a^{*,\text{rat}} : \kappa(W) \rightarrow \kappa(V).$$

Note that since $\kappa(W)$ is a field, the map $a^{*,\text{rat}}$ is injective. Also, if $a : V \rightarrow W$ is the inclusion of an open subvariety of V into W , it follows from the definitions that the map $a^{*,\text{rat}}$ is a bijection (check!). Finally, the construction of $a^{*,\text{rat}}$ is clearly compatible with compositions of dominant morphisms (ie if $b : W \rightarrow W_1$ is another dominant morphism of irreducible varieties, then $(b \circ a)^{*,\text{rat}} = a^{*,\text{rat}} \circ b^{*,\text{rat}}$). We conclude from all this that the homomorphism $a^{*,\text{rat}}$ only depends on the element of $\text{Rat}(V, W)$ defined by a . In turn, any dominant representative $g : O \rightarrow W$ of an element of $\text{Rat}(V, W)$ defines a map of k -algebras $g^{*,\text{rat}} : \kappa(W) \rightarrow \kappa(V) \simeq \kappa(O)$ and again it follows from the definitions that this map only depends on the class of g in $\text{Rat}(V, W)$. So all in all, any dominant rational map $\rho \in \text{Rat}_{\text{dom}}(V, W)$ gives rise to an injection of fields $\rho^{*,\text{rat}} : \kappa(W) \rightarrow \kappa(V)$.

Lemma 9.3. *Let X be an irreducible affine variety. Let $V \subseteq k^n$ be an algebraic set giving rise to X . Then there is a canonical isomorphism of k -algebras $\kappa(X) \rightarrow \text{Frac}(\mathcal{C}(V))$. This isomorphism is compatible with dominant regular maps between irreducible algebraic sets and the corresponding morphisms of varieties.*

Note that by question 2.5, the fact that V irreducible implies that the ring $\mathcal{C}(V)$ is an integral domain. So it makes sense to talk about the fraction field $\text{Frac}(\mathcal{C}(V))$ of $\mathcal{C}(V)$.

Proof. Define a map

$$\tau : \mathcal{C}(V) \rightarrow \kappa(X)$$

by sending an element of $\mathcal{C}(V)$ to the equivalence class of the corresponding morphism $X \rightarrow k$. By construction, this is a map of k -algebras. Now suppose that $f : V \rightarrow k$ is a regular map and suppose that $\tau(f) = 0$. Then by definition f vanishes on an open subset of V . However the vanishing set of f is equal to $Z(f)$ and so is closed. Hence $Z(f)$ contains the closure in V of an open subset of V and thus $Z(f) = V$ (because V is irreducible). Hence f is the map with constant value 0 and thus $\tau(f) = 0$. We thus see that τ is injective. Hence τ extends to a (necessarily injective) map of fields $\text{Frac}(\mathcal{C}(V)) \rightarrow \kappa(X)$ by the universal property of localisation (see Lemma-Definition 5.1 in CA). We have to show that this last map is surjective. To see this, let O be an open subset of V and let $g : O \rightarrow k$ be a representative of an element of $\kappa(X)$. By Lemma 4.1, we may assume without restriction of generality that $O = V \setminus Z(f)$, where $f \in \mathcal{C}(V)$. By Corollary 4.4, we know that $g = \frac{g_1}{g_2}|_O$, where $g_1, g_2 \in \mathcal{C}(V)$. Hence $\tau(g_1/g_2) = g$. Since g was arbitrary, we have shown that the map $\text{Frac}(\mathcal{C}(V)) \rightarrow \kappa(X)$ is surjective, and thus an isomorphism.

The fact that this isomorphism is compatible with dominant regular maps between irreducible algebraic sets follows directly from the definition of τ . \square

Proposition 9.4. *Let V be an irreducible variety. Then $\kappa(V)$ is finitely generated over k as a field and the dimension of V is equal to the transcendence degree of $\kappa(V)$ over k .*

Recall that the transcendence degree of $\kappa(V)$ over k is the largest integer $n \geq 0$ such that there exists an injection of k -algebras

$$k[x_1, \dots, x_n] \hookrightarrow \kappa(V)$$

See section 11.1 of CA for details.

Proof. (of Proposition 9.4) Let $\{V_i\}$ be a finite open covering of V and suppose that each V_i is an affine variety. By a remark at the beginning of this section, the function field of V_i is isomorphic to the function field of V as a k -algebra. On the other hand, we have $\dim(V) = \sup_i \dim(V_i)$ by question 2.7. Hence it is sufficient to show that the transcendence degree of $\kappa(V_i)$ over k is equal to $\dim(V_i)$ for all i . So we may suppose without restriction of generality that V is affine. In that case, the statement is a consequence of Lemma 8.3, Lemma 9.3 and Cor. 11.28 in CA (which follows from the computation of the dimension of polynomial rings and the Noether normalisation lemma). \square

Proposition 9.5. *Let $a : V \rightarrow W$ be a dominant morphism of irreducible subvarieties. Then $a^{*,\text{rat}} : \kappa(W) \rightarrow \kappa(V)$ is an isomorphism iff there exist open subvarieties $V_0 \subseteq V$ and $W_0 \subseteq W$ such that $a(V_0) \subseteq W_0$ and such that the induced morphism $a|_{V_0} : V_0 \rightarrow W_0$ is an isomorphism.*

Proof. The \Leftarrow direction of the equivalence is clear (why?) so we only have to establish the \Rightarrow direction. Let $W_0 \subseteq W$ be an open affine subvariety and let V_0 be an open affine subvariety of $a^{-1}(W_0)$ (this exists by Proposition 5.1). We claim that the map $V_0 \rightarrow W_0$ induced by a is also dominant. To prove this claim, suppose for contradiction that the map $V_0 \rightarrow W_0$ is not dominant. Then there is a non empty subset O of W_0 such that $O \subseteq W_0 \setminus a(V_0)$. Hence $a^{-1}(O) \cap V_0 = \emptyset$. Now $a^{-1}(O) \neq \emptyset$ since a is dominant, so this contradicts the irreducibility of V . We have thus established the claim. Since the inclusions $V_0 \rightarrow V$ and

$W_{00} \rightarrow W$ induce isomorphisms of function fields, we may thus assume without restriction of generality that V and W are affine to begin with. In view of Lemma 9.3 and question 2.5 (3), it is thus sufficient to prove the following statement of commutative algebra.

Let $\phi : A \rightarrow B$ be a homomorphism of finitely generated integral k -algebras. Suppose that $\text{Spm}(\phi)(\text{Spm}(B))$ is dense in $\text{Spm}(A)$ and suppose that the induced map $\text{Frac}(\phi) : \text{Frac}(A) \rightarrow \text{Frac}(B)$ is an isomorphism. Then there is an element $f \in A$ such that the induced map $A[f^{-1}] \rightarrow B[\phi(f)^{-1}]$ is an isomorphism.

To prove this assertion, note that by question 1.5 we already know that under the given assumptions, ϕ must be injective. Note also that since we have a commutative diagram

$$\begin{array}{ccc} \text{Frac}(A) & \xrightarrow{\text{Frac}(\phi)} & \text{Frac}(B) \\ \uparrow & & \uparrow \\ A & \xrightarrow{\phi} & B \end{array}$$

all whose maps are injective, the induced map $A[f^{-1}] \rightarrow B[\phi(f)^{-1}]$ is injective for any choice of $f \in A \setminus \{0\}$ (remember that A and B are integral domains). Thus we only have to show that there is $f \in A \setminus \{0\}$ such that the induced map $A[f^{-1}] \rightarrow B[\phi(f)^{-1}]$ is surjective. Now let b_1, \dots, b_l be generators of B as a k -algebra. Let $a_1/c_1, \dots, a_l/c_l \in \text{Frac}(A)$ such that

$$b_i/1 = \phi(a_i)/\phi(c_i) =: \text{Frac}(\phi)(a_i/c_i)$$

for all $i \in \{1, \dots, l\}$. Let $f := \prod_i c_i$. Then $b_i/1 = \text{Frac}(\phi)(a_i(\prod_{j \neq i} c_j)/f)$. Hence the image of

$$A[f^{-1}] \rightarrow B[\phi(f)^{-1}]$$

contains $b_i/1$ for all $i \in \{1, \dots, l\}$ and also contains $1/\phi(f) = \text{Frac}(\phi)(1/f)$. Since $B[\phi(f)^{-1}]$ is generated as a k -algebra by $1/\phi(f)$ and by the elements $b_i/1$ (use Lemma 5.3 in CA), we see that $A[f^{-1}] \rightarrow B[\phi(f)^{-1}]$ is surjective. \square

If V and W are irreducible varieties, and $V_0 \subseteq V$ and $W_0 \subseteq W$ are open subvarieties such that $V_0 \simeq W_0$, we shall say that V and W are *birational*, or *rationally isomorphic*.

A *birational map* from V to W is a rational map from V to W which has a representative $f : O \rightarrow W$, such that $f(O)$ is open and such that the induced map $O \rightarrow f(O)$ is an isomorphism (where $f(O)$ is endowed with its structure of open subvariety of W). A *birational morphism* from V to W is a morphism $V \rightarrow W$ which induces a birational map.

Proposition 9.5 implies that a dominant rational map $\rho \in \text{Rat}_{\text{dom}}(V, W)$ is birational iff $a^{, \text{rat}} : \kappa(W) \rightarrow \kappa(V)$ is bijective.*

Proposition 9.6. *Let V, W be irreducible varieties. Let $\kappa(W) \hookrightarrow \kappa(V)$ be a field extension compatible with the k -algebra structures. Then there is an open subvariety V_0 of V and a dominant morphism $a : V_0 \rightarrow W$ such that the extension $a^{*, \text{rat}} : \kappa(W) \rightarrow \kappa(V_0)$ is isomorphic to $\kappa(W) \hookrightarrow \kappa(V)$ as a $\kappa(W)$ -extension.*

A different wording of the conclusion of the proposition is that there is an isomorphism of rings between $\kappa(V)$ and $\kappa(V_0)$ compatible with the given $\kappa(W)$ -algebra structures.

Proof. We may suppose without restriction of generality that V and W are affine varieties (why?). Let B (resp. A) be the coordinate ring of V (resp. W). Let $\iota : \text{Frac}(A) \simeq \kappa(W) \hookrightarrow \kappa(V) \simeq \text{Frac}(B)$ be the given field extension.

We claim that there is an $g \in B \setminus \{0\}$ such that $\iota(A) \subseteq B[g^{-1}] \subseteq \text{Frac}(B)$ (where A is identified with its image in $\text{Frac}(A)$). To prove this, let a_1, \dots, a_l be generators of A as a k -algebra. For all $i \in \{1, \dots, l\}$ let $b_i, c_i \in B$ be such that $b_i/c_i = \iota(a_i/1)$. Let $g := \prod_i c_i$. We then have $\iota(a_i/1) \in B[g^{-1}]$ and thus $\iota(A) \subseteq B[g^{-1}]$, proving the claim.

Now let V_0 be the open affine subvariety associated with $B[g^{-1}]$. Let $\iota_0 : A \rightarrow B[g^{-1}]$ be the map induced by ι and the natural map from A to $\text{Frac}(A)$. Since the map ι_0 is injective, it induces a dominant map $V_0 \rightarrow W$ by question 1.5. Hence V_0 and the map $V_0 \rightarrow W$ satisfy the requirements of the proposition. \square

Finally, note the following. Let V and W be irreducible varieties. Consider the map

$$\text{Rat}_{\text{dom}}(V, W) \rightarrow \text{homomorphisms of } k\text{-algebras } \kappa(W) \rightarrow \kappa(V) \quad (*)$$

which sends $a \in \text{Rat}_{\text{dom}}(V, W)$ to $a^{*, \text{rat}} : \kappa(W) \rightarrow \kappa(V)$. Proposition 9.6 implies that this map is surjective.

On the other hand we have

Lemma 9.7. *The map $(*)$ is injective.*

Proof. Let $a_1, a_2 \in \text{Rat}_{\text{dom}}(V, W)$ and suppose that $a_1^{*, \text{rat}} = a_2^{*, \text{rat}}$. We have to show that $a_1 = a_2$.

Now there is a by construction an open subset $O \subseteq V$ and morphisms $\alpha_1, \alpha_2 : O \rightarrow W$ which represent a_1 and a_2 , respectively. Replacing W by one of its open affine subvarieties O' and replacing V by an open affine subvariety of $\alpha_1^{-1}(O')$, we may assume that both V and W are affine and that a_1 (resp. a_2) is represented by a morphism. Recycling notation, call $\alpha_1 : V \rightarrow W$ (resp. $\alpha_2 : V \rightarrow W$) a morphism representing a_1 (resp. a_2).

Now let B (resp. A) be the coordinate ring of V (resp. W).

Let $\iota : \text{Frac}(A) \simeq \kappa(W) \hookrightarrow \kappa(V) \simeq \text{Frac}(B)$ be the field extension given by $a_1^{*, \text{rat}} = a_2^{*, \text{rat}}$. We have by construction a commutative diagram

$$\begin{array}{ccc} \text{Frac}(A) & \xrightarrow{\iota} & \text{Frac}(B) \\ \uparrow & & \uparrow \\ A & \xrightarrow{\alpha_i^*} & B \end{array}$$

for $i \in \{1, 2\}$. Since the vertical maps are injective we thus have $\alpha_1^* = \alpha_2^*$. \square

In view of the last lemma and the comment preceding it, we thus see that *the map $(*)$ is bijective*. In particular, there is a one-to-one correspondence between dominant rational maps from V to W and $\kappa(W)$ -algebra structures on the field $\kappa(V)$.

We shall from now on often write a^* for $a^{*, \text{rat}}$ when V and W are irreducible varieties and $a \in \text{Rat}_{\text{dom}}(V, W)$. This is justified by the proof of Lemma 9.7.

10 Products

We wish to endow the cartesian product of two varieties with the structure of a variety. We shall do this for quasi-projective varieties.

Definition 10.1. Let V and W be varieties. A product of V and W is a triple $(V \amalg W, \pi_V, \pi_W)$, where $V \amalg W$ is a variety and $\pi_V : V \amalg W \rightarrow V$ and $\pi_W : V \amalg W \rightarrow W$ are morphisms of varieties. This triple is required to have the following property (PROD).

(PROD) If X is a variety and $a : X \rightarrow V$ and $b : X \rightarrow W$ are morphisms of varieties, then there is a unique morphism of varieties $a \amalg b : X \rightarrow V \amalg W$ such that $\pi_V \circ (a \amalg b) = a$ and $\pi_W \circ (a \amalg b) = b$.

Note that property (PROD) in Definition 10.1 characterises the triple $(V \amalg W, \pi_V, \pi_W)$ uniquely up to unique isomorphism of triples (an isomorphism of triples is an isomorphism of the underlying varieties which is compatible with the morphisms in play). This is an example of categorical product. Note that if V and W are varieties, it is not clear *a priori* that they have a product. However, if the product of V and W exists, it is uniquely defined. Abusing language, we shall often say that $V \amalg W$ is the product of V and W without writing the associated morphisms π_V and π_W . We first show

Theorem 10.2. Let $m, n \geq 0$. The product $\mathbb{P}^m(k) \amalg \mathbb{P}^n(k)$ exists.

Before starting with the proof, we make a construction. We shall consider the projective space \mathbb{P}^{mn+m+n} . The space \mathbb{P}^{mn+m+n} is by definition the set of lines through the origin in the vector space $k^{mn+m+n+1} = (m+1)(n+1)$ and we shall index its standard basis using double indices.

Let $b_1, \dots, b_{mn+m+n+1}$ be the standard basis of $k^{mn+m+n+1}$, indexed in the usual manner. If $i \in \{0, \dots, m\}$ and $j \in \{0, \dots, n\}$, we shall write b_{ij} for the element $b_{j(m+1)+i+1}$. With this convention, each b_l corresponds to precisely one b_{ij} . Since we shall exclusively work with double indices, this formula for b_{ij} is actually not important. One only needs to know that the b_{ij} form a basis of $k^{mn+m+n+1}$.

Let $\sigma : \mathbb{P}^m(k) \times \mathbb{P}^n(k) \rightarrow \mathbb{P}^{mn+m+n}$ be the map given by the formula

$$\sigma([X_0, \dots, X_m], [Y_0, \dots, Y_n]) = [(X_i Y_j)_{ij}]$$

where $(\cdot)_{ij}$ means that we put (\cdot) in the coordinate ij (corresponding to b_{ij}). We will write Z_{ij} for a quantity in the coordinate ij and we shall write z_{ij} for the homogenous variables of \mathbb{P}^{mn+m+n} .

Lemma 10.3. The map σ is injective and its image is closed in \mathbb{P}^{mn+m+n} .

Proof. (of Lemma 10.3). For each $[Z_{ij}] \in \sigma(\mathbb{P}^m(k) \times \mathbb{P}^n(k))$ let $i_0 j_0 = i_0([Z_{ij}]) j_0([Z_{ij}])$ be a pair of indices such that $Z_{i_0 j_0} \neq 0$. The map $\tau : \sigma(\mathbb{P}^m(k) \times \mathbb{P}^n(k)) \rightarrow \mathbb{P}^m(k) \times \mathbb{P}^n(k)$ sending $[Z_{ij}]$ to

$$([Z_{0j_0}, Z_{1j_0}, \dots, Z_{mj_0}], [Z_{i_0 0}, Z_{i_0 1}, \dots, Z_{i_0 n}])$$

has the property that $\tau \circ \sigma = \text{Id}_{\mathbb{P}^m(k) \times \mathbb{P}^n(k)}$ (why?). Hence σ is injective. To show that $\sigma(\mathbb{P}^m(k) \times \mathbb{P}^n(k))$ is closed, consider the subvariety of \mathbb{P}^{mn+m+n} described by the homogenous equations $Z_{ij} Z_{rs} - Z_{is} Z_{rj}$ (for all $i, r \in \{0, \dots, m\}$ and $j, s \in \{0, \dots, n\}$). We clearly have

$$X_i X_j X_r X_s = X_i X_s X_r X_j$$

and so $Z((Z_{ij} Z_{rs} - Z_{is} Z_{rj})) \supseteq \sigma(\mathbb{P}^m(k) \times \mathbb{P}^n(k))$. On the other hand, if we let $[Z_{ij}] \in Z((Z_{ij} Z_{rs} - Z_{is} Z_{rj}))$ and $Z_{i_0 j_0} \neq 0$ (say) then

$$\sigma([Z_{0j_0}, Z_{1j_0}, \dots, Z_{mj_0}], [Z_{i_0 0}, Z_{i_0 1}, \dots, Z_{i_0 n}]) = [(Z_{i_0 j_0} Z_{i_0 j})_{ij}] = [(Z_{ij} Z_{i_0 j_0})_{ij}] = [Z_{ij}]$$

so we also have $Z((Z_{ij} Z_{rs} - Z_{is} Z_{rj})) \subseteq \sigma(\mathbb{P}^m(k) \times \mathbb{P}^n(k))$. \square

We record one output of the proof of the last lemma: the image of the map σ is the zero set of the set of quadratic equations $z_{ij}z_{rs} = z_{is}z_{rj}$. The map σ is called the *Segre embedding*. Its image is called the *Segre variety* (which is a closed subvariety of \mathbb{P}^{mn+m+n}).

Proof. (of Theorem 10.2). Endow $\mathbb{P}^m(k) \times \mathbb{P}^n(k)$ with the variety structure inherited from the Segre variety via the Segre embedding. We will show that $\mathbb{P}^m(k) \times \mathbb{P}^n(k)$ has the properties listed in Definition 10.1. We first show that the projections $\pi_1 : \mathbb{P}^m(k) \times \mathbb{P}^n(k) \rightarrow \mathbb{P}^m(k)$ and $\pi_2 : \mathbb{P}^m(k) \times \mathbb{P}^n(k) \rightarrow \mathbb{P}^n(k)$ are morphisms of varieties. For any $i_0 \in \{0, \dots, m\}$ and any $j_0 \in \{0, \dots, n\}$, let $U_{i_0 j_0} \subseteq \mathbb{P}^{mn+m+n}$ be the open subset of the elements $[Z_{ij}]$ such that $Z_{i_0 j_0} \neq 0$ (this is a standard coordinate chart of \mathbb{P}^{mn+m+n}). Let $\pi_{i_0 j_0, 1} : U_{i_0 j_0} \rightarrow \mathbb{P}^m(k)$ be given by the formula

$$\pi_{i_0 j_0, 1}([Z_{ij}]) := [Z_{0j_0}, Z_{1j_0}, \dots, Z_{mj_0}]$$

By question 2.3, this defines a morphism from $U_{i_0 j_0}$ to $\mathbb{P}^m(k)$. Now suppose that

$$\sigma([X_0, \dots, X_m], [Y_0, \dots, Y_n]) = [(X_i Y_j)_{ij}] \in U_{i_0 j_0}$$

In other words, $X_{i_0}, Y_{j_0} \neq 0$. Then

$$\begin{aligned} \pi_{i_0 j_0, 1}(\sigma([X_0, \dots, X_m], [Y_0, \dots, Y_n])) &= \pi_{i_0 j_0, 1}([(X_i Y_j)_{ij}]) = [X_0 Y_{j_0}, X_1 Y_{j_0}, \dots, X_m Y_{j_0}] \\ &= [X_0, X_1, \dots, X_m] = \pi_1([X_0, \dots, X_m], [Y_0, \dots, Y_n]) \end{aligned}$$

Hence π_1 is a morphism on the open subset $\sigma^{-1}(U_{i_0 j_0})$ of $\mathbb{P}^m(k) \times \mathbb{P}^n(k)$. Now if we vary the indices i_0 and j_0 , the open subsets $\sigma^{-1}(U_{i_0 j_0})$ cover all of $\mathbb{P}^m(k) \times \mathbb{P}^n(k)$ and hence π_1 is a morphism (by Definition 4.7 and the fact that functions on open subsets of varieties are regular iff there are regular locally (see Definition 4.6)). Similarly π_2 is a morphism.

Choosing $\pi_{\mathbb{P}^m(k)} := \pi_1$ and $\pi_{\mathbb{P}^n(k)} := \pi_2$, we shall now verify (PROD) in Definition 10.1. So let X be a variety and $a : X \rightarrow \mathbb{P}^m(k)$ and $b : X \rightarrow \mathbb{P}^n(k)$ be morphisms of varieties. We have to show that there is a unique morphism of varieties $c : X \rightarrow \mathbb{P}^m(k) \times \mathbb{P}^n(k)$ such that $\pi_1 \circ c = a$ and $\pi_2 \circ c = b$. Now note that the set $\mathbb{P}^m(k) \times \mathbb{P}^n(k)$ is the cartesian product of the sets $\mathbb{P}^m(k)$ and $\mathbb{P}^n(k)$. Hence, if the morphism c exists, it must be given by the formula $c(x) = (a(x), b(x))$ for all $x \in X$. Hence we only have to verify that c is a morphism of varieties. Since by the definition of a Topskf, a morphism is a morphism iff it is every locally a morphism, we may assume that X is affine and that $a(X) \subseteq U_{\mathbb{P}^m(k), i_0}$ and $b(X) \subseteq U_{\mathbb{P}^n(k), j_0}$ for some indices i_0 and j_0 . Here $U_{\mathbb{P}^m(k), i_0}$ is the i_0 -th standard coordinate chart of $\mathbb{P}^m(k)$ (resp. $U_{\mathbb{P}^n(k), j_0}$ is the j_0 -th standard coordinate chart of $\mathbb{P}^n(k)$). So let us suppose that X is associated with an algebraic set $V \subseteq k^t$. The map a is then the restriction to V of a map $k^t \rightarrow U_{\mathbb{P}^m(k), i_0}$ of the form

$$\bar{v} \in k^t \mapsto [P_0(\bar{v}), \dots, P_{i_0-1}(\bar{v}), 1, P_{i_0+1}(\bar{v}), \dots, P_m(\bar{v})]$$

where the P_h are polynomials in the entries v_1, \dots, v_t of the vector \bar{v} . Similarly, the map b is the restriction to V of a map $k^t \rightarrow U_{\mathbb{P}^n(k), j_0}$ of the form

$$\bar{v} \in k^t \mapsto [Q_0(\bar{v}), \dots, Q_{j_0-1}(\bar{v}), 1, Q_{j_0+1}(\bar{v}), \dots, Q_n(\bar{v})]$$

where the P_l are polynomials in the entries v_1, \dots, v_t of the vector \bar{v} . We now compute

$$\sigma(c(\bar{v})) = [(P_i(\bar{v})Q_j(\bar{v}))_{ij}]$$

and since $P_{i_0}(\bar{v})Q_{j_0}(\bar{v}) = 1$, we see that $\sigma \circ c$ factors through a morphism $V \rightarrow U_{i_0 j_0}$ and in particular is a morphism from V to $\mathbb{P}^{m+n+m+n}$. We conclude from Lemma 5.3 that the morphism c is a morphism of varieties. \square

In the proof above, we have shown that $\mathbb{P}^m(k) \amalg \mathbb{P}^n(k)$ can be realised as the Cartesian product $\mathbb{P}^m(k) \times \mathbb{P}^n(k)$ endowed with a certain variety structure. Furthermore, the projections $\pi_{\mathbb{P}^m(k)}$ and $\pi_{\mathbb{P}^n(k)}$ are then simply the ordinary projections on the two factors. We shall thus often write $\mathbb{P}^m(k) \times \mathbb{P}^n(k)$ instead of $\mathbb{P}^m(k) \amalg \mathbb{P}^n(k)$.

We shall now use Theorem 10.2 to prove that any two quasi-projective varieties have a product.

We start with the

Lemma 10.4. *Let $C_1 \subseteq \mathbb{P}^m(k)$ and $C_2 \subseteq \mathbb{P}^n(k)$ be closed subsets. Let $V_1 \subseteq \mathbb{P}^m(k)$ and $V_2 \subseteq \mathbb{P}^n(k)$ be open subsets. Then the Cartesian product $C_1 \times C_2$ is closed in $\mathbb{P}^m(k) \amalg \mathbb{P}^n(k)$ and the Cartesian product $V_1 \times V_2$ is open in $\mathbb{P}^m(k) \amalg \mathbb{P}^n(k)$.*

Proof. Note that the second statement is a consequence of the first, because the complement of $V_1 \times V_2$ is $(\mathbb{P}^m(k) \setminus V_1) \times \mathbb{P}^n(k) \cup \mathbb{P}^m(k) \times (\mathbb{P}^n(k) \setminus V_2)$, which is closed according to the first statement.

For the proof of the first statement, suppose that C_1 (resp. C_2) is defined by homogenous polynomials $P_1(x_0, \dots, x_m), \dots, P_a(x_0, \dots, x_m)$ (resp. $Q_1(y_0, \dots, y_n), \dots, Q_b(y_0, \dots, y_n)$). Then we have

$$\sigma(C_1 \times C_2) = \sigma(\mathbb{P}^m(k) \times \mathbb{P}^n(k)) \cap \bigcap_{i=0, \dots, m} \bigcap_{j=0, \dots, n} Z \left(P_1(z_{0j}, \dots, z_{mj}), \dots, P_a(z_{0j}, \dots, z_{mj}), Q_1(z_{i0}, \dots, z_{in}), \dots, Q_b(z_{i0}, \dots, z_{in}) \right)$$

and thus $C_1 \times C_2$ is closed in $\mathbb{P}^m(k) \amalg \mathbb{P}^n(k)$. \square

Corollary 10.5. *Let V and W be two quasi-projective varieties. Then the product $V \amalg W$ exists.*

Proof. By assumption, there are integers $m, n \geq 0$ and open subvarieties $O_1 \subseteq \mathbb{P}^m(k)$ and $O_2 \subseteq \mathbb{P}^n(k)$ such that V is isomorphic to a closed subvariety of O_1 and W is isomorphic to a closed subvariety of O_2 . We may thus assume that V is a closed subvariety of O_1 and that W is a closed subvariety of O_2 , where O_1 and O_2 are as above. Let $C_1 \subseteq \mathbb{P}^m(k)$ and $C_2 \subseteq \mathbb{P}^n(k)$ be closed subsets such that $C_1 \cap O_1 = V$ and $C_2 \cap O_2 = W$. We then have $V \times W = (C_1 \times C_2) \cap (O_1 \times O_2)$ and hence $V \times W$ is closed in the open set $O_1 \times O_2$ by Lemma 10.4. We endow the set $V \times W$ with the structure of variety which comes from its inclusion into $O_1 \times O_2$ as a closed subset. We now claim that $V \times W$ is a product of V and W . To see this, let X be a variety and let $a : X \rightarrow V, b : X \rightarrow W$ be two morphisms of varieties. Since the set $V \times W$ is the Cartesian product of V and W , we see as before that if the morphism $a \amalg b$ exists, it must be given by the unique map $a \times b : X \rightarrow V \times W$ sending $x \in X$ to $(a(x), b(x))$. So we only have to verify that this map is a morphism. To see this, let $a' : X \rightarrow O_1$ be the map obtained by composing a with the inclusion $V \rightarrow O_1$ (resp. $b' : X \rightarrow O_2$ be the map obtained by composing b with the inclusion $W \rightarrow O_2$). Let $a'' : X \rightarrow \mathbb{P}^m(k)$ be the map obtained by composing a' with the inclusion $O_1 \rightarrow \mathbb{P}^m(k)$ (resp. $b'' : X \rightarrow \mathbb{P}^n(k)$ be the map obtained by composing b' with the inclusion $O_2 \rightarrow \mathbb{P}^n(k)$). We know that $a'' \times b''$ is a morphism by Theorem 10.2. Next, we know that $a' \times b'$ is a morphism because $(a' \times b')(X) \subseteq O_1 \times O_2$ and because $O_1 \times O_2$ is open in $\mathbb{P}^m(k) \times \mathbb{P}^n(k)$ by Lemma 10.4. Finally, by Lemma 5.3, we know that $a \times b$ is a morphism, as $(a \times b)(X) \subseteq V \times W$, and $V \times W$ is closed in $O_1 \times O_2$ by the above reasoning. So we have verified that $a \times b$ is a morphism. This completes the proof. \square

An outcome of the proof of Corollary 10.5 is the following. Let $m, n \geq 0$ and let $O_1 \subseteq \mathbb{P}^m(k)$ and $O_2 \subseteq \mathbb{P}^n(k)$ be open subvarieties. Suppose that V is a closed subvariety of O_1 and that W is a closed subvariety of O_2 . Then $O_1 \times O_2$ is open in $\mathbb{P}^m(k) \times \mathbb{P}^n(k)$, the Cartesian product $V \times W$ is closed in $O_1 \times O_2$ and the product of V and W is the set $V \times W$ endowed with the variety structure it inherits from $O_1 \times O_2$ as a closed subvariety. The projections π_V and π_W are then the ordinary projections on the two factors.

Again, this justifies simply writing $V \times W$ instead of $V \amalg W$.

Corollary 10.6. *Let V_1, V_2 be quasi-projective varieties. Let $C_1 \subseteq V_1$ and $C_2 \subseteq V_2$ be closed subsets. Let $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$ be open subsets. Then the set theoretic product $C_1 \times C_2$ (resp. the set theoretic product $U_1 \times U_2$) is closed (resp. open) in $V \times W = V \amalg W$. If $C_1 \times C_2$ (resp. $U_1 \times U_2$) is endowed with its structure of closed (resp. open) subvariety of $V_1 \amalg V_2$ and with the natural projection maps on the two factors, then $C_1 \times C_2$ (resp. $U_1 \times U_2$) is a product of C_1 and C_2 (resp. U_1 and U_2).*

Proof. Left to the reader. The proof is completely similar to the proof of Corollary 10.5. \square

The next lemma is needed for the following proposition.

Lemma 10.7. *Let $I \subseteq k[x_1, \dots, x_n]$ (resp. $J \subseteq k[y_1, \dots, y_t]$) be an ideal. Let \bar{I} (resp. \bar{J}) be the ideal generated by I (resp. J) in $k[x_1, \dots, x_n, y_1, \dots, y_t]$. If I and J are radical (resp. prime) then $\bar{I} + \bar{J}$ is radical (resp. prime).*

Proof. Suppose first that I and J are prime. Let $P, Q \in k[x_1, \dots, x_n, y_1, \dots, y_t]$. Suppose that $P \cdot Q \in \bar{I} + \bar{J}$. Suppose for contradiction that $P \notin \bar{I} + \bar{J}$ and $Q \notin \bar{I} + \bar{J}$. Note that we may without restriction of generality replace P (resp. Q) by $P + P_1$, where $P_1 \in \bar{I} + \bar{J}$ (resp. by $Q + Q_1$, where $Q_1 \in \bar{I} + \bar{J}$) without affecting the conclusion. Write

$$P = \sum_i A_i(x_1, \dots, x_n) B_i(y_1, \dots, y_t)$$

and

$$Q = \sum_j C_j(x_1, \dots, x_n) D_j(y_1, \dots, y_t)$$

where both sums are finite. We may assume the elements $B_i \pmod{J}$ (resp. the elements $D_j \pmod{J}$) are linearly independent over k . Indeed, if $B_{i_0} = R + \sum_{i \neq i_0} \lambda_i B_i$ where $R \in J$ and $\lambda_i \in k$, then we have

$$P = \sum_{i \neq i_0} A_i B_i + A_{i_0} (R + \sum_{i \neq i_0} \lambda_i B_i) = \sum_{i \neq i_0} (A_i + \lambda_i A_{i_0}) B_i + A_{i_0} R$$

where $A_{i_0} R \in \bar{J}$. By the preceding remark, we may thus replace P by $\sum_{i \neq i_0} (A_i + \lambda_i A_{i_0}) B_i$ and thus assume that $B_{i_0} = 0$. We can now repeat this process until all the elements $B_i \pmod{J}$ are linearly independent over k . The same construction applies to the elements $D_j \pmod{J}$.

Further, note that we may suppose that for some A_i , we have $A_i \notin I$, otherwise $P \in \bar{I}$ and there is nothing to prove. Similarly, we may suppose that for some C_j , we have $C_j \notin I$. So choose indices i_1, j_1 such that $A_{i_1}, C_{j_1} \notin I$.

Now let $\mathfrak{m}_0 \subseteq k[x_1, \dots, x_n]/I$ be a maximal ideal such that $A_{i_1} \pmod{I} \notin \mathfrak{m}_0$ and $C_{j_1} \pmod{I} \notin \mathfrak{m}_0$.

Such a maximal ideal exists. Indeed, suppose there is no such ideal. Then every maximal ideal of $k[x_1, \dots, n]/I$ contains $(A_{i_1} \pmod{I}) \cdot (C_{j_1} \pmod{I})$ and hence $(A_{i_1} \pmod{I}) \cdot (C_{j_1} \pmod{I})$ is contained in the Jacobson radical of $k[x_1, \dots, n]/I$. But $k[x_1, \dots, n]/I$ is Jacobson since it is finitely generated over k .

Since $k[x_1, \dots, n]/I$ is a domain we thus conclude that $(A_{i_1} \pmod{I}) \cdot (C_{j_1} \pmod{I}) = 0$. This implies that that either $A_{i_1} \pmod{I} = 0$ or $C_{j_1} \pmod{I} = 0$, which is a contradiction.

Now let $\mathfrak{m} \in \text{Spm}(k[x_1, \dots, x_n])$ be the maximal ideal corresponding to \mathfrak{m}_0 . By the weak Nullstellensatz, we have an isomorphism of k -algebras $k[x_1, \dots, x_n]/\mathfrak{m} \simeq k$ and thus we obtain a map of k -algebras $\phi : k[x_1, \dots, x_n] \rightarrow k$ whose kernel is \mathfrak{m} . Let $\Phi : k[x_1, \dots, x_n, y_1, \dots, y_t] \rightarrow k[y_1, \dots, y_t]$ be the induced map. The map Φ sends any polynomial in the y variable on itself and any polynomial H in the x variable on $\phi(H) \in k$. In particular, we have $\Phi(\bar{J}) \subseteq J$ and $\Phi(A_{i_1}) = \phi(A_{i_1}) \neq 0$ and $\Phi(C_{j_1}) = \phi(C_{j_1}) \neq 0$. Note also that the kernel of Φ contains $\bar{\mathfrak{m}} := \mathfrak{m} \cdot k[x_1, \dots, x_n, y_1, \dots, y_t]$ and that $\bar{\mathfrak{m}} \supseteq \bar{I}$. Thus we have $\Phi(PQ) \in J$.

Now we compute

$$0 = \Phi(PQ) \pmod{J} = \left(\sum_i \phi(A_i(x_1, \dots, x_n))(B_i(y_1, \dots, y_t) \pmod{J}) \right) \left(\sum_j \phi(C_j(x_1, \dots, x_n))(D_j(y_1, \dots, y_t) \pmod{J}) \right).$$

Since the elements $B_i(y_1, \dots, y_t) \pmod{J}$ are linearly independent over k and $\phi(A_{i_0}) \neq 0$ we see that

$$\sum_i \phi(A_i(x_1, \dots, x_n))B_i(y_1, \dots, y_t) \notin J.$$

Similarly

$$\sum_j \phi(C_j(x_1, \dots, x_n))D_j(y_1, \dots, y_t) \notin J.$$

This is a contradiction, since J is prime.

The proof that $\bar{I} + \bar{J}$ is radical if I and J are radical is completely similar and is left to the reader. \square

Proposition 10.8. *Let V and W be irreducible quasi-projective varieties. Then $V \times W = V \amalg W$ is also irreducible.*

Proof. We first prove the result in the situation where V and W are affine. So suppose that $V \subseteq k^n$ and $W \subseteq k^t$ are algebraic sets in k^n and k^t , respectively. By question 3.4, we know that the subset $V \times W$ of $k^n \times k^t = k^{n+t}$ is an algebraic subset in k^{n+t} and is a product of V and W . So we have to show that $V \times W$ is irreducible, when endowed with the topology induced from k^{n+t} . Write $k[x_1, \dots, x_n]$ for the coordinate ring of k^n and $k[y_1, \dots, y_t]$ for the coordinate ring of k^t . Let

$$\bar{\mathcal{I}}(V) = \mathcal{I}(V) \cdot k[x_1, \dots, x_n, y_1, \dots, y_t]$$

and

$$\bar{\mathcal{I}}(W) = \mathcal{I}(W) \cdot k[x_1, \dots, x_n, y_1, \dots, y_t].$$

By construction we have $Z(\bar{\mathcal{I}}(V) + \bar{\mathcal{I}}(W)) = V \times W$. Furthermore, by Lemma 10.7 the ideal $\bar{\mathcal{I}}(V) + \bar{\mathcal{I}}(W)$ is prime. Hence $\mathcal{I}(V \times W) = \bar{\mathcal{I}}(V) + \bar{\mathcal{I}}(W)$ and thus $V \times W$ is irreducible.

Now suppose that V and W are quasi-projective. Suppose for contradiction that $V \times W$ is not irreducible. Let T_1, \dots, T_l be the irreducible components of $V \times W$. By assumption, we have $l \geq 2$. Let $(v_1, w_1) \in T_1$ and $(v_2, w_2) \in T_2$. Let U_{v_1} be an open affine neighbourhood of v_1 in V and let U_{w_1} be an open affine neighbourhood of w_1 in W . Define U_{v_2} and U_{w_2} similarly. Then we have $(v_1, w_1) \in U_{v_1} \times U_{w_1}$ and $(v_2, w_2) \in U_{v_2} \times U_{w_2}$. Now from the first part and Lemma 10.6, we know that $U_{v_1} \times U_{w_1}$ and $U_{v_2} \times U_{w_2}$ are open irreducible subsets of $V \times W$. Hence $U_{v_1} \times U_{w_1} \subseteq T_1$ and $U_{v_2} \times U_{w_2} \subseteq T_2$ (why?). Also, we

have $U_{v_1} \times U_{w_1} \cap U_{v_2} \times U_{w_2} = \emptyset$, for otherwise $T_1 \setminus (T_1 \cap T_2)$ is not dense in T_1 . However, since V and W are irreducible there is a point $z_v \in U_{v_1} \cap U_{v_2}$ and a point $z_w \in U_{w_1} \cap U_{w_2}$. We have $(z_v, z_w) \in U_{v_1} \times U_{w_1} \cap U_{v_2} \times U_{w_2}$, which is a contradiction. So $V \times W$ is irreducible. \square

In the next proposition, we shall need Noether's normalisation lemma. This says the following: if R is a non zero finitely generated algebra over a field L , then there exists an injective map of L -algebras $L[x_1, \dots, x_l] \hookrightarrow R$ for some $l \geq 0$, such that R is finite as a $L[x_1, \dots, x_l]$ -module. See Theor. 9.1 in CA.

Proposition 10.9. *Let V and W be irreducible quasi-projective varieties. Then*

$$\dim(V \times W) = \dim(V) + \dim(W).$$

Proof. First suppose that V and W are affine. So we may suppose that V (resp. W) is an algebraic set in k^n (resp. k^t). Let $I := \mathcal{I}(V) \subseteq k[x_1, \dots, x_n]$ (resp. $J := \mathcal{I}(W) \subseteq k[y_1, \dots, y_t]$). We saw in the proof of Proposition 10.8 that the product of V and W can be realised as the closed subset $V \times W$ of k^{n+t} and that the ideal of $V \times W$ in $k[x_1, \dots, x_n, y_1, \dots, y_t]$ is $\bar{I} + \bar{J}$, where

$$\bar{I} = \mathcal{I}(V) \cdot k[x_1, \dots, x_n, y_1, \dots, y_t]$$

and

$$\bar{J} = \mathcal{I}(W) \cdot k[x_1, \dots, x_n, y_1, \dots, y_t].$$

Now use Noether's normalisation lemma to obtain an injective map of k -algebras

$$\phi_V : k[X_1, \dots, X_v] \hookrightarrow \mathcal{C}(V) = k[x_1, \dots, x_n]/I$$

making $\mathcal{C}(V)$ into a finite $k[X_1, \dots, X_v]$ -module. This corresponding map of algebraic sets $\text{Spm}(\phi_V) : V \rightarrow k^n$, is then surjective. This follows from Theorem 8.8 and Cor. 8.10 in CA. Incidentally, Proposition 8.12 in CA also implies that the map $\text{Spm}(\phi_V)$ has finite fibres. Similarly let

$$\phi_W : k[Y_1, \dots, Y_w] \hookrightarrow \mathcal{C}(W) = k[y_1, \dots, y_t]/J.$$

be an injective map of k -algebras making $\mathcal{C}(W)$ into a finite $k[Y_1, \dots, Y_w]$ -module.

Now let

$$\phi_{VW} : k[X_1, \dots, X_v, Y_1, \dots, Y_w] \rightarrow k[x_1, \dots, x_n, y_1, \dots, y_t]/(\bar{I} + \bar{J})$$

be the map of k -algebras sending X_i to $\phi_V(X_i)$ and Y_j to $\phi_W(Y_j)$.

We claim that the map Φ_{VW} is injective. To see this, notice that by unrolling the definitions, we have that $\Phi_{VW} = (\text{Spm}(\phi_V) \times \text{Spm}(\phi_W))^*$, where

$$\text{Spm}(\phi_V) \times \text{Spm}(\phi_W) : V \times W \rightarrow k^n \times k^t = k^{n+t}$$

is the map given by the formula $(\text{Spm}(\phi_V) \times \text{Spm}(\phi_W))(a \times b) = \text{Spm}(\phi_V)(a) \times \text{Spm}(\phi_W)(b)$. In particular, the map $\text{Spm}(\Phi_{VW})$ is surjective. Thus Φ_{VW} is injective by question 1.5 (1).

We also claim that Φ_{VW} makes $k[x_1, \dots, x_n, y_1, \dots, y_t]/(\bar{I} + \bar{J})$ into a finite $k[X_1, \dots, X_v, Y_1, \dots, Y_w]$ -module. To see this, note that by definition, each x_i is integral over $k[X_1, \dots, X_v]$ via ϕ_V (see section 8 in CA). This means that there are polynomials $P_{0,i}(X_1, \dots, X_v), P_{1,i}(X_1, \dots, X_v), \dots, P_{\delta(i)-1,i}(X_1, \dots, X_v)$ such that

$$x_i^{\delta(i)} + \sum_{s=0}^{\delta(i)-1} \phi_V(P_{s,i}) x_i^s = 0$$

in $k[x_1, \dots, x_n]/I$. In particular, we have

$$x_i^{\delta_i} + \sum_{s=0}^{\delta(i)-1} \phi_{VW}(P_{s,i})x_i^s = 0$$

in $k[x_1, \dots, x_n, y_1, \dots, y_t]/(\bar{I} + \bar{J})$. So x_i is also integral over $k[X_1, \dots, X_v, Y_1, \dots, Y_w]$ via ϕ_{VW} . The same reasoning applies to each y_j . Since the x_i and the y_j generate $k[x_1, \dots, x_n, y_1, \dots, y_t]/(\bar{I} + \bar{J})$ as a k -algebra and hence as a $k[X_1, \dots, X_v, Y_1, \dots, Y_w]$ -algebra, we deduce that $k[x_1, \dots, x_n, y_1, \dots, y_t]/(\bar{I} + \bar{J})$ is finitely generated as a $k[X_1, \dots, X_v, Y_1, \dots, Y_w]$ -module (about this see the discussion after Corollary 8.5 in CA).

Appealing to Lemma 11.29 in CA, we deduce that

$$\dim(k[x_1, \dots, x_n, y_1, \dots, y_t]/(\bar{I} + \bar{J})) = \dim(V \times W) = \dim(k[X_1, \dots, X_v, Y_1, \dots, Y_w]) = v + w$$

On the other hand, by Lemma 11.29 again, we have

$$v = \dim(\mathcal{C}(V)) = \dim(V)$$

and

$$w = \dim(\mathcal{C}(W)) = \dim(W).$$

Hence $\dim(V \times W) = \dim(V) + \dim(W)$.

Now we turn to the general case. Let V_1 (resp. W_1) be an open affine subvariety of V (resp. W). The set $V_1 \times W_1$ is open in $V \times W$ and is a product of V_1 and W_1 when considered as an open subvariety of $V \times W$ (by Lemma 10.6). Also, $V_1 \times W_1$ and $V \times W$ are irreducible by Proposition 10.8. Now we apply Proposition 9.4 and the above to obtain

$$\begin{aligned} \dim(V \times W) &= \text{tr. deg. of } \kappa(V \times W) \text{ over } k = \text{tr. deg. of } \kappa(V_1 \times W_1) \text{ over } k \\ &= \dim(V_1 \times W_1) = \dim(V_1) + \dim(W_1) = (\text{tr. deg. of } \kappa(V_1) \text{ over } k) + (\text{tr. deg. of } \kappa(W_1) \text{ over } k) \\ &= (\text{tr. deg. of } \kappa(V) \text{ over } k) + (\text{tr. deg. of } \kappa(W) \text{ over } k) = \dim(V) + \dim(W). \end{aligned}$$

□

We end with the following important remark. *One can show that for any varieties V, W the product $V \amalg W$ exists.* The proof uses different methods. It proceeds roughly as follows. One covers V and W with open affine varieties V_i and W_j , respectively. It can be shown using commutative algebra that the products $V_i \amalg W_j$ exist (see question 3.5). One then constructs the product $V \amalg W$ by glueing the varieties $V_i \amalg W_j$. The advantage of the above construction of the product of quasi-projective varieties is that it bypasses the need for such a glueing procedure, which is combinatorially cumbersome.

11 Intersections in affine and projective space

Proposition 11.1 (affine dimension theorem). *Let $n \geq 0$ and let $V, W \subseteq k^n$ be irreducible algebraic sets. Then every irreducible component of $V \cap W$ has dimension $\geq \dim(V) + \dim(W) - n$.*

Proof. Note that the Cartesian product $V \times W \subseteq k^{2n}$ is closed and is a product of V and W (see question 3.4). Let

$$\Delta := \{(a_1, \dots, a_n, a_1, \dots, a_n) \mid a_1, \dots, a_n \in k\}$$

be the diagonal of k^{2n} . Note that we have

$$\Delta = Z(x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)$$

where we write $\mathcal{C}(k^{2n}) = k[x_1, \dots, x_n, y_1, \dots, y_n]$. We have a k -algebra map

$$\phi : k[x_1, \dots, x_n, y_1, \dots, y_n] / (x_1 - y_1, x_2 - y_2, \dots, x_n - y_n) \rightarrow k[z_1, \dots, z_n]$$

such that $\phi(x_i) = \phi(y_i) = z_i$ for all $i \in \{1, \dots, n\}$. The map ϕ has an inverse given by the map sending z_i to $x_i \pmod{(x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)}$. In particular $\text{Spm}(\phi) : k^n \rightarrow \Delta$ is an isomorphism of algebraic sets. By construction, we have

$$\text{Spm}(\phi)^{-1}(V \times W \cap \Delta) = V \cap W.$$

Thus we only have to prove that every irreducible component of $V \times W \cap \Delta$ has dimension $\geq \dim(V) + \dim(W) - n$. Now by construction we have

$$V \times W \cap \Delta = Z(x_1 - y_1) \cap Z(x_2 - y_2) \cap \dots \cap Z(x_n - y_n) \cap V \times W.$$

Applying Theorem 8.9, we see that for any irreducible component C of $V \times W \cap \Delta$ we have

$$\text{cod}(C, V \times W) \leq n$$

and by Corollary 8.8, Proposition 10.8 and Proposition 10.9, this translates as

$$\dim(V \times W) - \dim(C) = \dim(V) + \dim(W) - \dim(C) \leq n$$

which is equivalent to the conclusion of the proposition. \square

Proposition 11.2 (projective dimension theorem). *Let $n \geq 0$ and let $V, W \subseteq \mathbb{P}^n(k)$ be closed irreducible subvarieties. Then every irreducible component of $V \cap W$ has dimension $\geq \dim(V) + \dim(W) - n$. Furthermore, we have $V \cap W \neq \emptyset$ if $\dim(V) + \dim(W) - n \geq 0$.*

Proof. We first prove the first assertion. Let C be an irreducible component of $V \cap W$. Let U_i be a standard coordinate chart of $\mathbb{P}^n(k)$ such that $C \cap U_i \neq \emptyset$.

We claim that $C \cap U_i$ is an irreducible component of $(V \cap W) \cap U_i$. To see this, note that since $C \cap U_i$ is irreducible (because $C \cap U_i$ is non empty and open in C), there is an irreducible component T of $(V \cap W) \cap U_i$, which contains $C \cap U_i$. Write \bar{T} for the closure of T in $V \cap W$. Then \bar{T} is also irreducible by question 2.5 (1) and hence $\bar{T} \subseteq C$. On the other hand, by construction, we also have $\bar{T} \supseteq C$ so that $C = \bar{T}$. Hence $T = \bar{T} \cap U_i = C \cap U_i$ so that $C \cap U_i$ is an irreducible component of $V \cap W$.

Now by Proposition 11.1, we have

$$\dim(C \cap U_i) \geq \dim(V \cap U_i) + \dim(W \cap U_i) - n$$

and by Proposition 9.4, we have $\dim(V \cap U_i) = \dim(V)$, $\dim(W \cap U_i) = \dim(W)$ and $\dim(C \cap U_i) = \dim(C)$. This proves the first assertion.

For the second assertion, consider again the map $q : k^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n(k)$ such that $q(\bar{v}) = [\bar{v}]$ for all $\bar{v} \in k^{n+1} \setminus \{0\}$. Let $V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_{\dim(V)} = V$ be an ascending sequence of irreducible closed subsets of V , which is of maximal length. The closed subvarieties $q^{-1}(V_i)$ of $k^{n+1} \setminus \{0\}$ are all irreducible by question

3.6 (2). Write $\overline{q^{-1}(V_i)}$ for the closure of $q^{-1}(V_i)$ in k^{n+1} . The closed subsets $\overline{q^{-1}(V_i)}$ of k^{n+1} are then all irreducible by question **3.6** (2) and question **2.5** (1). We thus get an ascending sequence

$$\overline{q^{-1}(V_0)} \subsetneq \overline{q^{-1}(V_1)} \subsetneq \cdots \subsetneq \overline{q^{-1}(V_{\dim(V)})} = \overline{q^{-1}(V)}$$

of closed irreducible subsets of k^{n+1} . Now note that by maximality the variety V_0 is a point (otherwise one could extend the sequence $V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{\dim(V)}$ one step further). We thus have

$$q^{-1}(V_0) = \{\lambda \bar{v}_0 \mid \lambda \in k\} \cap (k^{n+1} \setminus \{0\})$$

for some $\bar{v}_0 \in k^{n+1} \setminus \{0\}$.

We claim that the closure of $\{\lambda \bar{v}_0 \mid \lambda \in k\} \cap (k^{n+1} \setminus \{0\})$ in k^{n+1} is $\{\lambda \bar{v}_0 \mid \lambda \in k\}$. To see this, let $P(x_0, \dots, x_n)$ be some (not assumed to be homogenous) polynomial such that $P(\lambda v_0) = 0$ for all $\lambda \in k \setminus \{0\}$. We then have

$$P(\lambda \bar{v}_0) = \sum_i P_{[i]}(\lambda \bar{v}_0) = \sum_i \lambda^i P_{[i]}(\bar{v}_0) = 0$$

for all $\lambda \neq 0$. Considering $\deg(P) + 1$ different values of λ we obtain a system of linear equations with the unique solution 0 (arising from a Vandermonde matrix). In other words, we have $P_{[i]}(\bar{v}_0) = 0$ for all i and in particular $P_{[0]} = P(0) = 0$. Now note that for all $i \in \{1, \dots, \deg(P)\}$, the polynomial $P_{[i]}$ is a non constant homogenous polynomial and hence vanishes at 0. We conclude that P also vanishes at 0. In particular, the closure of $\{\lambda \bar{v}_0 \mid \lambda \in k\} \cap (k^{n+1} \setminus \{0\})$ in k^n contains 0 and is thus equal to $\{\lambda \bar{v}_0 \mid \lambda \in k\}$.

We thus obtain an ascending sequence of irreducible closed subsets

$$\{0\} \subsetneq \{\lambda \bar{v}_0 \mid \lambda \in k\} = \overline{q^{-1}(V_0)} \subsetneq \overline{q^{-1}(V_1)} \subsetneq \cdots \subsetneq \overline{q^{-1}(V_{\dim(V)})} = \overline{q^{-1}(V)}$$

and we thus see that $\overline{q^{-1}(V)}$ has dimension $\geq \dim(V) + 1$. Similarly, $\overline{q^{-1}(W)}$ is irreducible in k^{n+1} and has dimension $\geq \dim(W) + 1$. We conclude from Proposition **11.1** that every irreducible component of $\overline{q^{-1}(V)} \cap \overline{q^{-1}(W)}$ has dimension larger or equal to

$$\dim(\overline{q^{-1}(V)}) + \dim(\overline{q^{-1}(W)}) - (n + 1) \geq \dim(V) + \dim(W) + 2 - (n + 1) = \dim(V) + \dim(W) - n + 1.$$

Hence, if $\dim(V) + \dim(W) - n \geq 0$ then every irreducible component of $\overline{q^{-1}(V)} \cap \overline{q^{-1}(W)}$ has dimension ≥ 1 . On the other hand, both $\overline{q^{-1}(V)}$ and $\overline{q^{-1}(W)}$ contain the point 0, so $\overline{q^{-1}(V)} \cap \overline{q^{-1}(W)}$ is not empty. We conclude that $\overline{q^{-1}(V)} \cap \overline{q^{-1}(W)}$ contains points other than 0, or in other words that $q^{-1}(V) \cap q^{-1}(W) \neq \emptyset$. This implies that $V \cap W \neq \emptyset$. \square

Corollary 11.3. *Let $n \geq 0$ and let $V \subseteq \mathbb{P}^n(k)$ be a closed irreducible subset. Let H be a closed irreducible subset such that $\text{cod}(H, \mathbb{P}^n(k)) = 1$. If $\dim(V) \geq 1$ then $H \cap V \neq \emptyset$.*

Proof. Left to the reader. \square

12 Separatedness and completeness

Separatedness is an algebraic analogue of the Hausdorff property in topology. Completeness is an algebraic analogue of the notion of compactness in topology.

If X is a quasi-projective variety. Write $\delta_X : X \rightarrow X \amalg X$ for the map $\text{Id}_X \amalg \text{Id}_X$. We shall write $\Delta_X \subseteq X \amalg X$ for the image of δ_X . We call it the *diagonal* in $X \amalg X$.

Definition 12.1. Let X be a quasi-projective variety. We say that X is separated if the diagonal in $X \amalg X$ is closed.

Note that if Δ_X is closed in $X \amalg X$ then δ_X induces an isomorphism between X and Δ_X , where Δ_X is seen as a closed subvariety of $X \amalg X$. Indeed, the map δ_X induces a morphism $X \rightarrow \Delta_X$ by Lemma 5.3 and this map has an inverse, given by the projection on the first factor.

To understand this definition, note that if T is a topological space and $T \times T$ is endowed with the product topology, then T is Hausdorff iff the diagonal $\Delta_T \subseteq T \times T$ is closed. Indeed, let $a, b \in T$ and $a \neq b$. Then $(a, b) \notin \Delta_T$. If Δ_T is closed then there are open subsets $U, V \subseteq T$ such that $U \times V \cap \Delta_T = \emptyset$ and such that $(a, b) \in U \times V$. In particular, $a \in U$, $b \in V$ and $U \cap V = \emptyset$. So a and b have disjoint neighbourhoods. On the other hand, if a and b have disjoint neighbourhoods U and V , respectively, then $U \times V \cap \Delta_T = \emptyset$ and $(a, b) \in U \times V$. So $(T \times T) \setminus \Delta_T$ is open, ie Δ_T is closed.

Note that in the definition of separatedness given above, the variety $X \amalg X$ does not in general carry the product topology induced by its natural identification with the Cartesian product $X \times X$ so that the algebraic definition differs significantly from the topological one (but serves the same purpose from a structural point of view).

Remark 12.2. As remarked at the end of section 10, any two varieties have a product. It is therefore possible to extend to definition of separatedness to all varieties. We shall see below that all quasi-projective are separated, but it is possible to give examples of non quasi-projective varieties, which are not separated.

Lemma 12.3. Let X be a separated quasi-projective variety. Let V be a closed (resp. open) subvariety of X . Then V is separated.

Proof. Suppose that V is a closed subvariety of X . The Cartesian product $V \times V \subseteq X \times X$ is closed and represents the product of V with itself as a closed subvariety of $X \times X$ (by Corollary 10.6). On the other hand, we have $\Delta_V = \Delta_X \cap V \times V$ so Δ_V is closed in $V \times V$ since Δ_X is closed. In other words, V is separated. The proof in the situation where V is an open subvariety of X is similar. \square

Lemma 12.4. Affine varieties are separated.

Proof. We first prove that the varieties k^t are separated for $t \geq 0$. Recall that by question 3.5, $k^t \amalg k^t \simeq k^{2t}$. Write $\mathcal{C}(k^{2t}) = k[x_1, \dots, x_t, y_1, \dots, y_t]$. Now note that

$$\Delta_{k^t} = Z(x_1 - y_1, x_2 - y_2, \dots, x_t - y_t).$$

Hence Δ_{k^t} is closed. The general case now follows from Lemma 12.3. \square

Lemma 12.5. Let X be a quasi-projective variety. Suppose that for any two points $a, b \in X$ there exists an open affine subvariety $U \subseteq X$ such that $a, b \in U$. Then X is separated.

Proof. Let $(a, b) \in X \times X \setminus \Delta_X$ (ie $a, b \in X$ and $a \neq b$). Let $U_{a,b}$ be an open affine subvariety of X such that $a, b \in U_{a,b}$. Then $(a, b) \in U_{a,b} \times U_{a,b}$. Furthermore, $\Delta_{U_{a,b}} = \Delta_X \cap (U_{a,b} \times U_{a,b})$ and the Cartesian product $U_{a,b} \times U_{a,b}$ is a product of $U_{a,b}$ with itself as an open subvariety of $X \times X$ (by Corollary 10.6). Hence $\Delta_{U_{a,b}}$ is closed as a subset of $U_{a,b} \times U_{a,b}$ by Lemma 12.4. In particular, (a, b) is contained in an open subset of $X \times X$, which is disjoint from Δ_X . Since $(a, b) \in X \times X \setminus \Delta_X$ was arbitrary, we conclude that $X \times X \setminus \Delta_X$ is open, ie Δ_X is closed. \square

Proposition 12.6. *Any quasi-projective variety is separated.*

Proof. Suppose first that $X = \mathbb{P}^n(k)$ for some $n \geq 0$. Then X is separated by Lemma 12.5 and question 2.8. The general case follows from this and Lemma 12.3. \square

Proposition-Definition 12.7 (The graph of a morphism). *Let X and Y be quasi-projective varieties. Let $\gamma : X \rightarrow Y$ be a morphism. Let*

$$\Gamma_\gamma := \{(x, \gamma(x)) \mid x \in X\} \subseteq X \times Y$$

be the graph of γ . Then Γ_γ is closed in $X \times Y$.

Proof. Let $\tilde{\gamma} : X \times Y \rightarrow Y \times Y$ be the morphism such that $\tilde{\gamma}(x, y) := (\gamma(x), y)$ for all $(x, y) \in X \times Y$ (this is a morphism by the definition of products). We have

$$\Gamma_\gamma = \tilde{\gamma}^{-1}(\Delta_Y)$$

and so Γ_γ is closed since Δ_Y is closed by Proposition 12.6. \square

Definition 12.8. *Let X be a quasi-projective variety. We say that X is complete if for any quasi-projective variety B and any closed subset $C \subseteq X \times B$, the set $\pi_B(C)$ is closed.*

Here $\pi_B : X \times B \rightarrow B$ is the projection on the second factor.

Lemma 12.9. *Let X be a complete quasi-projective variety. Then any closed subvariety of X is also complete.*

Proof. Left to the reader. Unroll the definitions and use Corollary 10.6. \square

Theorem 12.10. *Projective varieties are complete.*

Proof. By Lemma 12.9, we only need to prove this for $X = \mathbb{P}^n(k)$.

So let B be a quasi-projective variety and let $\{B_i\}$ be an open affine covering of B . Let $C \subseteq \mathbb{P}^n(k) \times B$ be a closed subset. By Corollary 10.6, the Cartesian product $\mathbb{P}^n(k) \times B_i$ is open in $\mathbb{P}^n(k) \times B$ and if $\mathbb{P}^n(k) \times B_i$ is viewed as an open subvariety of $\mathbb{P}^n(k) \times B$ it is a product of $\mathbb{P}^n(k)$ and B_i . Now $\pi_B(C)$ is closed iff $\pi_B(C) \cap B_i$ is closed in B_i for all i and we have $\pi_B(C) \cap B_i = \pi_{B_i}(C \cap (\mathbb{P}^n(k) \times B_i))$. Hence we may suppose from the start that B is affine. In that case B is a closed subvariety of k^t for some $t \geq 0$. By Corollary 10.6 again, the subset $\mathbb{P}^n(k) \times B \subseteq \mathbb{P}^n(k) \times k^t$ is closed and is a product of $\mathbb{P}^n(k)$ and B if $\mathbb{P}^n(k) \times B$ is viewed as a closed subvariety of $\mathbb{P}^n(k) \times k^t$. Furthermore, $\pi_B(C)$ is closed in B iff it is closed in k^t . So we might suppose that $B = k^t$.

Now let $i \in \{0, \dots, n\}$ and let $U_i \subseteq \mathbb{P}^n(k)$ be the well-known coordinate chart. Recall that there is an isomorphism $u_i : k^n \rightarrow U_i$ given by the formula

$$u_i(\langle X_0, \dots, \check{X}_i, \dots, X_n \rangle) = [X_0, \dots, X_{i-1}, 1, X_{i+1}, \dots, X_n] \in \mathbb{P}^n(k).$$

By question 3.5, the variety $U_i \times k^t$ is affine and we have

$$\mathcal{C}(k^n \times k^t) = k[x_0, \dots, \check{x}_i, \dots, x_n, y_1, \dots, y_t]$$

where the x_j are the coordinates of k^n and the y_j are the coordinates of k^t .

Write

$$\phi_i : k[x_0, \dots, x_n, y_1, \dots, y_t] \rightarrow k[x_0, \dots, \check{x}_i, \dots, x_n, y_1, \dots, y_t]$$

for the map of k -algebras such that $\phi(x_j) = x_j$ for all $j \neq i$, $\phi(x_i) = 1$ and $\phi(y_j) = y_j$ for all j .

Let $I_i := \mathcal{I}((u_i \times \text{Id}_{k^t})^{-1}(C)) \subseteq k[x_0, \dots, \check{x}_i, \dots, x_n, y_1, \dots, y_t]$.

Note the following. Suppose that $H \in k[x_0, \dots, x_n, y_1, \dots, y_t]$ and that H is homogenous in the x -variables. Then $H \in \phi_i^{-1}(I_i)$ iff $H(X_0, \dots, X_n, Y_1, \dots, Y_t) = 0$ for all $[X_0, \dots, X_n] \times \langle Y_1, \dots, Y_t \rangle \in C \cap (U_i \times k^t)$. This follows directly from the definitions.

In particular a polynomial $H \in k[x_0, \dots, x_n, y_1, \dots, y_t]$ which is homogenous in the x -variables lies in $\cap_i \phi_i^{-1}(I_i)$ iff $H(X_0, \dots, X_n, Y_1, \dots, Y_t) = 0$ for all $[X_0, \dots, X_n] \times \langle Y_1, \dots, Y_t \rangle \in C$.

For any $N \geq 0$, write $S_N \subseteq k[x_0, \dots, x_n, y_1, \dots, y_t]$ for the polynomials, which are homogenous in the x -variable and which are of degree N in the x -variable. This gives $k[x_0, \dots, x_n, y_1, \dots, y_t]$ the structure of a graded ring with $S_0 = k[y_1, \dots, y_t]$. In particular S_N is a $S_0 = k[y_1, \dots, y_t]$ -submodule of $k[x_0, \dots, x_n, y_1, \dots, y_t]$. We also write $A_N = S_N \cap (\cap_i \phi_i^{-1}(I_i))$. It follows from the definitions that $\oplus_{l \geq 0} A_l$ is then a graded ideal in (= graded sub- $k[x_0, \dots, x_n, y_1, \dots, y_t]$ -module of) $k[x_0, \dots, x_n, y_1, \dots, y_t]$. In particular, A_N is a $S_0 = k[y_1, \dots, y_t]$ -submodule of S_N .

Now let $\bar{w} = \langle W_1, \dots, W_t \rangle \in k^t$ and suppose that $\bar{w} \notin \pi_B(C)$. Let $\bar{m} = (y_1 - W_1, \dots, y_t - W_t) \subseteq k[y_1, \dots, y_t]$ be the maximal ideal associated with \bar{w} . Let $i \in \{0, \dots, n\}$. By assumption, we have

$$I_i + \mathfrak{m} \cdot k[x_0, \dots, \check{x}_i, \dots, x_n, y_1, \dots, y_t] = k[x_0, \dots, \check{x}_i, \dots, x_n, y_1, \dots, y_t]$$

(since the zero set of $\mathfrak{m} \cdot k[x_0, \dots, \check{x}_i, \dots, x_n, y_1, \dots, y_t]$ is $k^n \times \{w\}$ and by assumption $u_i^{-1}(C) = Z(I_i)$, which does not meet $k^n \times \{w\}$). In particular, there is a polynomial $P_i \in I_i$ and polynomials $M_{il} \in \mathfrak{m}$ and $G_{il} \in k[x_0, \dots, \check{x}_i, \dots, x_n, y_1, \dots, y_t]$ such that

$$1 = P_i + \sum_l M_{il} \cdot G_{il}$$

Hence, for any $N \geq 0$ we have

$$\begin{aligned} x_i^N &= x_i^{N - \deg_x(P_i)} (x_i^{\deg_x(P_i)} P_i(x_0/x_i, \dots, \check{x}_i, \dots, x_n/x_i, y_1, \dots, y_t)) \\ &+ \sum_l M_{il}(y_1, \dots, y_t) [x_i^{N - \deg_x(G_{il})} (x_i^{\deg_x(G_{il})} G_{il}(x_0/x_i, \dots, \check{x}_i, \dots, x_n/x_i, y_1, \dots, y_t))] \end{aligned}$$

Now note that the polynomial $x_i^{\deg_x(P_i)} P_i(x_0/x_i, \dots, \check{x}_i, \dots, x_n/x_i, y_1, \dots, y_t)$ is by construction homogenous in the x -variable and of x -degree $\deg_x(P_i)$; the same polynomial also lies in $\phi_i^{-1}(I_i)$ since

$$\phi_i(x_i^{\deg_x(P_i)} P_i(x_0/x_i, \dots, \check{x}_i, \dots, x_n/x_i, y_1, \dots, y_t)) = P_i.$$

Furthermore, by definition, the polynomial

$$x_i^{\deg_x(P_i)+1} P_i(x_0/x_i, \dots, \check{x}_i, \dots, x_n/x_i, y_1, \dots, y_t)$$

vanishes when evaluated on $\langle X_0, \dots, X_n, Y_1, \dots, Y_t \rangle$ whenever $[X_0, \dots, X_n] \times \langle Y_1, \dots, Y_t \rangle \in C$ (remember that x_i vanishes on $(\mathbb{P}^n(k) \setminus U_i) \times k^t$). Hence

$$x_i^{\deg_x(P_i)+1} P_i(x_0/x_i, \dots, \check{x}_i, \dots, x_n/x_i, y_1, \dots, y_t) \in A_{\deg_x(P_i)+1}$$

by the above discussion.

Similarly, the polynomial $x_i^{\deg_x(G_{il})}G_{il}(x_0/x_i, \dots, \tilde{x}_i, \dots, x_n/x_i, y_1, \dots, y_t)$ is also homogenous in the x -variable and is of x -degree $\deg_x(G_{il})$.

So if N is larger than $\deg_x(P_i) + 1$ and also larger than $\deg_x(G_{il})$ for all l , we have an equality

$$x_i^N = T_i + \sum_l M_{il}H_{il}$$

where $T_i \in A_N$ and $H_{il} \in S_N$. Since there is only a finite number of indices i , there is thus a natural number N_0 such that

$$x_i^N \in A_N + \mathfrak{m}S_N$$

for all $N \geq N_0$ and all $i \in \{0, \dots, n\}$.

Now note that if N_1 is sufficiently large, any monomial of degree $\geq N_1$ in the x_i becomes divisible by $x_j^{N_0}$ for some x_j . So if N_1 is sufficiently large then for all $N \geq N_1$ we have

$$S_N \subseteq (\oplus_{s \geq 0} S_s)(A_{N_0} + \mathfrak{m}S_{N_0})$$

Since $\oplus_{s \geq 0} S_s$ is a graded ideal, we then have

$$S_N \subseteq S_{N-N_0}(A_{N_0} + \mathfrak{m}S_{N_0}) \subseteq A_N + \mathfrak{m}S_N.$$

In particular, we have $(S_N/A_N) = \mathfrak{m}(S_N/A_N)$ where the quotient S_N/A_N is quotient of $k[y_1, \dots, y_t]$ -modules.

We conclude from the generalised form of Nakayama's lemma (see Q4 in Sheet 1 of CA) that there is $Q \in 1 + \mathfrak{m}$ such that $Q \cdot (S_N/A_N) = 0$. In particular $Q \cdot x_i^N \in A_N$ for all $i \in \{0, \dots, n\}$. In other words, for any i we have

$$X_i^N Q(X_0, \dots, X_n, Y_1, \dots, Y_t) = X_i^N Q(Y_1, \dots, Y_t) = 0$$

for all $[X_0, \dots, X_n] \times \langle Y_1, \dots, Y_t \rangle \in C$ (see the discussion above). In particular, whenever $Q(Y_1, \dots, Y_t) \neq 0$ the set $C \cap (U_i \times \{\langle Y_1, \dots, Y_t \rangle\})$ is empty. Since this holds for all $i \in \{0, \dots, n\}$, the set $C \cap (\mathbb{P}^n(k) \times \{\langle Y_1, \dots, Y_t \rangle\})$ is empty whenever $Q(Y_1, \dots, Y_t) \neq 0$. Said differently, if $\langle Y_1, \dots, Y_t \rangle \in k^t \setminus Z(Q)$ then $\langle Y_1, \dots, Y_t \rangle \notin \pi_B(C)$. Finally, we have $Q(\bar{w}) \neq 0$ since $Q \in 1 + \mathfrak{m}$, so $k^t \setminus Z(Q)$ is a neighbourhood of \bar{w} . Since $\bar{w} \in k^t \setminus \pi_B(C)$ was arbitrary, we conclude that $k^t \setminus \pi_B(C)$ is open, ie $\pi_B(C)$ is closed. \square

Remark 12.11. Let $n, t \geq 0$ and consider the variety $\mathbb{P}^n(k) \times k^t$.

Suppose given polynomials $H_1, \dots, H_l \in k[x_0, \dots, x_n, y_1, \dots, y_t]$. Suppose that the H_j are homogenous in the variable x . Let

$$C := \{[X_0, \dots, X_n] \times \langle Y_1, \dots, Y_t \rangle \in \mathbb{P}^n(k) \times k^t \mid \forall j \in \{1, \dots, l\} : H_j(X_0, \dots, X_n, Y_1, \dots, Y_t) = 0\}.$$

It can be shown that C is a closed subset of $\mathbb{P}^n(k) \times k^t$ (prove this!). By Theorem 12.10, the set

$$\pi_{k^t}(C) := \{\langle Y_1, \dots, Y_t \rangle \in k^t \mid \exists [X_0, \dots, X_n] \in \mathbb{P}^n(k) : \forall j \in \{1, \dots, l\} : H_j(X_0, \dots, X_n, Y_1, \dots, Y_t) = 0\}$$

is then closed. In other words, there are polynomials $Q_1, \dots, Q_a \in k[y_1, \dots, y_t]$ such that

$$Q_1(Y_1, \dots, Y_t) = Q_2(Y_1, \dots, Y_t) = \dots = Q_a(Y_1, \dots, Y_t) = 0$$

iff there is $X_0, \dots, X_n \in k^{n+1} \setminus \{0\}$ such that

$$H_1(X_0, \dots, X_n, Y_1, \dots, Y_t) = H_2(X_0, \dots, X_n, Y_1, \dots, Y_t) = \dots = H_l(X_0, \dots, X_n, Y_1, \dots, Y_t) = 0.$$

Writing $R = k[y_1, \dots, y_t]$, one can rephrase this result in terms of commutative algebra:

Theorem. Let $H_1, \dots, H_l \in R[x_0, \dots, x_n]$ be homogenous polynomials. Then there are elements $Q_1, \dots, Q_a \in R$ with the following property. If $\phi : R \rightarrow L$ is a ring homomorphism from R to an algebraically closed field L , then the equations

$$\phi(H_1)(X_0, \dots, X_n) = \phi(H_2)(X_0, \dots, X_n) = \dots = \phi(H_l)(X_0, \dots, X_n) = 0$$

have a non vanishing solution in L iff $Q_1, \dots, Q_a \in \ker(\phi)$. The radical of the ideal (Q_1, \dots, Q_a) depends only on the polynomials H_1, \dots, H_l .

This theorem is called the *main theorem of elimination theory*. If $t = 1$, the unique monic generator of $\mathfrak{r}((Q_1, \dots, Q_a))$ is called the *resultant* of the polynomials H_1, \dots, H_l .

One can show that the theorem holds for any noetherian commutative ring R . Theorem 12.10 proves this when $R = k[y_1, \dots, y_t]$ (and more generally if R is any finitely generated reduced k -algebra). The scheme-theoretic generalisation of Theorem 12.10 implies the general form of the main theorem of elimination theory.

Corollary 12.12 (of Theorem 12.10). *Let X, Y be quasi-projective varieties and suppose that X is complete. Let $\phi : X \rightarrow Y$ be a morphism. Then $\phi(X)$ is closed.*

Proof. The image of $\phi(X)$ is the projection of the graph $\Gamma_\phi \subseteq X \times Y$ by the projection to Y . Hence Proposition-Definition 12.7 implies the result. \square

Proposition 12.13. *A complete quasi-projective variety is projective.*

Proof. Let X be a quasi-projective complete variety. By definition, we may suppose that there is an open subvariety U of $\mathbb{P}^n(k)$ such that X is a closed subvariety of U . By Corollary 12.12, X is closed in $\mathbb{P}^n(k)$. Hence, from the definition of subvarieties, X is a closed subvariety of $\mathbb{P}^n(k)$. Hence X is projective. \square

Lemma 12.14. *Let X be an affine complete variety. Then X consists of a finite number of points.*

Proof. By question 3.3, $\mathcal{C}(X)$ is a finite dimensional k vector space. In particular, $\mathcal{C}(X)$ is finite over k . We deduce from Prop. 8.12 in CA that $\mathcal{C}(X)$ has only finitely maximal ideals. Hence X has only finitely many points by the discussion before Lemma 2.8. Alternatively, we can reason as follows. We see from Lemma 12.9 that all the irreducible components of X are complete. Let V be an irreducible component of X . Then $\mathcal{C}(V)$ is an integral domain by Lemma 2.5 and it is finite over k by question 3.3. Hence $\mathcal{C}(V)$ is a field by the (elementary) Lemma 8.9 in CA. Hence V is a point. Since X has only finitely many irreducible components, we conclude again that X only has a finite number of points. \square

13 Smoothness

A variety is smooth if it has "no kinks". For a curve C in the plane given by one equation $f(x, y) = 0$, this can be analysed by looking at its gradient $\text{grad}(f) = \langle \frac{\partial}{\partial x} f, \frac{\partial}{\partial y} f \rangle$. The curve will be smooth if $\text{grad}(f)$ does not vanish for any point of C . The general definition has a similar flavour.

Definition 13.1. Let $V \subseteq k^n$ be an algebraic set. Suppose that $\mathcal{I}(V) = (P_1, \dots, P_t) \subseteq k[x_1, \dots, x_n]$. Let $\bar{v} \in V$. We say that V is nonsingular at \bar{v} if the matrix $[(\frac{\partial}{\partial x_j} P_i)(\bar{v})]_{ij}$ has rank $n - \text{cod}(\{\bar{v}\}, V)$. If V is irreducible then $\text{cod}(\{\bar{v}\}, V) = \dim(V)$ so that in that case V is nonsingular at \bar{v} iff the matrix $[(\frac{\partial}{\partial x_j} P_i)(\bar{v})]_{ij}$ has rank $n - \dim(V)$.

Note that when C is a curve in the plane, we recover the definition given above. To make sense of this definition, we need to show that it does not depend on the polynomials P_i . In fact, we will show that the definition only depends on the coordinate ring $\mathcal{C}(V)$.

On the way to this result, we first make another definition.

Definition 13.2. Let R be a noetherian local ring with maximal ideal \mathfrak{m} and residue field $k_0 := R/\mathfrak{m}$. We say that R is a regular local ring if $\dim(R) = \dim_{k_0} \mathfrak{m}/\mathfrak{m}^2$.

A few comments are in order. Note that with the notation of the last definition, we have $\dim(R) = \text{ht}(\mathfrak{m})$ (this follows from the definition of dimension and the fact that R is local). On the other hand, by Nakayama's lemma (see Cor. 3.6 in CA), the ideal \mathfrak{m} can be generated by $\dim_{k_0} \mathfrak{m}/\mathfrak{m}^2$ elements. Hence by a corollary of Krull's theorem (see CA Cor. 11.15), we have

$$\dim(R) = \text{ht}(\mathfrak{m}) \leq \dim_{k_0} \mathfrak{m}/\mathfrak{m}^2.$$

The local ring R is regular iff this last inequality is an equality.

Proposition 13.3. Let $V \subseteq k^n$ be an algebraic set. Then V is nonsingular at $\bar{v} \in V$ (for some and hence any choice of generators P_i of $\mathcal{I}(V)$) iff the local ring $\mathcal{O}_{V,\bar{v}} \simeq \mathcal{C}(V)_{\mathcal{I}(\{\bar{v}\})}$ is regular.

For the proof, we shall need the

Lemma 13.4. Let R be a ring and let $\mathfrak{m} \subseteq R$ be a maximal ideal. Let $\phi : R \rightarrow R_{\mathfrak{m}}$ be the natural map of rings. Let $n \geq 0$. Then the unique maximal ideal $\underline{\mathfrak{m}}$ of $R_{\mathfrak{m}}$ is the ideal of $R_{\mathfrak{m}}$ generated by $\phi(\mathfrak{m})$. Furthermore, we have $\phi^{-1}(\underline{\mathfrak{m}}^n) = \mathfrak{m}^n$ and the map of R -modules induced by ϕ

$$\mathfrak{m}^n/\mathfrak{m}^{n+1} \rightarrow \underline{\mathfrak{m}}^n/\underline{\mathfrak{m}}^{n+1}$$

is an isomorphism.

Note that the lemma is obviously false if \mathfrak{m} is not maximal (look eg at the case $n = 0$).

Proof. (of Lemma 13.4) The first assertion is contained in Lemma 5.6 in CA (standard properties of localisations - you can also prove this directly). This also implies that $\underline{\mathfrak{m}}^n$ is the ideal generated by $\phi(\mathfrak{m}^n)$. In particular, any element of $\underline{\mathfrak{m}}^n$ can be written in the form r/u , where $r \in \mathfrak{m}^n$ and $u \in R \setminus \mathfrak{m}$ (prove this directly or refer to Lemma 5.6 in CA).

We now prove that $\phi^{-1}(\underline{\mathfrak{m}}^n) = \mathfrak{m}^n$. To see this, note that by the definition of localisation, the ideal $\phi^{-1}(\underline{\mathfrak{m}}^n)$ is the set of elements $r \in R$, such that for some $t \in \mathfrak{m}^n$ and for some $u, v \in R \setminus \mathfrak{m}$, we have $v(ur - t) = 0$ (use the definition of localisation). Hence if $r \in R$ and there is $u \in R \setminus \mathfrak{m}$ such that $ur \in \mathfrak{m}^n$ then $r \in \phi^{-1}(\underline{\mathfrak{m}}^n)$. On the other hand, if $r \in R$ and for some $t \in \mathfrak{m}^n$ and $u, v \in R \setminus \mathfrak{m}$ we have $vur = vt$, then $(vu)r \in \mathfrak{m}^n$. Thus

$$\phi^{-1}(\underline{\mathfrak{m}}^n) = \{r \in R \mid \exists u \in R \setminus \mathfrak{m} : ur \in \mathfrak{m}^n\}.$$

Now suppose that $r \in R$ and that $u \in R \setminus \mathfrak{m}$ is such that $ur \in \mathfrak{m}^n$. Recall that \mathfrak{m}^n is \mathfrak{m} -primary (see Lemma 6.4 in CA). Since $u \notin \mathfrak{m}^n$, we deduce that either $r \in \mathfrak{m}^n$ or both $u \pmod{\mathfrak{m}^n}$ and $r \pmod{\mathfrak{m}^n}$ are nilpotent in R/\mathfrak{m}^n . The second possibility cannot occur because all the powers of u lie in $R \setminus \mathfrak{m}$ (since \mathfrak{m} is prime). Hence we must have $r \in \mathfrak{m}^n$. In other words we have $\phi^{-1}(\underline{\mathfrak{m}}^n) = \mathfrak{m}^n$.

We now show that the natural map $\mathfrak{m}^n/\mathfrak{m}^{n+1} \rightarrow \underline{\mathfrak{m}}^n/\underline{\mathfrak{m}}^{n+1}$ is an isomorphism. Since $\phi^{-1}(\underline{\mathfrak{m}}^{n+1}) = \mathfrak{m}^{n+1}$, we see that the map is injective. To prove surjectivity, let $r/u \in \underline{\mathfrak{m}}^n$, where $r \in \mathfrak{m}^n$ and $u \in R \setminus \mathfrak{m}$. Let $v \in R \setminus \mathfrak{m}$ be such that $uv = 1 \pmod{\mathfrak{m}}$ (such a v exists because R/\mathfrak{m} is a field). Then there is an $a \in \mathfrak{m}$ such that

$$rv/1 = ruv/u = (r + ra)/u = r/u + (a/1)(r/u) = r/u \pmod{\underline{\mathfrak{m}}^{n+1}}$$

and thus $r/u \pmod{\underline{\mathfrak{m}}^{n+1}}$ is in the image of \mathfrak{m}^n in $\underline{\mathfrak{m}}^n/\underline{\mathfrak{m}}^{n+1}$. \square

Proof. (of Proposition 13.3) Let $\bar{v} = \langle v_1, \dots, v_n \rangle \in V \subseteq k^n$. Suppose that $\mathcal{I}(V) = (P_1, \dots, P_t)$. Let

$$\mathfrak{m} := \mathcal{I}(\{\bar{v}\}) = (x_1 - v_1, \dots, x_n - v_n)$$

be the maximal ideal of $k[x_1, \dots, x_n]$ associated with \bar{v} . Let $\mathfrak{n} = \mathfrak{m} \pmod{\mathcal{I}(V)} \subseteq \mathcal{C}(V)$ be the maximal ideal of $\mathcal{C}(V)$ associated with \bar{v} . Define a map of k -vector spaces $\phi : \mathfrak{m} \rightarrow k^n$ by the formula

$$\phi(Q) = \langle (\frac{\partial}{\partial x_1} Q)(\bar{v}), \dots, (\frac{\partial}{\partial x_n} Q)(\bar{v}) \rangle.$$

Since \mathfrak{m}^2 is generated by the elements $(x_i - v_i)(x_j - v_j)$, we see that $\phi(\mathfrak{m}^2) = 0$ (apply the Leibniz rule). We thus obtain a k -linear map $\mathfrak{m}/\mathfrak{m}^2 \rightarrow k^n$. This map is surjective because $\phi(x_i - v_i)$ is the i -th element of the standard basis of k^n . On the other hand, $\mathfrak{m}/\mathfrak{m}^2$ is generated by n elements as a $R/\mathfrak{m} = k$ -vector space and so is of dimension $\leq n$. Hence the map $\mathfrak{m}/\mathfrak{m}^2 \rightarrow k^n$ is an isomorphism of k -vector spaces. Now the image $(\mathcal{I}(V) + \mathfrak{m}^2)/\mathfrak{m}^2$ of $\mathcal{I}(V) \subseteq \mathfrak{m}$ in $\mathfrak{m}/\mathfrak{m}^2$ is generated by $P_1 \pmod{\mathfrak{m}^2}, \dots, P_t \pmod{\mathfrak{m}^2}$ as a $R/\mathfrak{m} = k$ -vector space. Hence

$$\dim_k((\mathcal{I}(V) + \mathfrak{m}^2)/\mathfrak{m}^2) = \dim_k(\phi(\mathcal{I}(V))) = \text{rk} \begin{pmatrix} (\frac{\partial}{\partial x_1} P_1)(\bar{v}) & \dots & (\frac{\partial}{\partial x_n} P_1)(\bar{v}) \\ (\frac{\partial}{\partial x_1} P_2)(\bar{v}) & \dots & (\frac{\partial}{\partial x_n} P_2)(\bar{v}) \\ \vdots & \vdots & \vdots \\ (\frac{\partial}{\partial x_1} P_t)(\bar{v}) & \dots & (\frac{\partial}{\partial x_n} P_t)(\bar{v}) \end{pmatrix} =: \text{rk}[(\frac{\partial}{\partial x_j} P_i)(\bar{v})]_{ij}.$$

On the other hand, we have by construction a complex of $R/\mathfrak{m} = k$ -vector spaces

$$0 \rightarrow (\mathcal{I}(V) + \mathfrak{m}^2)/\mathfrak{m}^2 \rightarrow \mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{n}/\mathfrak{n}^2 \rightarrow 0$$

We claim that this complex is exact. The second arrow from the left is injective by definition and likewise it follows from the definitions that the third arrow from the left is surjective. So we only have to show that the complex is exact at $\mathfrak{m}/\mathfrak{m}^2$. To see this, suppose that $P \in \mathfrak{m}$ and that $P \pmod{\mathcal{I}(V)} \in \mathfrak{n}^2$. Since $\mathfrak{n}^2 = (\mathfrak{m}^2 + \mathcal{I}(V))/\mathcal{I}(V)$, there is $Q \in \mathfrak{m}^2 + \mathcal{I}(V)$ such that $P \pmod{\mathcal{I}(V)} = Q \pmod{\mathcal{I}(V)}$. We then have $(P - Q) \pmod{\mathcal{I}(V)} = 0$, or in other words $P - Q \in \mathcal{I}(V)$. Hence P is the sum of an element of $\mathcal{I}(V)$ and an element of \mathfrak{m}^2 . This shows that the complex is exact at $\mathfrak{m}/\mathfrak{m}^2$ and is thus an exact complex.

We conclude that

$$\text{rk}[(\frac{\partial}{\partial x_j} P_i)(\bar{v})]_{ij} + \dim_k(\mathfrak{n}/\mathfrak{n}^2) = n. \quad (2)$$

Now we have $\text{cod}(V, \{\bar{v}\}) = \text{ht}(\mathfrak{n}) = \dim(\mathcal{C}(V)_{\mathcal{I}(\{\bar{v}\})})$ (follows from the definition of dimension - see Lemma 11.2 in CA). Using Lemma 13.4, we see that the local ring $\mathcal{C}(V)_{\mathcal{I}(\{\bar{v}\})}$ is regular iff

$$\text{rk}\left[\left(\frac{\partial}{\partial x_j} P_i\right)(\bar{v})\right]_{ij} = n - \text{cod}(V, \{\bar{v}\}).$$

This proves the proposition. \square

Remark 13.5. (1) Keep the notation of the proof of Proposition 13.3. From the remark preceding the proposition, we have $\dim_k(\mathfrak{n}/\mathfrak{n}^2) \geq \text{cod}(V, \{\bar{v}\})$ and so we always have

$$\text{rk}\left[\left(\frac{\partial}{\partial x_j} P_i\right)(\bar{v})\right]_{ij} = n - \dim_k(\mathfrak{n}/\mathfrak{n}^2) \leq n - \text{cod}(V, \{\bar{v}\})$$

even if V is singular at \bar{v} .

(2) Note that equation (2) gives an effective way to compute $\dim_k(\mathfrak{n}/\mathfrak{n}^2)$.

We also record the following lemma, which will be useful in calculations.

Lemma 13.6. *Keep the assumptions and notation of Proposition 13.3. Let $Q_1, \dots, Q_s \in \mathcal{I}(V)$. Suppose that $\left[\left(\frac{\partial}{\partial x_j} Q_i\right)(\bar{v})\right]_{ij}$ has rank $n - \text{cod}(V, \{\bar{v}\})$. Then V is nonsingular at \bar{v} .*

This lemma will allow us to check nonsingularity in situations where it is difficult to find generators of $\mathcal{I}(V)$.

Proof. We use the notation of the proof of Proposition 13.3. Let $J \subseteq \mathcal{I}(V)$ be the ideal generated by Q_1, \dots, Q_s . It was shown in the proof of Proposition 13.3 that

$$\text{rk}\left[\left(\frac{\partial}{\partial x_j} Q_i\right)(\bar{v})\right]_{ij} = \dim_k(\phi(J))$$

and in particular that $\text{rk}\left[\left(\frac{\partial}{\partial x_j} P_i\right)(\bar{v})\right]_{ij} = \dim_k(\phi(\mathcal{I}(V)))$. On the other hand, we have $\dim_k(\phi(\mathcal{I}(V))) \geq \dim_k(\phi(J))$ since $J \subseteq \mathcal{I}(V)$. Hence by the remark preceding the lemma, we have

$$\text{rk}\left[\left(\frac{\partial}{\partial x_j} Q_i\right)(\bar{v})\right]_{ij} \leq \text{rk}\left[\left(\frac{\partial}{\partial x_j} P_i\right)(\bar{v})\right]_{ij} \leq n - \text{cod}(V, \{\bar{v}\}).$$

The assumptions of the lemma now imply that the two last inequalities are equalities, hence the conclusion. \square

Let now X be any variety. We shall write $\text{Sing}(X)$ for the set of points $x \in X$ such that the local ring $\mathcal{O}_{X,x}$ is not a regular local ring. This clearly specialises to Definition 13.1 when X is an affine variety.

A variety X is *nonsingular* or *smooth* if $\text{Sing}(X) = \emptyset$.

Proposition 13.7. *Let X be a non empty irreducible variety. Then the set $\text{Sing}(X)$ is closed and $\text{Sing}(X) \neq X$.*

For the proof, we shall need the following definition and the subsequent proposition. Let R be a UFD with fraction field K . If

$$Q(x) = x^m + r_{m-1}x^{m-1} + \dots + r_0 \in R[x],$$

we define the *content* $\text{cont}(Q)$ to be the gcd of the coefficients of Q (note that the gcd is only well-defined up to multiplication by a unit of R). If $Q(x) \in K[x]$, we define the content of $Q(x)$ to be $\text{cont}(d \cdot Q)/d$, where $d \in R$ is such that $d \cdot Q(x) \in R[x]$. One can show that this last definition does not depend on the choice of

d. Moreover, one can show that $\text{cont}(Q_1 \cdot Q_2) = \text{cont}(Q_1) \cdot \text{cont}(Q_2)$ for any two $Q_1, Q_2 \in K[x]$. Note that if $Q(x) \in K[x]$ and $\text{cont}(Q)$ is a unit, then $Q(x) \in R[x]$ (why?). The all-important result concerning the content function is the

Lemma (generalisation of Gauss's lemma). The irreducible elements of $R[x]$ are the irreducible elements of R and the polynomials $P(x) \in R[x]$, whose content is a unit and which are irreducible (and hence non constant) in $K[x]$.

The proofs of all these statements are similar to the ones considered in the Rings and Modules course in the situation where $R = \mathbb{Z}$ (but they are not examinable). See IV, §2 in S. Lang's book *Algebra* (Springer) for more details.

Proposition 13.8. *Let X be a non empty irreducible variety. Then X is birational to an algebraic set $V \subseteq k^n$ such that $\mathcal{I}(V) \subseteq k[x_1, \dots, x_n]$ is prime and principal.*

Proof. (of Proposition 13.8) We shall only prove this in the situation where $\text{char}(k) = 0$ (but the result holds without this assumption). So suppose that $\text{char}(k) = 0$. Restricting to an open affine subset of X , we may assume wlog that X is an irreducible affine variety. Let $K := \text{Frac}(\mathcal{C}(X))$ be the function field of X .

Since the k -algebra $\mathcal{C}(X)$ is finitely generated over k , the field K is finitely generated as a field over k . Let $b_1, \dots, b_t \in K$ be a transcendence basis for K over k . By definition, this means that the b_i are algebraically independent over k (ie the map of k -algebras $k[y_1, \dots, y_t] \rightarrow K$ sending y_i to b_i is injective) and that the field extension $K|k(b_1, \dots, b_t)$ is algebraic. A transcendence basis always exists. See Prop. 11.3 in CA for this. Since $\text{char}(k) = 0$, the extension $K|k(b_1, \dots, b_t)$ is a separable extension. This extension is also a finite extension because K is finitely generated as a field over $k(b_1, \dots, b_t)$ (since K is finitely generated as a field over k). Hence the extension $K|k(b_1, \dots, b_t)$ is a simple extension by the "primitive element theorem" (see the course on Galois theory) and so there is an element $b \in K$, such that $K = k(b_1, \dots, b_t)(b)$ and an irreducible polynomial $Q(x) \in k(b_1, \dots, b_t)[x]$ such that $Q(b) = 0$.

Now note that every element of $k(b_1, \dots, b_t)$ can be written as quotient c/d , where $c, d \in k[b_1, \dots, b_t]$ (here $k[b_1, \dots, b_t]$ is the k -subalgebra of K generated by the b_i). Write

$$Q(x) = x^m + \frac{c_{m-1}}{d_{m-1}}x^{m-1} + \dots + \frac{c_1}{d_1}x + \frac{c_0}{d_0}$$

where $c_i, d_i \in k[b_1, \dots, b_t]$. Let $d = \prod_i d_i$. Consider the polynomial $dQ \in k[b_1, \dots, b_t][x]$ and let

$$P := dQ/\text{cont}(dQ) \in k[b_1, \dots, b_t][x],$$

where (abusing language) $\text{cont}(dQ) \in k[b_1, \dots, b_t]$ is an arbitrary representative of the content of dQ . By construction, the polynomial $P(x)$ is irreducible in $k(b_1, \dots, b_t)[x]$ (since it is a constant multiple of a minimal polynomial) and its content is a unit. By the generalised Gauss lemma (see the discussion above), $P(x)$ is thus irreducible in $k[b_1, \dots, b_t][x]$.

Now let

$$\phi : k[b_1, \dots, b_t][x] \rightarrow K$$

be the homomorphism of k -algebras sending the b_i to themselves and x to b .

The kernel $\ker(\phi)$ is then a prime ideal (since the image of ϕ is a domain) and by construction we have $P(x) \in \ker(\phi)$. Now the ideal $(P) \subseteq k[b_1, \dots, b_t][x]$ is also prime, since P is irreducible. Hence

$\text{cod}((P), k[b_1, \dots, b_t][x]) = 1$ by Krull's principal ideal theorem (see Th. 11.13 in CA). On the other hand, the fraction field of

$$\text{Im}(\phi) = k[b_1, \dots, b_t, b] \simeq k[b_1, \dots, b_t][x]/\ker(\phi)$$

is the field K and K has transcendence degree t by assumption. Thus

$$\dim(k[b_1, \dots, b_t][x]/\ker(\phi)) = t$$

by Corollary 11.28 in CA. Using Theorem 8.7, we deduce that

$$\text{cod}(\ker(\phi), k[b_1, \dots, b_t][x]) = \dim(k[b_1, \dots, b_t][x]) - t = t + 1 - t = 1.$$

Hence we must have $\ker(\phi) = (P)$, for otherwise we would have $\text{cod}(P, k[b_1, \dots, b_t][x]) \geq 2$.

So we conclude that $k[b_1, \dots, b_t][x]/(P) \simeq k[b_1, \dots, b_t, b]$.

Now the b_i are algebraically independent and thus the k -algebra $k[b_1, \dots, b_t][x]$ can be viewed as the coordinate ring of k^{t+1} . The ring $k[b_1, \dots, b_t][x]/(P)$ is thus isomorphic to the coordinate ring of an irreducible algebraic set V in k^{t+1} , whose (prime) radical ideal is generated by a single irreducible polynomial. Since the function field of V is isomorphic to K as a K -algebra, it satisfies the conclusion of the proposition (by Proposition 9.5). \square

Proof. (of Proposition 13.7) We first show that $\text{Sing}(X)$ is closed. Let $\{U_i\}$ be an open affine covering of X . By Proposition 13.3, a point $x \in U_i$ is nonsingular in X iff it is nonsingular in U_i , ie we have $\text{Sing}(X) \cap U_i = \text{Sing}(U_i)$. On the other hand, the set $\text{Sing}(X)$ is closed iff $\text{Sing}(X) \cap U_i$ is closed for all i (why?). Hence we may assume that X is isomorphic to an algebraic set $V \subseteq k^n$ for some n .

Let P_1, \dots, P_t be generators of $\mathcal{I}(V) \subseteq k[x_1, \dots, x_n]$. From the remark following the proof of Proposition 13.3, we have

$$\text{Sing}(V) = \{\bar{v} \in V \mid \text{rk}[(\frac{\partial}{\partial x_j} P_i)(\bar{v})]_{ij} < n - \dim(V)\}.$$

Now recall that

$$\text{rk}[(\frac{\partial}{\partial x_j} P_i)(\bar{v})]_{ij} = \max\{h \in \mathbb{N} \mid \text{there exists a } h \times h\text{-submatrix } M \text{ in } [(\frac{\partial}{\partial x_j} P_i)(\bar{v})]_{ij} \text{ such that } \det(M) \neq 0\}$$

and hence

$$\text{Sing}(V) = \{\bar{v} \in V \mid \det(M) = 0 \text{ for all the } (n - \dim(V)) \times (n - \dim(V))\text{-submatrices } M \text{ in } [(\frac{\partial}{\partial x_j} P_i)(\bar{v})]_{ij}\}$$

and hence $\text{Sing}(V)$ is the zero set of a set of polynomials and is thus closed.

We now prove that $\text{Sing}(X) \neq X$. Again, we only show this when $\text{char}(k) = 0$ (but the statement holds without that assumption). We may replace wlog X by any of its open subsets and so thanks to Proposition 13.8 we may suppose that X is an algebraic set $V \subseteq k^n$ such that $\mathcal{I}(V) = (P)$, where $P \in k[x_1, \dots, x_n]$ is an irreducible polynomial. In this situation, we have to show that

$$\text{Sing}(V) = \{\bar{v} \in V \mid (\frac{\partial}{\partial x_1} P)(\bar{v}) = (\frac{\partial}{\partial x_2} P)(\bar{v}) = \dots = (\frac{\partial}{\partial x_n} P)(\bar{v}) = 0\} \neq V.$$

Suppose for contradiction that $\text{Sing}(V) = V$. Then $P \mid \frac{\partial}{\partial x_i} P$ for all i since P is irreducible. Now let i_0 be such that P has a monomial divisible by x_{i_0} . This exists since P is irreducible and in particular not constant. In that case $\frac{\partial}{\partial x_{i_0}} P \neq 0$ (note that we use the fact that $\text{char}(k) = 0$ here) and $\deg_{x_{i_0}}(\frac{\partial}{\partial x_{i_0}} P) < \deg_{x_{i_0}}(P)$. In particular, $\frac{\partial}{\partial x_{i_0}} P$ is not divisible by P . This is a contradiction, so $\text{Sing}(V) \neq V$. \square

14 Blowing up

The blow-up construction is a geometric construction, which replaces the ambient variety of a closed subvariety by a new variety, which lies over it and such that the inverse image of the closed subvariety is locally defined by one equation. This new variety often has better properties than the old one - eg the blow-up of a variety at a singular point tends to be "less" singular than the original variety. This construction is best understood in the language of schemes. In this section, we explain in the language of varieties how to blow up an affine variety at a point. We can only establish few properties of such blow-ups in our setting.

Let $n \geq 1$. Let x_1, \dots, x_n be variables for k^n and let y_1, \dots, y_n be homogenous variables for $\mathbb{P}^{n-1}(k)$. Note that contrary to what is customary, the index of the homogenous variables runs between 1 and n here (not 0 and $n-1$). Let Z be the subset of $k^n \times \mathbb{P}^{n-1}(k)$ defined by the equations $\{x_i y_j - x_j y_i = 0\}_{i,j \in \{1, \dots, n\}}$ (note that this makes sense because the polynomials are homogenous in the y -variables). The subset Z of $k^n \times \mathbb{P}^{n-1}(k)$ is called the *blow-up* of k^n at the origin of k^n . Let $\phi : Z \rightarrow k^n$ the map obtained by restricting the projection $k^n \times \mathbb{P}^{n-1}(k) \rightarrow k^n$ to Z .

Proposition 14.1. (1) *The set Z is a closed subvariety of $k^n \times \mathbb{P}^{n-1}(k)$.*

(2) *The closed subvariety $\phi^{-1}(\{0\})$ of Z is canonically isomorphic to $\mathbb{P}^{n-1}(k)$. The points of $\phi^{-1}(0)$ are in one-to-one correspondence with the lines going through the origin of k^n .*

(3) *The restriction of ϕ to the open subvariety $\phi^{-1}(k^n \setminus \{0\})$ of Z induces an isomorphism $\phi^{-1}(k^n \setminus \{0\}) \simeq k^n \setminus \{0\}$.*

Proof. (1) On the open affine subset $k^n \times U_{j_0}^{n-1}$, Z is given by the equations

$$\{x_i y_j - x_j y_i = 0, x_i - x_{j_0} y_i = 0\}_{i \in \{1, \dots, n\}, j \in \{1, \dots, j_0-1, j_0+1, \dots, n\}}.$$

The set $Z \cap k^n \times U_{j_0}^{n-1}$ is thus closed in $k^n \times U_{j_0}^{n-1}$. Since the $k^n \times U_{j_0}^{n-1}$ cover $k^n \times \mathbb{P}^{n-1}(k)$, we see that Z is closed.

(2) It follows from the definitions that $\phi^{-1}(\{0\}) = \{0\} \times \mathbb{P}^{n-1}(k)$.

(3) Suppose that $\langle X_1, \dots, X_n \rangle \neq 0$. Then there is an i_0 such that $X_{i_0} \neq 0$. The equations for Z then give $Y_j = X_j(Y_{i_0}/X_{i_0})$ for all j . Up to multiplication of all the Y_j by a non zero scalar factor, the only solution to this set of equations is $\langle X_1, \dots, X_n \rangle$. In particular, we have

$$\phi^{-1}(\langle X_1, \dots, X_n \rangle) = \{\langle X_1, \dots, X_n \rangle\} \times \{[X_1, \dots, X_n]\}.$$

This shows that the morphism $\phi^{-1}(k^n \setminus \{0\}) \rightarrow k^n \setminus \{0\}$ is a bijection. To show that it is an isomorphism, we shall provide an inverse morphism. For this, consider the morphism $q : k^n \setminus \{0\} \rightarrow \mathbb{P}^{n-1}(k)$ introduced in question 3.7. We define a map $k^n \setminus \{0\} \rightarrow Z$ by the formula $g := \text{Id}_{k^n \setminus \{0\}} \prod q$. By construction, this gives an inverse of the morphism $\phi^{-1}(k^n \setminus \{0\}) \rightarrow k^n \setminus \{0\}$. \square

Let now $X \subseteq k^n$ be a closed subvariety (ie an algebraic set). Let $\bar{v} := \langle v_1, \dots, v_n \rangle \in X$ and suppose that $\{\bar{v}\}$ is not an irreducible component of X . Let $\tau_{\bar{v}} : k^n \rightarrow k^n$ be the map such that

$$\tau_{\bar{v}}(\langle w_1, \dots, w_n \rangle) = \langle w_1 + v_1, \dots, w_n + v_n \rangle$$

for all $\bar{w} = \langle w_1, \dots, w_n \rangle \in k^n$ (note that this is an automorphism of the variety k^n). Let $Y := \tau_{-\bar{v}}(X)$. Note that by construction we have $0 \in Y$.

We define the *blow-up* $\text{Bl}(X, \bar{v})$ of X at \bar{v} to be the closure of $\phi^{-1}(Y \setminus \{0\})$ in Z . Let $b : \text{Bl}(X, \bar{v}) \rightarrow X$ be the morphism $\tau_{\bar{v}} \circ \phi|_{\text{Bl}(X, \bar{v})}$.

Proposition 14.2. (1) We have $\phi(\text{Bl}(X, \bar{v})) = Y$. In particular, b is surjective.

(2) Suppose that X is irreducible. Then $\text{Bl}(X, \bar{v})$ is an irreducible component of $\phi^{-1}(Y) \subseteq k^n \times \mathbb{P}^{n-1}(k)$. The morphism b is birational. If $X \neq k^n$, the irreducible components of $\phi^{-1}(Y)$ are $\text{Bl}(X, \bar{v})$ and $\{0\} \times \mathbb{P}^{n-1}(k)$.

The closed set $b^{-1}(\{v\}) = \text{Bl}(X, \bar{v}) \cap (\{0\} \times \mathbb{P}^{n-1}(k))$ is called the *exceptional divisor* of $\text{Bl}(X, \bar{v})$.

Proof. (1) Note first that \bar{v} lies in the closure of $X \setminus \{\bar{v}\}$. To see this, let C be the irreducible component of X containing \bar{v} . Then $C \setminus \{\bar{v}\}$ is non-empty (by assumption) and it is open in C (since $\{\bar{v}\}$ is closed). Furthermore, $C \setminus \{\bar{v}\}$ is not closed in C , for otherwise C would be disconnected and hence reducible. Thus \bar{v} lies in the closure of $C \setminus \{\bar{v}\}$ in C (which must be C) and hence \bar{v} lies in the closure of $X \setminus \{\bar{v}\}$ in X .

Now since $\mathbb{P}^{n-1}(k)$ is complete (see Theorem 12.10), we know that $\phi(\text{Bl}(X, \bar{v}))$ is closed. By (3) of Proposition 14.1, we know that $\phi(\text{Bl}(X, \bar{v})) \setminus \{0\} = Y \setminus \{0\}$ and thus by the reasoning in the last paragraph, we see that $0 \in \phi(\text{Bl}(X, \bar{v}))$. In particular, $\phi(\text{Bl}(X, \bar{v})) = Y$.

(2) From (3) of Proposition 14.1 we know that the natural morphism $\phi^{-1}(Y \setminus \{0\}) \rightarrow Y \setminus \{0\}$ is an isomorphism. Now if X is irreducible, so is Y and so is $Y \setminus \{0\}$. Hence $\text{Bl}(X, \bar{v})$ is irreducible by question 2.5 (1). On the other hand, $\text{Bl}(X, \bar{v}) \subseteq \phi^{-1}(Y)$ since $\phi^{-1}(Y)$ is closed in Z . Since $\text{Bl}(X, \bar{v})$ contains the non empty open subset $\phi^{-1}(Y \setminus \{0\})$ of $\phi^{-1}(Y)$, we see that $\text{Bl}(X, \bar{v})$ is an irreducible component of $\phi^{-1}(Y)$. Since $\phi^{-1}(Y \setminus \{0\}) \rightarrow Y \setminus \{0\}$ is an isomorphism, the morphism b is birational.

On the other hand, we have by construction $\phi^{-1}(Y) = \text{Bl}(X, \bar{v}) \cup (\{0\} \times \mathbb{P}^{n-1}(k))$. Now suppose that $X \neq k^n$. We then have $\{0\} \times \mathbb{P}^{n-1}(k) \not\subseteq \text{Bl}(X, \bar{v})$ because

$$\dim(\{0\} \times \mathbb{P}^{n-1}(k)) = n - 1 \geq \dim(\text{Bl}(X, \bar{v})) = \dim(X) \leq n - 1$$

(use Proposition 9.4, question 2.7 and Theorem 8.7). Since $\{0\} \times \mathbb{P}^{n-1}(k)$ is irreducible (since it is isomorphic to $\mathbb{P}^{n-1}(k)$) we see that the irreducible components of $\phi^{-1}(Y)$ are $\text{Bl}(X, \bar{v})$ and $\{0\} \times \mathbb{P}^{n-1}(k)$. \square

Example. Let C be the curve $y^2 = x^3$ in k^2 . Let $b : \text{Bl}(C, 0) \rightarrow C$ of C be the blow-up of C at the origin.

(1) We have $\text{Bl}(C, 0) \simeq k$.

(2) The map b is a homeomorphism but is not an isomorphism.

Use the terminology of the last two propositions, letting $n = 2$ and $X = Z(x_2^2 - x_1^3) = Y$ (note that the point to blow-up is the origin, so we do not have to translate X). We first compute $\phi^{-1}(X)$. Let $\pi : k^2 \times \mathbb{P}^1(k) \rightarrow k^2$ be the natural projection. By definition

$$\phi^{-1}(X) = \pi^{-1}(X) \cap Z = Z(x_1y_2 - x_2y_1, x_2^2 - x_1^3)$$

Let $U_1 := \{[1, Y_2] \mid Y_2 \in k\} \subset \mathbb{P}^1(k)$. In $k^2 \times U_1$, we have

$$\begin{aligned} \phi^{-1}(X) \cap (k^2 \times U_1) &= Z(x_1y_2 - x_2, x_2^2 - x_1^3) = Z(x_1y_2 - x_2, x_1^2y_2^2 - x_1^3) \\ &= Z(x_1y_2 - x_2, x_1) \cup Z(x_1y_2 - x_2, y_2^2 - x_1) = (\{0\} \times U_1) \cup Z(x_1y_2 - x_2, y_2^2 - x_1) \end{aligned}$$

The closed set $Z(x_1y_2 - x_2, y_2^2 - x_1)$ does not contain $\{0\} \times U_1$. Also $\phi^{-1}(X) \cap (k^2 \times U_1)$ has at most two irreducible components by Proposition 14.2 (2) so we conclude that

$$Z(x_1y_2 - x_2, y_2^2 - x_1) = \text{Bl}(X, 0) \cap (k^2 \times U_1).$$

On the other hand, $Z(x_1y_2 - x_2, y_2^2 - x_1) \cap (\{0\} \times U_1) = \{0\} \times \{[1, 0]\}$.

We now repeat the above reasoning for $U_2 := \{[Y_1, 1] \mid Y_1 \in k\} \subseteq \mathbb{P}^1(k)$ instead of U_1 . We have

$$\begin{aligned} \phi^{-1}(X) \cap (k^2 \times U_2) &= Z(x_1 - x_2y_1, x_2^2 - x_1^3) = Z(x_1 - x_2y_1, x_2^2 - x_2^3y_1^3) \\ &= Z(x_1 - x_2y_1, x_2) \cup Z(x_1 - x_2y_1, 1 - x_2y_2^3) = (\{0\} \times U_2) \cup Z(x_1 - x_2y_1, 1 - x_2y_2^3) \end{aligned}$$

As before, we have $Z(x_1 - x_2y_1, 1 - x_2y_2^3) \cap (k^2 \times U_2) = \text{Bl}(X, 0) \cap (k^2 \times U_2)$. On the other hand, a simple calculation shows that $Z(x_1 - x_2y_1, 1 - x_2y_2^3) \cap (\{0\} \times U_2) = \emptyset$.

So we conclude that the exceptional divisor of $\text{Bl}(X, 0)$ consists of the one point $\{0\} \times \{[1, 0]\}$. In particular, the map $b : \text{Bl}(X, 0) \rightarrow X$ is bijective. Since $\mathbb{P}^1(k)$ is complete, the morphism b sends closed sets to closed sets (see Theorem 12.10 and Corollary 12.12) and thus (since b is bijective), b sends open sets to open sets. Hence b is a homeomorphism. Taking into account (1), which we will establish below, we see that b is not an isomorphism because k is smooth whereas X has a singularity at 0. Indeed, X is irreducible and of dimension 1 (use Theorem 8.4, Corollary 8.8 and Theorem 8.9) and the gradient $\langle -3x_1^2, 2x_2 \rangle$ of the polynomial $x_2^2 - x_1^3$ vanishes at $\langle 0, 0 \rangle \in X$. This establishes (2).

We now turn to (1). We have

$$\phi^{-1}(X) \cap (k^2 \times (\mathbb{P}^1 \setminus U_1)) = Z(x_1y_2 - x_2y_1, x_2^2 - x_1^3, y_1) = Z(x_1, y_1, x_2) = \{0\} \times \{[0, 1]\}$$

and this set is not in $\text{Bl}(X, 0)$ by the above. Hence

$$\text{Bl}(X, 0) = Z(x_1y_2 - x_2, y_2^2 - x_1) \subseteq k^2 \times U_1 \subseteq k^3$$

We claim that the map $A(t) = \langle t^2, t^3, t \rangle$ gives an isomorphism between k and $Z(x_1y_2 - x_2, y_2^2 - x_1)$. Indeed this map has an inverse, which is the restriction to $Z(x_1y_2 - x_2, y_2^2 - x_1)$ of the map $B : k^3 \rightarrow k$ given by the formula $B(X_1, X_2, Y_2) = Y_2$. To verify this, note first that we clearly have $A(t) \in Z(x_1y_2 - x_2, y_2^2 - x_1)$ and $B(A(t)) = t$. Secondly, for $\langle X_1, X_2, Y_2 \rangle \in Z(x_1y_2 - x_2, y_2^2 - x_1)$ we have

$$A(B(X_1, X_2, Y_2)) = (Y_2^2, Y_2^3, Y_2)$$

and we have $Y_2^2 = X_1$, $Y_2^3 = X_1Y_2 = X_2$. We conclude that $\text{Bl}(X, 0) \simeq k$. This establishes (1).

15 Appendix. Proof of Theorem 8.7.

The material in this appendix is not examinable.

Let R be a ring and let \mathfrak{p} be a prime ideal of R . From the definition of dimension and height, we have the inequality

$$\dim(R/\mathfrak{p}) + \text{ht}(\mathfrak{p}) \leq \dim(R).$$

This inequality is not an equality in general. However, it is an equality if R is a finitely generated domain over a field.

If R is a finite-dimensional noetherian ring and $n \geq 0$, we shall say that R satisfies $\text{DE}(n)$ if

$$\dim(R/\mathfrak{p}) + \text{ht}(\mathfrak{p}) = \dim(R)$$

for all prime ideals \mathfrak{p} of R such that $\text{ht}(\mathfrak{p}) = n$.

Theorem A(=Theorem 8.7). *Let R be a finitely generated domain over k . Then R satisfies $\text{DE}(n)$ for all $n \geq 0$.*

The assumption that k is algebraically closed will not be used in the proof, so the result holds without that assumption.

Proceed as follows.

(1) Show that to prove Theorem A, it is sufficient to show that any finitely generated domain over a field satisfies $\text{DE}(1)$. [Hint: induction on n .]

Solution. Suppose that any finitely generated domain over k satisfies $\text{DE}(1)$. Let R be a finitely generated domain over k . We prove that

$$\dim(R/\mathfrak{p}) + \text{ht}(\mathfrak{p}) = \dim(R)$$

by induction on $\text{ht}(\mathfrak{p})$. The case $\text{ht}(\mathfrak{p}) = 0$ is clear, since in that case $\mathfrak{p} = (0)$ (since R is a domain) so we may suppose that $\text{ht}(\mathfrak{p}) > 1$. Let \mathfrak{p} be a prime ideal and let

$$\mathfrak{p} = \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_\delta$$

be a descending chain of prime ideals of length $\delta := \text{ht}(\mathfrak{p})$. We suppose that any finitely generated domain over k satisfies $\text{DE}(0), \text{DE}(1), \dots, \text{DE}(\delta - 1)$. Since \mathfrak{p}_0 is minimal over \mathfrak{p}_1 , we have $\text{ht}(\mathfrak{p}/\mathfrak{p}_1) = 1$. We thus have

$$\text{ht}(\mathfrak{p}/\mathfrak{p}_1) + \dim(R/\mathfrak{p}) = \dim(R/\mathfrak{p}_1)$$

Also, we have $\text{ht}(\mathfrak{p}_1) = \delta - 1 \geq 1$ by construction. So we have

$$\dim(R/\mathfrak{p}_1) + \text{ht}(\mathfrak{p}_1) = \dim(R)$$

We conclude that

$$1 + \dim(R/\mathfrak{p}) = \dim(R) - (\delta - 1)$$

or in other words that

$$\text{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim(R)$$

as required.

(2) Show that to prove that any finitely generated domain over a field satisfies $\text{DE}(1)$, it is sufficient to prove that any polynomial ring $k[x_1, \dots, x_d]$ ($d \geq 0$) satisfies $\text{DE}(1)$. [Hint: apply Noether's normalisation lemma and the Going-up theorem.]

Solution. Suppose that any polynomial ring over k satisfies $\text{DE}(1)$. Let R be a finitely generated domain over k . By Noether's normalisation lemma, there is an injection $k[x_1, \dots, x_d] \hookrightarrow R$ ($d \geq 0$) making R into a finite $k[x_1, \dots, x_d]$ -algebra. Let $P := k[x_1, \dots, x_d]$. Let \mathfrak{p} be a prime ideal of height one in R . By the Going-up theorem and Q1 of Sheet 3 in CA, the height of \mathfrak{p} and the height of $\mathfrak{p} \cap P$ is the same. Hence

$$1 + \dim(P/\mathfrak{p} \cap P) = \dim(P).$$

By Lemma 11.28 in CA, we have $\dim(P) = \dim(R)$. Also R/\mathfrak{p} is an integral extension of $P/\mathfrak{p} \cap P$ and thus $\dim(R/\mathfrak{p}) = \dim(P/\mathfrak{p} \cap P)$ by the same lemma. We conclude that $1 + \dim(R/\mathfrak{p}) = \dim(R)$, as required.

(3) Prove that the height of a maximal ideal in $k[x_1, \dots, x_d]$ is d for any $d \geq 1$. [Hint: use the method of Sheet 3, Q6 in CA.]

Solution. Let \mathfrak{m} be a maximal ideal of $k[x_1, \dots, x_d] =: P$. In Sheet 3, Q6 of CA it is shown that \mathfrak{m} is generated by polynomials $P_1(x_1), P_2(x_1, x_2), P_3(x_1, x_2, x_3), \dots, P_d(x_1, \dots, x_d)$. In the course of the proof (this can be seen in the solution, which is available), it is also shown that the ideals

$$\mathfrak{a}_i := (P_1(x_1), P_2(x_1, x_2), P_3(x_1, x_2, x_3), \dots, P_i(x_1, \dots, x_i))$$

are maximal in $k[x_1, \dots, x_i]$ for all $i = 1, \dots, d$. It was shown at the beginning of section 11.4 of CA that we have $\mathfrak{a}_i P = \mathfrak{a}_i[x_{i+1}, \dots, x_d]$. This implies that $\mathfrak{a}_i P \neq \mathfrak{a}_{i+1} P$ for otherwise

$$\mathfrak{a}_i[x_{i+1}] = \mathfrak{a}_i P \cap k[x_1, \dots, x_{i+1}] = \mathfrak{a}_{i+1} P \cap k[x_1, \dots, x_{i+1}] = \mathfrak{a}_{i+1}.$$

This is not possible because \mathfrak{a}_{i+1} is maximal in $k[x_1, \dots, x_{i+1}]$ and $\mathfrak{a}_i[x_{i+1}]$ is not, because

$$k[x_1, \dots, x_{i+1}]/\mathfrak{a}_i[x_{i+1}] = (k[x_1, \dots, x_i]/\mathfrak{a}_i)[x_{i+1}],$$

which is a domain but not a field.

Hence we obtain a chain of prime ideals

$$\mathfrak{m} \supsetneq \mathfrak{a}_{d-1} P \supsetneq \mathfrak{a}_{d-2} P \supsetneq \dots \supsetneq \mathfrak{a}_1 P \supsetneq (0)$$

and thus $\text{ht}(\mathfrak{m}) \geq d$. Since we also have $\text{ht}(\mathfrak{m}) \leq d = \dim(P)$ (by Corollary 11.26 in CA), we thus have $\text{ht}(\mathfrak{m}) = d$.

(4) Prove that the height of a maximal ideal in a finitely generated domain R over k is $\dim(R)$. [Hint: apply Noether's normalisation lemma and (3). When k is algebraically closed, this is done in [1], Theorem 11.25.]

Solution. Let \mathfrak{m} be a maximal ideal of R . By Noether's normalisation lemma, there is an injection $k[x_1, \dots, x_d] \hookrightarrow R$ ($d \geq 0$) making R into a finite $k[x_1, \dots, x_d]$ -algebra. Let $P := k[x_1, \dots, x_d]$. Let \mathfrak{p} be a prime ideal of P , which lies over \mathfrak{m} (this exists by Th. 8.8). By the Going-down theorem (Q1 of Sheet 4 in CA) and question 3.2, the height of \mathfrak{p} and the height of $\mathfrak{p} \cap P$ is the same and by Corollary 8.20 of CA, \mathfrak{p} is also a maximal ideal of P . By Lemma 11.28 in CA, we have $\dim(P) = \dim(R)$ and thus the height of \mathfrak{p} is $\dim(R)$ by (3).

(5) Let R be a local noetherian ring with maximal ideal \mathfrak{m} . Let $f \in \mathfrak{m}$. Prove that $\dim(R/(f)) \geq \dim(R) - 1$. [Hint: this is a consequence of Krull's principal ideal theorem; you may follow the proof given in [3], Th. 2.5.15, p. 72.]

Solution. In this case, it is only a matter of reproducing [3], Th. 2.5.15, p. 72.

(6) Show that if R is noetherian and a UFD, then any prime ideal of height 1 is principal. [Hint: this is a classical statement, which can be deduced from Krull's principal ideal theorem; you may follow the proof given in Cor. 10.6, [2], p. 236.]

Solution. The argument is given in the proof of Cor. 10.6, [2], p. 236.

(7) Deduce that any polynomial ring $k[x_1, \dots, x_d]$ ($d \geq 0$) satisfies DE(1). [Hint: localise at a maximal ideal and apply (4), (5), (6).]

Solution. Let $P := k[x_1, \dots, x_d]$ and let $\mathfrak{p} \in \text{Spec}(P)$ be a prime ideal of height 1. Let \mathfrak{m} be a maximal ideal of P , which contains \mathfrak{p} . The equality

$$\dim(P) = 1 + \dim(P/\mathfrak{p})$$

is equivalent to the equality

$$\dim(P_{\mathfrak{m}}) = 1 + \dim((P/\mathfrak{p})_{\mathfrak{m}/\mathfrak{p}})$$

by (4). Note also that we have $(P/\mathfrak{p})_{\mathfrak{m}/\mathfrak{p}} = P_{\mathfrak{m}}/\mathfrak{p}P_{\mathfrak{m}}$ by Lemma 5.5. in CA.

On the other hand, by (6) and the fact that P is a UFD, we have $\mathfrak{p} = (f)$ for some $f \in \mathfrak{p}$ and so $P_{\mathfrak{m}}/\mathfrak{p}P_{\mathfrak{m}} = P_{\mathfrak{m}}/(f)$. By (5) and (4), we thus have

$$\dim(P_{\mathfrak{m}}/\mathfrak{p}P_{\mathfrak{m}}) \geq \dim(P_{\mathfrak{m}}) - 1 = \dim(P) - 1.$$

Hence

$$1 + \dim(P/\mathfrak{p}) \geq \dim(P)$$

Since we have $1 + \dim(P/\mathfrak{p}) \leq \dim(P)$ by the definition of dimension and height, we thus have $1 + \dim(P/\mathfrak{p}) = \dim(P)$, as required.

(8) Prove the theorem.

Solution: (7), (2), (1).

References

- [1] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Student economy edition, Addison-Wesley Series in Mathematics, Westview Press, Boulder, CO, 2016.
- [2] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995. With a view toward algebraic geometry.
- [3] Qing Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, Oxford, 2002. Translated from the French by Reinie Ern ; Oxford Science Publications.

Exercise sheet 1. Chapters 1-4.

Part A

Question 1.1. (1) Describe the Zariski topology of k .

(2) Show that the Zariski topology of k^2 is not the product topology of $k \times k = k^2$.

Solution. (1) The non empty closed sets of k are the vanishing sets of polynomials in one variable, so the non empty closed sets are precisely the finite subsets of k .

(2) Let $W \subseteq k^2$ the vanishing set of the polynomial $P(x, y) := xy - 1$. We will show that if U and V are open sets in k , then $U \times V \cap W \neq \emptyset$. This shows that the open subset $k^2 \setminus W$ is not a union of open subsets of the type $U \times V$, and so the Zariski topology on k^2 is not the product topology. By (1), we have $U = k \setminus A$, where A is finite (resp. $V = k \setminus B$, where B is finite). We need to show that $xy - 1$ vanishes at some point of $U \times V$. For any $a \neq 0$, we have $P(a, 1/a) = 0$. As a runs through $U \setminus \{0\}$, the function $1/a$ takes infinitely many values (since U is infinite), so for some $a_0 \in U$ we have $1/a_0 \in V$. We then have $(a_0, 1/a_0) \in W$ and $(a_0, 1/a_0) \in U \times V$. Hence $W \cap U \times V \neq \emptyset$.

Question 1.2. Let $V \subseteq k^n$ be an algebraic set. Show that V is the disjoint union of two non empty algebraic sets in k^n iff there are two non-zero finitely generated reduced k -algebras T_1 and T_2 and an isomorphism of k -algebras $T_1 \oplus T_2 \simeq \mathcal{C}(V)$.

Solution. Suppose that V is the disjoint union of two non empty algebraic sets V_1 and V_2 in k^n . Let $I_1 := \mathcal{I}(V_1) \subseteq \mathcal{C}(V)$ and let $I_2 := \mathcal{I}(V_2) \subseteq \mathcal{C}(V)$ be the radical ideals in $\mathcal{C}(V)$ corresponding to V_1 and V_2 . The intersection $I_1 \cap I_2$ consists of the regular functions on V which vanish on both V_1 and V_2 and thus on all of V . Thus $I_1 \cap I_2 = 0$. On the other hand, if $f : V \rightarrow k$ is a regular function, then the function f_1 which is 0 on V_2 and equal to f on V_1 is a regular function by Proposition 4.5. Similarly, the function f_2 which is 0 on V_1 and equal to f on V_2 is a regular function. By construction, we have $f_1 \in I_2$, $f_2 \in I_1$ and $f_1 + f_2 = f$. We conclude that $I_1 + I_2 = \mathcal{C}(V)$. We might now apply the Chinese remainder theorem to conclude that $\mathcal{C}(V) \simeq \mathcal{C}(V)/I_1 \oplus \mathcal{C}(V)/I_2$.

Conversely, suppose that $\mathcal{C}(V) \simeq R_1 \oplus R_2$, where R_1 and R_2 are finitely generated reduced k -algebras. Let $I_1 := \{(0, a) \mid a \in R_1\}$ and $I_2 := \{(0, a) \mid a \in R_2\}$. We clearly have $I_1 \cap I_2 = 0$ and $I_1 + I_2 = \mathcal{C}(V)$. Also, I_1 and I_2 are easily seen to be radical. Hence V is the disjoint union of $Z(I_1)$ and $Z(I_2)$ by Lemma 2.8 and the following discussion.

Part B

Question 1.3. Let $V \subseteq k^3$ be the set

$$V := \{(t, t^2, t^3) \mid t \in k\}.$$

Show that V is an algebraic set and that it is isomorphic to k as an algebraic set. Provide generators for $\mathcal{I}(V)$.

Solution. We have $V = Z((x_2 - x_1^2, x_3 - x_1^3))$ so V is an algebraic set. If we let $A : k \rightarrow k^3$ be the polynomial map such that $A(t) := (t, t^2, t^3)$ and $B : k^3 \rightarrow k$ be the polynomial map such that $B(x_1, x_2, x_3) = x_1$, then $A(k) \subseteq V$, $B|_V \circ A = \text{Id}_k$, $A \circ B|_V = \text{Id}_V$ so A and $B|_V$ are regular maps from k to V and from V to

k respectively, which are inverse to each other. So they give an isomorphism between V and k . We still have to provide generators for $\mathcal{I}(V)$. For this, consider that $(x_2 - x_1^2, x_3 - x_1^3)$ is prime by the above so $\mathcal{I}(V) = (x_2 - x_1^2, x_3 - x_1^3)$.

Question 1.4. (1) Let $V \subseteq k^2$ be the set of solutions of the equation $y = x^2$. Show that V is isomorphic to k as an algebraic set.

(2) Let $V \subseteq k^2$ be the set of solutions of the equation $xy = 1$. Show that V is not isomorphic to k as an algebraic set.

(3) [difficult] (optional) Let $P(x, y) \in k[x, y]$ be an irreducible quadratic polynomial and let $V \subseteq k^2$ be the set of zeroes of $P(x, y)$. Show that V is isomorphic to one of the algebraic sets defined in (1) and (2).

Solution. (1) This is similar to question 1.3, with $\{(t, t^2) \mid t \in k\}$ instead of V .

(2) If V is isomorphic to k then $k[x, y]/(xy - 1) \simeq k[t]$ by Theorem 3.7. Now note that x is a unit in $k[x, y]/(xy - 1)$ by construction and that x is not in the image of k in $k[x, y]/(xy - 1)$. Since the only units of $k[t]$ lie in the image of k , there is thus no isomorphism $k[x, y]/(xy - 1) \simeq k[t]$.

(3) See

math.stackexchange.com/questions/3577406/affine-conics-over-an-algebraically-closed-field-of-char-2

Question 1.5. Let $V \subseteq k^n$ and $W \subseteq k^t$ be two algebraic sets. Let $\psi : V \rightarrow W$ be a regular map.

(1) Show that $\psi(V)$ is dense in W iff the map of rings $\psi^* : \mathcal{C}(W) \rightarrow \mathcal{C}(V)$ is injective.

(2) Show that ψ^* is surjective iff $\psi(V)$ is closed and the induced map $V \rightarrow \psi(V)$ is an isomorphism of algebraic sets.

Solution. (1) By definition of the closure of a set, $\psi(V)$ is dense in W iff any closed subset of W containing $\psi(V)$ is W . In view of Lemma 2.8 (and the following discussion) and Lemma 3.4, we thus see that $\psi(V)$ is dense in W iff I is any radical ideal of $\mathcal{C}(W)$ contained in $\psi^*(\mathfrak{m})$ for all $\mathfrak{m} \in \text{Spm}(\mathcal{C}(V))$ then I is the 0 ideal. Since $\bigcap_{\mathfrak{m} \in \text{Spm}(\mathcal{C}(V))} \psi^*(\mathfrak{m})$ is a radical ideal by construction, we thus see that $\psi(V)$ is dense in W iff $\bigcap_{\mathfrak{m} \in \text{Spm}(\mathcal{C}(V))} \psi^*(\mathfrak{m}) = 0$. Now we have

$$\bigcap_{\mathfrak{m} \in \text{Spm}(\mathcal{C}(V))} \psi^*(\mathfrak{m}) = \psi^*(\bigcap_{\mathfrak{m} \in \text{Spm}(\mathcal{C}(V))} \mathfrak{m}) = \psi^*(0) = \ker(\psi^*)$$

where the equality before last holds because $\mathcal{C}(V)$ is a reduced Jacobson ring. The equivalence follows.

(2) Suppose that ψ^* is surjective. Then ψ^* induces an isomorphism $\mathcal{C}(W)/\ker(\psi^*) \simeq \mathcal{C}(V)$ and thus the map $\text{Spm}(\psi^*)$ is injective and its image is the set of maximal ideals of $\mathcal{C}(W)$ which contain $\ker(\psi^*)$. Let Z be the closed subset of W corresponding to $\ker(\psi^*)$ via Lemma 2.8. Then Z is also an algebraic set in k^t and the image of $\psi = \text{Spm}(\psi^*)$ is Z . Hence there is a map $\psi' : V \rightarrow Z$ of algebraic sets such that ψ is the composition of ψ' with the inclusion map $Z \rightarrow W$. Translating this into the language of coordinate rings, we see that ψ^* is the composition of $\mathcal{C}(W) \rightarrow \mathcal{C}(W)/\ker(\psi^*)$ with $(\psi')^* : \mathcal{C}(W)/\ker(\psi^*) \rightarrow \mathcal{C}(V)$. By (1), we have that $(\psi')^*$ is injective. On the other hand, $(\psi')^*$ is surjective because ψ^* is surjective, so $(\psi')^*$ is an isomorphism. Hence ψ' is an isomorphism.

Now suppose that $\psi(V)$ is closed and that the induced map $V \rightarrow \psi(V)$ is an isomorphism of algebraic sets. Let $I := \mathcal{I}(\psi(V)) \subseteq \mathcal{C}(W)$. By Theorem 3.7 the map ψ^* factors through $\mathcal{C}(W)/I$ and the induced map $\mathcal{C}(W)/I \rightarrow \mathcal{C}(V)$ is an isomorphism. In particular ψ^* is surjective.

Question 1.6. Let $V \subseteq k^3$ be the algebraic set described by the ideal $(x^2 - yz, xz - x)$. Show that V has three irreducible components. Find generators for the radical (actually prime) ideals associated with these components.

Solution. Treat x, y, z as variable elements of k . If $x \neq 0$, then $z = 1$ and $y = x^2$. Also we have $\langle 0, 0, 1 \rangle \in V$ and hence $Z((x^2 - yz, xz - x)) \supseteq \{\langle x, x^2, 1 \rangle \mid x \in k\}$. We have $\{\langle x, x^2, 1 \rangle \mid x \in k\} = Z((y - x^2, z - 1))$ and the first projection gives an isomorphism between this algebraic set and k . Hence $\{\langle x, x^2, 1 \rangle \mid x \in k\}$ is an irreducible algebraic set in k^3 , which is contained in V . If $x = 0$, then the simultaneous solutions of the equation $x^2 = yz$ and $xz = x$ are precisely the solutions of the equation $yz = 0$. The simultaneous solutions of $yz = 0$ and $x = 0$ consist of the union of the z -axis and the y -axis. So V is the union of the z -axis, the y -axis and $\{\langle x, x^2, 1 \rangle \mid x \in k\}$, which are all three irreducible. One easily checks that none of these three sets are contained in the union of the two others, so they are the irreducible components of V .

Question 1.7. Let $V \subseteq k^n$ and $W \subseteq k^t$ be algebraic subsets. Let $V_0 \subseteq V$ and $W_0 \subseteq W$ be open subsets. View V_0 and W_0 as open subvarieties of V and W respectively. For $i \in \{1, \dots, t\}$ let $\pi_i : k^t \rightarrow k$ be the projection on the i -coordinate. Let $\psi : V_0 \rightarrow W_0$ be a map. Show that ψ is a morphism of varieties iff $\pi_i \circ \psi$ is a regular function on V_0 for all $i \in \{1, \dots, t\}$.

Solution. The direction \Rightarrow of the stated equivalence is clear, since compositions of regular maps are regular and the projections π_i are polynomial maps (and regular functions are regular maps with target k). For the direction \Leftarrow of the stated equivalence, recall that by Proposition 4.5 a function is regular on V_0 iff it is regular when restricted to any member of an open covering of V_0 . Now choose an open covering of V_0 by open subsets of the form $V \setminus Z(f)$, where $f \in \mathcal{C}(V)$ (this exists by Lemma 4.1). The set $V \setminus Z(f)$ is the image of a regular injective map of algebraic sets $T \rightarrow V$, such that a function on $V \setminus Z(f)$ is regular iff its composition with the map $T \rightarrow V$ is regular (by Lemma 4.3). Hence we may suppose wlog that $V_0 = V$ (effectively replacing V_0 by T). If $V_0 = V$ and $\pi_i \circ \psi$ is a regular function on V for all $i \in \{1, \dots, t\}$, then the induced map $V \rightarrow W$ is by definition the restriction of a polynomial map $k^n \rightarrow k^t$ and is thus a regular map. Since $W_0 \subseteq W$ is open, the map ψ is thus automatically regular.

Part C

Question 1.8. Show that the open subvariety $k^2 \setminus \{0\}$ of k^2 is not affine.

Solution. Let $f : k^2 \setminus \{0\} \rightarrow k$ be a regular function. Then the restriction of f to the complement of the x -axis is of the form $P(x, y)/x^n$ by Corollary 4.4. Similarly, the restriction to the complement of the y -axis is of the form $Q(x, y)/y^m$. Dividing by powers of x (resp. y), we may assume that P is not divisible by x (resp. Q is not divisible by y). Now we have $P(x, y)/x^n = Q(x, y)/y^m$ for all $x, y \neq 0$ and thus $Q(x, y)x^n = P(x, y)y^m$ (as polynomials) since k is infinite. Since $k[x, y]$ is a UFD, we deduce that $m = 0$ and that $n = 0$ and hence that $Q = P$. Thus $f(a, b) = P(a, b)$ for all a, b . In other words, the regular maps $k^2 \setminus \{0\} \rightarrow k$ are all restrictions of polynomial maps $k^2 \rightarrow k$. Now suppose for contradiction that $k^2 \setminus \{0\}$ is affine. We have just seen that the natural map $\mathcal{C}(k^2) \rightarrow \mathcal{C}(k^2 \setminus \{0\})$ is surjective and so $k^2 \setminus \{0\}$ is closed in k^2 by question 1.5 (2). But this is a contradiction, so $k^2 \setminus \{0\}$ is not affine.

Exercise sheet 2. Chapters 1-8.

Part A

Question 2.1. Give an example of a noetherian topological space of infinite dimension.

Solution. Second part. Consider the natural numbers \mathbb{N} with the topology in which the closed subsets are the subsets

$$C_n := \{1, 2, 3, \dots, n\}$$

($n \geq 0$) (where we set $C_0 = \emptyset$). In this topology, the sets C_n are irreducible. Indeed, if $C_n = C_m \cup C_t$, then either $C_m \subseteq C_t$ or $C_t \subseteq C_m$. So C_n cannot be written as the union of two proper closed subsets not contained in each other, ie C_n is irreducible. Also, this topology on \mathbb{N} is noetherian. Indeed, since all the closed subsets are finite, any descending sequence of closed subsets must stabilise for reasons of cardinality. Finally, we have

$$C_1 \subsetneq C_2 \subsetneq C_3 \subsetneq \dots$$

so that \mathbb{N} has infinite dimension.

Question 2.2. (1) Let $P(x_0, \dots, x_n)$ be a homogenous polynomial. Show that all the irreducible factors of P are also homogenous.

(2) Let $D \subseteq \mathbb{P}^n(k)$ be a closed subvariety. Suppose that D is irreducible and that $\text{cod}(D, \mathbb{P}^n(k)) = 1$. Show that there is a homogenous irreducible polynomial $P \in k[x_0, \dots, x_n]$ such that $D = Z(P)$.

Solution. (1) Let Q be an irreducible factor of P . Since P is homogenous, the polynomial $Q(tx_0, \dots, tx_n)$ is also an irreducible factor of P (up to a constant) for all $t \in k \setminus \{0\}$. Since there are only finitely many irreducible factors, we thus see that for infinitely many $t \in k$, we have $Q(tx_0, \dots, tx_n)/Q_0 \in k$ for some (irreducible) polynomial Q_0 . In other words, for infinitely many $t \in k$, we have

$$Q(tx_0, \dots, tx_n) = \sum_{i \geq 0} t^i Q_{[i]}(x_0, \dots, x_n) = c(t)Q_0$$

for some $c(t) \in k$. By the same Vandermonde argument used in question 2.4, we conclude that each $Q_{[i]}$ is a multiple of Q_0 by a constant. In particular, each non zero $Q_{[i]}$ has the same degree so that only one $Q_{[i]}$ can be non zero. In particular, Q is homogenous.

(2) Let $D_i := D \cap U_i$, where U_i a coordinate chart such that $D \cap U_i \neq \emptyset$. By question 2.7 (1), we may suppose that $\text{cod}(D \cap U_i, U_i) = 1$. Then $u_i^{-1}(D_i)$ corresponds to a prime ideal \mathfrak{p} of height one in $k[x_0, \dots, \check{x}_i, \dots, x_n]$ (use Lemma 2.5 and the fact that $D_i \cap U_i$ is irreducible).

We claim that \mathfrak{p} is principal. To see this, let $P \in \mathfrak{p}$. Since $k[x_0, \dots, \check{x}_i, \dots, x_n]$ is a UFD, we can write $P = \prod_j P_j$, where P_j is irreducible. Since \mathfrak{p} is prime we have $\mathfrak{p} \supseteq (P_{j_0})$ for some j_0 . However, since P_{j_0} is irreducible, the ideal (P_{j_0}) is prime. Since \mathfrak{p} has height one (and because the 0 ideal is prime), we thus have $\mathfrak{p} = (P_{j_0})$, proving the claim.

Now write $Q := P_{j_0}$. Note that since D is irreducible, the closure of $D \cap U_i$ in $\mathbb{P}^n(k)$ is D . On the other hand, by question 2.9 (2) the closure of $D \cap U_i$ in $\mathbb{P}^n(k)$ is

$$Z(x_i^{\deg(Q)} Q(x_0/x_i, \dots, x_{i-1}/x_i, \widetilde{x_i/x_i}, x_{i+1}/x_i, \dots, x_n/x_i)) =: Z(\beta_i(Q))$$

(see question 2.9 for the notation). We contend that $\beta_i(Q)$ is also irreducible. To see this, suppose for contradiction that $\beta_i(Q)$ is not irreducible. Then $\beta_i(Q) = S_1 S_2$, where S_1 and S_2 are non constant. By (1) we may assume that S_1 and S_2 are homogenous. Also $\beta_i(Q)$ is by construction not divisible by x_i and so neither are S_1 and S_2 . We then have

$$Q = \beta_i(Q)(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) = S_1(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) S_2(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)$$

so that either $S_1(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)$ or $S_2(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)$ is constant (since Q is irreducible). Say $S_1(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)$ is constant. Since we have

$$\beta_i(S_1(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)) = S_1$$

(because S_1 is homogenous and not divisible by x_i) we conclude that S_1 is a constant, which is a contradiction. So $\beta_i(Q)$ is irreducible.

So we have $D = Z(\beta_i(Q))$, where $\beta_i(Q)$ is irreducible, which is what we wanted to prove.

Part B

Question 2.3. Let V (resp. W) be a closed subvariety of $\mathbb{P}^n(k)$ (resp. $\mathbb{P}^t(k)$). Let $V_0 \subseteq V$ (resp. $W_0 \subseteq W$) be an open subset of V (resp. and open subset of W). View V_0 (resp. W_0) as an open subvariety of V (resp. W). Let $Q_0, \dots, Q_t \in k[x_0, \dots, x_n]$ be homogenous polynomials of the same degree. Suppose that $V_0 \cap Z((Q_0, \dots, Q_t)) = \emptyset$. Let $f : V_0 \rightarrow \mathbb{P}^t(k)$ be the map given by the formula $f(\bar{v}) := [Q_0(\bar{v}), \dots, Q_t(\bar{v})]$. Suppose finally that $f(V_0) \subseteq W_0$. Show that the induced map $V_0 \rightarrow W_0$ is a morphism of varieties.

Solution. By Lemma 5.3, we may and do assume that $W_0 = \mathbb{P}^t(k)$. Furthermore, we may and do assume that $V_0 = \mathbb{P}^n(k) \setminus Z((Q_0, \dots, Q_t))$ (since f arises as a restriction of a map defined on $\mathbb{P}^n(k) \setminus Z((Q_0, \dots, Q_t))$). Now note that for any $j \in \{0, \dots, t\}$, the image of $f|_{V_0 \setminus Z((Q_j))}$ lies in the coordinate chart U_j^t of $\mathbb{P}^t(k)$. Since a map is a morphism iff it is a morphism everywhere locally, we thus only have to check that the map $f|_{(V_0 \setminus Z((Q_j))) \cap U_i^n} : (V_0 \setminus Z((Q_j))) \cap U_i^n \rightarrow U_j^t$ is a morphism for any $i \in \{0, \dots, n\}$ and $j \in \{0, \dots, t\}$. In the coordinates of U_i^n and U_j^t , the map has the form

$$f(X_0, \dots, \widetilde{X_i}, \dots, X_n) = \left(\frac{Q_0(X_0, \dots, X_{i-1}, 1, X_{i+1}, \dots, X_n)}{Q_j(X_0, \dots, X_{i-1}, 1, X_{i+1}, \dots, X_n)}, \dots, \frac{Q_j(X_0, \dots, X_{i-1}, \widetilde{1}, X_{i+1}, \dots, X_n)}{Q_j(X_0, \dots, X_{i-1}, 1, X_{i+1}, \dots, X_n)}, \dots, \frac{Q_t(X_0, \dots, X_{i-1}, 1, X_{i+1}, \dots, X_n)}{Q_j(X_0, \dots, X_{i-1}, 1, X_{i+1}, \dots, X_n)} \right)$$

and thus by Proposition 4.5 and question 1.7, the map $f|_{(V_0 \setminus Z((Q_j))) \cap U_i^n} : (V_0 \setminus Z((Q_j))) \cap U_i^n \rightarrow U_j^t$ is a morphism.

Question 2.4. Prove Lemma 7.1.

Solution. Recall the statement. Let $I \subseteq k[x_0, \dots, x_n]$ be an ideal.

- (1) I is homogenous iff for all $P \in I$ and all $i \geq 0$, we have $P_{[i]} \in I$.
- (2) If I is homogenous then its radical $\mathfrak{r}(I)$ is also homogenous.

We prove (1). The \Leftarrow direction is clear so we only have to establish the \Rightarrow direction of the equivalence. Suppose that $I = (H_1, \dots, H_l)$ where the H_j are homogenous. If $P \in I$ then $P = Q_1 H_1 + \dots + Q_l H_l$ for some polynomials Q_i and we compute

$$P_{[i]} = \sum_j Q_{j, [i - \deg(H_j)]} H_j$$

so that $P_{[i]} \in I$.

(2) Let $t \in K \setminus \{0\}$ and let $\rho_t : k[x_0, \dots, x_n] \rightarrow k[x_0, \dots, x_n]$ be the map of k -algebras sending x_i to tx_i . Since $\rho_{1/t} \circ \rho_t = \text{Id}$, the map ρ_t is a bijection. Note that since I is homogenous, we have $\rho_t(I) = I$. Now

$$\rho_t(\mathfrak{r}(I)) = \rho_t(\cap_{\mathfrak{p} \in \text{Spec}(k[x_0, \dots, x_n]), \mathfrak{p} \supseteq I} \mathfrak{p}) = \cap_{\mathfrak{p} \in \text{Spec}(k[x_0, \dots, x_n]), \mathfrak{p} \supseteq I} \rho_t(\mathfrak{p}) = \cap_{\mathfrak{p} \in \text{Spec}(k[x_0, \dots, x_n]), \mathfrak{p} \supseteq I} \mathfrak{p} = \mathfrak{r}(I)$$

In particular, if $P \in \mathfrak{r}(I)$ then

$$\rho_t(P) = \sum_{i=0}^{\deg(P)} t^i P_{[i]} \in \mathfrak{r}(I)$$

for all $t \in K \setminus \{0\}$. If we let t run through $\deg(P) + 1$ different values, we obtain a system of linear equations with maximal rank (a Vandermonde matrix), with a unique solution. Hence $P_{[i]} \in \mathfrak{r}(I)$ for all $i \geq 0$.

Question 2.5. Let T be a topological space.

- (1) Let $S \subseteq T$ be a subset. Suppose that S is irreducible. Show that the closure of S in T is also irreducible.
- (2) Suppose that T is noetherian. Show that T is Hausdorff iff T is finite and discrete.
- (3) Let V be a variety. Show that V is irreducible iff the ring $\mathcal{O}_V(U)$ is an integral domain for all open subsets $U \subseteq V$.
- (4) Suppose T is noetherian. Show that T is quasi-compact.

Solution. (1) S is dense in its closure \bar{S} by construction. Hence any open subset U of \bar{S} (for the induced topology) meets S . By assumption $U \cap S$ is dense in S and hence U is also dense in \bar{S} .

(2) By Lemma 8.1 (= question 2.6 below), we may suppose wlog that T is irreducible (recall that any subset of a noetherian space is noetherian in the induced topology). In that case, T is Hausdorff iff T is one point. Indeed if U_1 and U_2 are neighbourhoods of two distinct points then they must meet (by irreducibility), so that T is not Hausdorff if it has more than one point.

(3) Suppose first that V is reducible. Then there are two disjoint non empty open subsets $O_1, O_2 \subseteq V$. Possibly choosing smaller open sets O_1 and O_2 , we may assume that O_1 and O_2 are affine. We have $\mathcal{O}_V(O_1 \cup O_2) = \mathcal{O}_V(O_1) \times \mathcal{O}_V(O_2)$ by the sheaf property and neither of the rings $\mathcal{O}_V(O_1)$ and $\mathcal{O}_V(O_2)$ is the 0-ring because O_1 and O_2 are affine and non-empty (use Theorem 3.7). Hence $\mathcal{O}_V(O_1 \cup O_2)$ is not an integral domain. Conversely, suppose that there is an open subset O , such that $\mathcal{O}_V(O)$ is not integral. Let $f_1, f_2 \in \mathcal{O}_V(O)$ be such that $f_1 f_2 = 0$ and $f_1, f_2 \neq 0$. Then there is by the sheaf property an open affine subset $O' \subseteq O$ such that $f_1|_{O'} \neq 0$. If O' arises from an algebraic set $V' \subseteq k^n$ then f_1 will vanish nowhere on $V' \setminus Z(f|_{O'})$, so there exists an open subset $O'' \subseteq V$ such that $f_1|_{O''}$ vanishes nowhere. Similarly there exists an open subset U'' such that $f_2|_{U''}$ vanishes nowhere. Since $f_1 f_2 = 0$, the sets O'' and U'' are disjoint, so V is reducible.

(4) Let $\{U_i\}_{i \in I}$ be an open covering of T . Let J be the set of all finite subcoverings of $\{U_i\}$ (so that J can be identified with the set of finite subsets of the indexing set I). If $A \in J$ then write $\text{Un}(A)$ for the union of all the open sets which appear in A . Suppose for contradiction that T is not quasi-compact so that $\text{Un}(A) \neq T$ for all $A \in J$. Construct a sequence $A_1, A_2, \dots \in J$ in the following way. Let $A_1 \in J$ be arbitrary. Since $\text{Un}(A_1) \neq T$ and $\cup_i U_i = T$, there is A_2 so that $\text{Un}(A_2) \supsetneq \text{Un}(A_1)$. Proceed in the same way for A_2, A_3 etc. The sequence will be infinite for otherwise $\{U_i\}_{i \in I}$ has a finite subcovering. Hence

$$T \setminus A_1 \supsetneq T \setminus A_2 \supsetneq \dots$$

is an infinite descending sequence of closed sets, contradicting noetherianity. Hence T is quasi-compact.

Question 2.6. Prove Lemma 8.1.

Solution. We recall the statement. Suppose T is noetherian and non empty. Then there is a unique finite collection $\{T_i\}$ of irreducible closed subsets of T such that

$$(a) \ T = \cup_i T_i.$$

$$(b) \ T_i \not\subseteq \cup_{j \neq i} T_j \text{ for all } i.$$

We first prove that T is the union of a finite collection of irreducible closed subsets. Denote this statement by $S(T)$. To prove $S(T)$, we first make the assumption (*) that for any proper closed subset C of T (ie such that $C \neq T$) we have $S(C)$. Now if T is irreducible then $S(T)$ holds tautologically. If T is not irreducible, then T has a non dense open subset and so T contains two disjoint non empty open subsets U_1 and U_2 , so that $T = (T \setminus U_1) \cup (T \setminus U_2)$, where $(T \setminus U_1)$ and $(T \setminus U_2)$ are not contained in each other. Since we know that $S(T \setminus U_1)$ and $S(T \setminus U_2)$ hold, we obtain a presentation of T as a union of finitely many irreducible closed subsets.

So the claim is proven under assumption (*). Now suppose for contradiction that (*) does not hold. Then T has a proper closed subset C_1 so that $S(C_1)$ does not hold. Hence C_1 is in particular not irreducible and thus can be written as union of two proper closed subsets C' and C'' . We know that either $S(C')$ or $S(C'')$ does not hold (otherwise $S(C_1)$ would hold), so suppose that $S(C')$ does not hold. Write $C_2 := C'$. Repeating the same process for C_2 in place of C_1 and continuing in the same way, we obtain a sequence

$$C_1 \supsetneq C_2 \supsetneq C_3 \supsetneq \dots$$

contradiction the fact that C_1 is noetherian. So we conclude that (*) holds and in particular we have established $S(T)$.

So T is a union of finitely many irreducible closed subsets. Since this set of irreducible closed subsets is finite, it has maximal elements for the relation of inclusion and also any element of it is contained in a maximal element. Let $\{T_i\}$ be the set of maximal elements. Then we have $T = \cup_i T_i$ (since any element is contained in a maximal element) and we have $T_i \not\subseteq \cup_{j \neq i} T_j$ for all i , because if $T_i \subseteq \cup_{j \neq i} T_j$ for some index i , then T_i is contained in T_j for some $j \neq i$ (because T_i is irreducible), which contradicts maximality. So we have established the existence of a collection $\{T_i\}$ with properties (a) and (b).

We now prove uniqueness. So suppose that we have a collection $\{T_i\}$ with properties (a) and (b). We note that the T_i are precisely the closed irreducible subsets of T , which are maximal among all irreducible closed subsets of T for the relation of inclusion. Indeed, let $C \subseteq T$ be an irreducible closed subset and suppose that $C \supseteq T_i$ for some i . Since $C \subseteq T_j$ for some j (because $C \subseteq \cup_i T_i$ and C is irreducible), we have $T_i \subseteq T_j$. This implies that $T_i = T_j$ by property (b). Hence $C = T_i$. So the T_i are maximal among all irreducible closed subsets of T for the relation of inclusion. On the other hand, suppose that C is a closed irreducible subset of T , which is maximal for the relation of inclusion. Then $C \subseteq T_i$ for some i (again just because C is irreducible) and thus $C = T_i$ by maximality. Hence the T_i are precisely the closed irreducible subsets of T , which are maximal among all irreducible closed subsets of T for the relation of inclusion. But this determines the T_i uniquely.

Question 2.7. Let T be a topological space. Let $\{V_i\}$ be an open covering of T . Let $C \subseteq T$ be an irreducible closed subset (hence non empty).

(1) Show that $C \cap V_i$ is irreducible if $C \cap V_i \neq \emptyset$ and that $\sup_{i, C \cap V_i \neq \emptyset} \text{cod}(C \cap V_i, V_i) = \text{cod}(C, T)$ and $\sup_i \dim(V_i) = \dim(T)$.

(2) Prove Proposition 8.6.

Solution. (1) First part. We first show that $C \cap V_i$ is irreducible if $C \cap V_i$ is not empty. If $U \subseteq C \cap V_i$ is open (for the induced topology) and non empty it is also open in C , and hence it is dense in C , and hence dense in $C \cap V_i$. So $C \cap V_i$ is also irreducible. To show that $\sup_{i, C \cap V_i \neq \emptyset} \text{cod}(C \cap V_i, V_i) = \text{cod}(C, T)$, note first that from the definitions, we have $\text{cod}(C \cap V_i, V_i) \leq \text{cod}(C, T)$ for all i such that $C \cap V_i \neq \emptyset$. So we only have to prove that there is an $i_0 := i_0(C)$ such that $\text{cod}(C \cap V_{i_0}, V_{i_0}) \geq \text{cod}(C, T)$. So let

$$C \subsetneq C_1 \subsetneq C_2 \subsetneq \cdots \subsetneq C_{\text{cod}(C, T)}$$

be an ascending sequence of closed irreducible subsets. Let i_0 be an index such that $C \cap V_{i_0} \neq \emptyset$. We then have $C_j \cap V_{i_0} \neq \emptyset$ for all $j \geq 1$ and moreover C_j is the closure of $C_j \cap V_{i_0}$ in T since C_j is irreducible and closed in T . Hence we also have

$$C \cap V_{i_0} \subsetneq C_1 \cap V_{i_0} \subsetneq C_2 \cap V_{i_0} \subsetneq \cdots \subsetneq C_{\text{cod}(C, T)} \cap V_{i_0}$$

So we have $\text{cod}(C \cap V_{i_0}, V_{i_0}) \geq \text{cod}(C, T)$ as required.

Second part. Using the first part, we have

$$\begin{aligned} \dim(T) &= \sup_{C \text{ irreducible closed in } T} \text{cod}(C, T) = \sup_{C \text{ irreducible closed in } T} \sup_{i, C \cap V_i \neq \emptyset} \text{cod}(C \cap V_i, V_i) \\ &= \sup_i \sup_{C \text{ irreducible closed in } T, C \cap V_i \neq \emptyset} \text{cod}(C \cap V_i, V_i) = \sup_i \sup_{\tilde{C} \text{ irreducible closed in } V_i} \text{cod}(\tilde{C}, V_i) = \sup_i \dim(V_i) \end{aligned}$$

where we have used question 2.5 (1) in the equality before last. So the second part follows from the first part.

(2) This follows from (1) and the fact that the dimension of a polynomial ring is finite (Theorem 8.4). Indeed an affine variety is topologically a closed subset of k^n for some $n \geq 0$ and thus all its closed subsets have finite dimension and codimension because k^n has finite dimension. The general case then follows from (1) and the fact that a variety has a finite covering by affine open subvarieties.

Question 2.8. (1) Show that any element of $\text{GL}_{n+1}(k)$ (= group of $(n+1) \times (n+1)$ -matrices with entries in k and with non zero determinant) defines an automorphism of $\mathbb{P}^n(k)$.

(2) Show that if V is a projective variety, then for any two points $v_1, v_2 \in V$, there is an open affine subvariety $V_0 \subseteq V$ such that $v_1, v_2 \in V_0$.

Solution. (1) Let $M \in \text{GL}_{n+1}(k)$. Define a map $\phi_M : \mathbb{P}^n(k) \rightarrow \mathbb{P}^n(k)$ by the formula

$$\phi_M([v]) = [M \cdot v] = \left[\sum_j M_{0j} v_j, \sum_j M_{1j} v_j, \dots, \sum_j M_{nj} v_j \right]$$

By question 2.3, this is a morphism from $\mathbb{P}^n(k)$ to $\mathbb{P}^n(k)$. Its inverse is by construction given by $\phi_{M^{-1}}$. This construction in fact gives a homomorphism of groups $\text{GL}_{n+1}(k) \rightarrow \text{Aut}(\mathbb{P}^n(k))$.

(2) It is sufficient to prove this for $V = \mathbb{P}^n(k)$.

Indeed, suppose that V is a closed subvariety of $\mathbb{P}^n(k)$. Suppose that $U \subseteq \mathbb{P}^n(k)$ is an open affine subvariety such that $v_1, v_2 \in U$. Note that $C \cap U$ is a closed subset of U and an open subset of C . By construction,

the sheaf of functions that $C \cap U$ inherits from C as an open subvariety of C is the same as the sheaf of functions it inherits from U as closed subvariety of U (this follows from the definitions). On the other hand, if one endows $C \cap U$ with the sheaf of functions it inherits from U as a closed subvariety, then the resulting Topskf is affine by Lemma 5.4. Hence $C \cap U$ is an affine open subvariety of C . Furthermore, $C \cap U$ contains both v_1 and v_2 .

So we prove the statement for $\mathbb{P}^n(k)$. Suppose that $v_1 = [\bar{v}_1]$ and $v_2 = [\bar{v}_2]$, where $\bar{v}_1, \bar{v}_2 \in k^{n+1} \setminus \{0\}$. We may suppose that $v_1 \neq v_2$ (otherwise take V_0 to be any open subvariety containing v_1) and so \bar{v}_1 and \bar{v}_2 are not multiples of each other. In particular, \bar{v}_1 and \bar{v}_2 are linearly independent. Now let $i \in \{0, \dots, n\}$ be such that $v_1 \in U_i$, where U_i is one of the standard coordinate charts. Let $\bar{w} \in k^{n+1} \setminus \{0\}$ be such that $[\bar{w}] \in U_i \setminus \{v_1\}$. Choose $M \in \text{GL}_{n+1}(k)$ such that $M(\bar{v}_1) = \bar{v}_1$ and $M(\bar{v}_2) = \bar{w}$. So in particular, $\phi_M(v_1) = v_1$ and $\phi_M(v_2) \in U_i$. Then we have $v_1, v_2 \in \phi_M^{-1}(U_i) = \phi_{M^{-1}}(U_i)$. Now $\phi_{M^{-1}}(U_i)$ is an open affine subvariety of $\mathbb{P}^n(k)$ because U_i is affine and $\phi_{M^{-1}}$ is an automorphism of varieties.

Part C

Question 2.9. Let $i \in \{0, \dots, n\}$ and let $u_i : k^n \rightarrow \mathbb{P}^n(k)$ be the standard map (with image the coordinate chart U_i). Let $C \subseteq k^n$ be a closed subvariety of k^n (ie an algebraic set in k^n). For any $P \in k[x_0, \dots, x_{i-1}, \check{x}_i, x_{i+1}, \dots, x_n]$ let

$$\beta_i(P) := x_i^{\deg(P)} P\left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{\check{x}_i}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right) \in k[x_0, \dots, x_n].$$

- (1) Let \bar{C} be the closure of $u_i(C)$ in $\mathbb{P}^n(k)$. Show that $(\beta_i(\mathcal{I}(C))) = \mathcal{I}(\bar{C})$ (where $(\beta_i(\mathcal{I}(C)))$ is the ideal of $k[x_0, \dots, x_n]$ generated by all the elements of $\beta_i(\mathcal{I}(C))$).
- (2) Suppose that $\mathcal{I}(C) = (J)$ (ie $\mathcal{I}(C)$ is a principal ideal with generator J). Show that $(\beta_i(J)) = \mathcal{I}(\bar{C})$.
- (3) Suppose that $n = 3$ and that C is the variety considered in question 1.3. Describe the closure of $u_0(C)$ in $\mathbb{P}^3(k)$. Find homogenous polynomials (H_1, \dots, H_h) such that $Z(H_1, \dots, H_h)$ is the closure of $u_0(C)$ in $\mathbb{P}^3(k)$.

Solution. (1) By construction we have $u_i^{-1}(Z(\beta_i(P))) = Z(P)$ for any P as above so the closed set $Z((\beta_i(\mathcal{I}(C))))$ contains the closure of $u_i(C)$ in $\mathbb{P}^n(k)$. In particular $(\beta_i(\mathcal{I}(C))) \subseteq \mathcal{I}(\bar{C})$. So we only have to show the opposite inclusion, ie we have to show that if Q is a homogenous polynomial which vanishes on $u_i(C)$ then $Q \in (\beta_i(\mathcal{I}(C)))$. So suppose that $Q = Q(x_0, \dots, x_n)$ vanishes on $u_i(C)$. Write $Q = x_i^\delta Q_0$, where Q_0 is not divisible by x_i . By construction, Q_0 is also homogenous and also vanishes on $u_i(C)$. In particular, $Q_0(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)$ vanishes on C (by the definition of the map u_i). But

$$\beta_i(Q_0(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)) = Q_0(x_0, \dots, x_n)$$

because $\deg(Q_0(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)) = \deg(Q_0(x_0, \dots, x_n))$. So $Q_0 \in (\beta_i(\mathcal{I}(C)))$ and hence $Q \in (\beta_i(\mathcal{I}(C)))$.

- (2) Notice that β_i is multiplicative. Hence $(\beta_i(\mathcal{I}(C))) = (\beta_i(J))$.
- (3) First note the following. Let T be a topological space and $\{T_i\}$ be an open covering of T . Let $S \subseteq T$ be a subset. Write $\text{cl}(S, T)$ for the closure of S in T . Then we have $\text{cl}(S, T) = \cup_i \text{cl}(S \cap T_i, T_i)$. Indeed, from the definitions, we have $\text{cl}(S, T) \supseteq \text{cl}(S \cap T_i, T_i)$ for all i and hence $\text{cl}(S, T) \supseteq \cup_i \text{cl}(S \cap T_i, T_i)$. We prove the opposite inclusion. So suppose that $t \in \text{cl}(S, T)$. Note that this is equivalent to saying that every

open neighbourhood of t meets S . For all the indices i such that $t \in T_i$, we then have $t \in \text{cl}(S \cap T_i, T_i)$. Indeed if V is an open neighbourhood of t in T_i , then V is also open in T (since T_i is open) and thus V meets S . Since V is contained in T_i , V thus meets $S \cap T_i$. Since V was arbitrary, we thus see that $t \in \text{cl}(S \cap T_i, T_i)$. Since t is contained in at least one T_i , we conclude that $\text{cl}(S, T) \subseteq \cup_i \text{cl}(S \cap T_i, T_i)$ and thus that $\text{cl}(S, T) = \cup_i \text{cl}(S \cap T_i, T_i)$.

Using this statement, we see that $\text{cl}(u_0(C), \mathbb{P}^3(k)) = \cup_{i=0}^3 u_i(\text{cl}(u_i^{-1}(u_0(C)), k^3))$.

Now the variety $u_1^{-1}(u_0(C))$ is $\{(1/t, t, t^2) \mid t \in k \setminus \{0\}\}$ in the coordinates x_0, x_2, x_3 . This is precisely the zero set of $(x_2x_0 - 1, x_3 - x_2^2)$ in k^3 and is thus closed in k^3 .

The variety $u_2^{-1}(u_0(C))$ is $\{(1/t^2, 1/t, t) \mid t \in k \setminus \{0\}\}$ in the coordinates x_0, x_1, x_3 . This is precisely the zero set of $(x_3x_1 - 1, x_1^2 - x_0)$ in k^3 and is thus closed in k^2 .

The variety $u_3^{-1}(u_0(C))$ is $\{(1/t^3, 1/t^2, 1/t) \mid t \in k \setminus \{0\}\}$ in the coordinates x_0, x_1, x_2 . This is precisely the zero set of $(x_2^2 - x_1, x_2^3 - x_0)$ in $k^3 \setminus \{0\}$. On the other hand, the zero set of $(x_2^2 - x_1, x_2^3 - x_0)$ in k^3 contains 0 and it is precisely the set $\{(u^3, u^2, u) \mid u \in k\}$. By the reasoning of question 1.3, this set is isomorphic to k via the third projection. Hence $u_3^{-1}(u_0(C))$ is an open set of a closed irreducible algebraic set (namely $Z(x_2^2 - x_1, x_2^3 - x_0)$) in k^3 and its closure is thus $Z(x_2^2 - x_1, x_2^3 - x_0)$.

As $u_3(\langle 0, 0, 0 \rangle) = [0, 0, 0, 1]$, we conclude that $\text{cl}(u_0(C), \mathbb{P}^3(k)) = u_0(C) \cup \{[0, 0, 0, 1]\}$. Now C is described by the equation $x_1^2 - x_2$ and $x_1^3 - x_3$ in k^3 and thus

$$\text{cl}(u_0(C), \mathbb{P}^3(k)) \subseteq Z(\beta_0(x_1^2 - x_2), \beta_0(x_1^3 - x_3)) = Z(x_1^2 - x_2x_0, x_1^3 - x_3x_0^2)$$

If $x_0 = 0$ then the equations $x_1^2 - x_2x_0$ and $x_1^3 - x_3x_0^2$ are equivalent to the equation $x_1 = 0$. So we need a third equation which vanishes on $u_0(C)$ and which forces the equation $x_2 = 0$ if $x_0 = x_1 = 0$. Consider $x_2^3 - x_3^2x_0 = \beta_0(x_2^3 - x_3^2) = 0$. By inspection, we see that $x_2^3 - x_3^2x_0$ vanishes on $u_0(C)$. On the other hand, if $x_0 = 0$ and $x_2^3 - x_3^2x_0 = 0$ then $x_2 = 0$.

We thus see that

$$\text{cl}(u_0(C), \mathbb{P}^3(k)) = Z(x_1^2 - x_2x_0, x_1^3 - x_3x_0^2, x_2^3 - x_3^2x_0).$$

Exercise sheet 3. Chapters 1-12.

Part A

Question 3.1. Show that k^2 is not homeomorphic to $\mathbb{P}^2(k)$.

Solution. Suppose for contradiction that there is a homeomorphism $h : k^2 \rightarrow \mathbb{P}^2(k)$. Let $V_0 := Z(x_1)$ and $V_1 = Z(x_1 - 1)$. We have $\dim(V_0) = \dim(V_1) = 1$ and V_0 and V_1 are irreducible. Noting that both dimension and irreducibility only depend on the underlying topology, we see that $h(V_0)$ and $h(V_1)$ are irreducible closed subsets of dimension 1 of $\mathbb{P}^2(k)$. Thus we have $h(V_0) \cap h(V_1) \neq \emptyset$ by Proposition 11.2. However, by construction we have $V_0 \cap V_1 = \emptyset$, so this is a contradiction.

Part B

Question 3.2. Let $V_0 = Z(x_0x_3 - x_1^2) \subseteq \mathbb{P}^3(k)$ and $V_1 = Z(x_1x_3 - x_2^2) \subseteq \mathbb{P}^3(k)$. Let $C := V_0 \cap V_1 \subseteq \mathbb{P}^3(k)$. Let $U := \mathbb{P}^3 \setminus Z(x_0, x_1, x_2)$ and endow U with its structure of open subvariety of $\mathbb{P}^3(k)$. Let $g : U \rightarrow \mathbb{P}^2(k)$ be the morphism such that $g([X_0, X_1, X_2, X_3]) = [X_0, X_1, X_2]$ for all $[X_0, X_1, X_2, X_3] \in U$ (see question 2.3).

- (1) Show that the morphism $g|_{C \cap U} : C \cap U \rightarrow \mathbb{P}^2(k)$ extends to a morphism $f : C \rightarrow \mathbb{P}^2(k)$.
- (2) Show that $f(C)$ is closed and that $f(C) = Z(z_0z_2^2 - z_1^3)$.
- (3) Show that the induced map $f : C \rightarrow f(C)$ is an isomorphism.

Solution. (1) We have $Z(x_0, x_1, x_2) = \{[0, 0, 0, 1]\} \subseteq C$. So $g|_{C \cap U}$ is only undefined at one point. Now assuming that $X_1, X_2, X_3 \neq 0$ and $[X_0, X_1, X_2, X_3] \in C$, we have

$$\begin{aligned} g([X_0, X_1, X_2, X_3]) &= [X_0, X_1, X_2] = [X_0X_3, X_1X_3, X_2X_3] = [X_1^2, X_2^2, X_2X_3] \\ &= [X_1^2X_2, X_2^3, X_2^2X_3] = [X_1^2X_2, X_1X_2X_3, X_1X_3^2] = [X_1X_2, X_2X_3, X_3^2]. \end{aligned}$$

So the map $h : C \setminus (Z(x_1x_2, x_2x_3, x_3^2) \cap C) \rightarrow \mathbb{P}^2(k)$ given by the formula

$$h([X_0, X_1, X_2, X_3]) = [X_1X_2, X_2X_3, X_3^2]$$

coincides with g on $C \setminus ((Z(x_1x_2x_3) \cup Z(x_0, x_1, x_2)) \cap C)$. Now $[0, 0, 0, 1] \notin Z(x_1x_2, x_2x_3, x_3^2)$ so h extends $g|_{U \cap C}$ in a neighborhood of $[0, 0, 0, 1]$.

- (2) The fact that $f(C) \subseteq Z(z_0z_2^2 - z_1^3)$ follows from the fact that if $[X_0, X_1, X_2, X_3] \in C$ then

$$X_1^3 = X_1X_0X_3 = X_0(X_1X_3) = X_0X_2^2.$$

The fact that $f(C)$ is closed follows from Corollary 12.12 (and the fact that projective varieties are complete). It also follows from (3) of this question.

- (3) We shall construct an inverse map. Suppose that $[Z_0, Z_1, Z_2] \in Z(z_0z_2^2 - z_1^3)$. Suppose first that $Z_0, Z_1 \neq 0$. Then we have

$$[Z_0, Z_1, Z_2, Z_2^2/Z_1] = [Z_0Z_1, Z_1^2, Z_2Z_1, Z_2^2] = [Z_0, Z_1, Z_2, Z_1^2/Z_0] = [Z_0^2, Z_1Z_0, Z_2Z_0, Z_1^2]$$

(because $Z_2^2/Z_1 = Z_1^2/Z_0$ since $[Z_0, Z_1, Z_2] \in Z(z_0z_2^2 - z_1^3)$). Next, plugging either member of this equality into the equations for C we see that

$$[Z_0Z_1, Z_1^2, Z_2Z_1, Z_2^2] \in C$$

if either $Z_0Z_1 \neq 0$ or $Z_1^2 \neq 0$ or $Z_2Z_1 \neq 0$ or $Z_2^2 \neq 0$ and

$$[Z_0^2, Z_1Z_0, Z_2Z_0, Z_1^2] \in C$$

if either $Z_0^2 \neq 0$ or $Z_1Z_0 \neq 0$ or $Z_2Z_0 \neq 0$ or $Z_1^2 \neq 0$.

Let $\iota_2 : \mathbb{P}^2(k) \setminus Z(z_0z_1, z_1^2, z_2z_1, z_2^2) \rightarrow \mathbb{P}^3(k)$ be the map such that

$$\iota_2([Z_0, Z_1, Z_2]) = [Z_0Z_1, Z_1^2, Z_2Z_1, Z_2^2]$$

and $\iota_1 : \mathbb{P}^2(k) \setminus Z(z_0^2, z_1z_0, z_2z_0, z_1^2) \rightarrow \mathbb{P}^3(k)$ be the map such that

$$\iota_1([Z_0, Z_1, Z_2]) = [Z_0^2, Z_1Z_0, Z_2Z_0, Z_1^2].$$

We have shown that these two maps coincide on $Z(z_0z_2^2 - z_1^3)$ whenever $Z_0, Z_1 \neq 0$. If $[Z_0, Z_1, Z_2] \in Z(z_0z_2^2 - z_1^3)$ and $Z_0 = 0$ then $Z_1 = 0$ so $[Z_0, Z_1, Z_2] = [0, 0, 1]$. By the above ι_2 is defined at $[0, 0, 1]$. On the other hand, if $Z_1 = 0$ then either $Z_0 = 0$ or $Z_2 = 0$ so that either $[Z_0, Z_1, Z_2] = [0, 0, 1]$ or $[Z_0, Z_1, Z_2] = [1, 0, 0]$. Again, ι_1 is defined at $[1, 0, 0]$. Hence ι_1 and ι_2 together define a map $\phi : Z(z_0z_2^2 - z_1^3) \rightarrow C$. Also, by construction, we have

$$\phi|_{Z(z_0z_2^2 - z_1^3) \setminus Z(z_0z_2^2 - z_1^3, z_0z_1)} \circ f|_{C \setminus Z(x_0x_1)} = \text{Id}_{C \setminus Z(x_0x_1)}$$

and

$$f|_{C \setminus Z(x_0x_1)} \circ \phi|_{Z(z_0z_2^2 - z_1^3) \setminus Z(z_0z_2^2 - z_1^3, z_0z_1)} = \text{Id}_{Z(z_0z_2^2 - z_1^3) \setminus Z(z_0z_2^2 - z_1^3, z_0z_1)}$$

(use the equalities at the beginning of the solution to (3)). Finally, we check by hand

$$f(\phi([1, 0, 0])) = f(\iota_1([1, 0, 0])) = f([1, 0, 0, 0]) = [1, 0, 0]$$

$$f(\phi([0, 0, 1])) = f(\iota_2([0, 0, 1])) = f([0, 0, 0, 1]) = h([0, 0, 0, 1]) = [0, 0, 1].$$

Suppose $[X_0, X_1, X_2, X_3] \in C$. If $X_0 = 0$ then $X_1 = X_2 = 0$. Also, if $X_1 = 0$ then $X_2 = 0$ and either $X_0 = 0$ or $X_3 = 0$. So we have either $[X_0, X_1, X_2, X_3] = [0, 0, 0, 1]$ or $[X_0, X_1, X_2, X_3] = [1, 0, 0, 0]$. Again we check

$$\phi(f([0, 0, 0, 1])) = \phi([0, 0, 1]) = \iota_2([0, 0, 1]) = [0, 0, 0, 1]$$

$$\phi(f([1, 0, 0, 0])) = \phi([1, 0, 0]) = \iota_1([1, 0, 0]) = [1, 0, 0, 0]$$

So we have shown that $f : C \rightarrow f(C)$ and ϕ are inverse to each other.

Question 3.3. (1) Let $f : X \rightarrow Y$ be a surjective morphism of quasi-projective varieties. Suppose that X is complete. Show that Y is also complete.

(2) Show that a noetherian topological space only has finitely many connected components.

(3) Let (V, \mathcal{O}_V) be a projective variety. Show that the k -vector space $\mathcal{O}_V(V)$ is finite-dimensional.

Solution. (1) We have to show that for any quasi-projective variety B and any closed subset $C \subseteq Y \times B$ the projection $\pi_B(C)$ of C on the second factor is closed. Now the natural map $f \times \text{Id}_B : X \times B \rightarrow Y \times B$ is surjective, so we have $\pi_B(C) = \pi_B((f \times \text{Id}_B)^{-1}(C))$ and $\pi_B((f \times \text{Id}_B)^{-1}(C))$ is closed since X is complete.

(2) Recall that the connected components of a topological space T are the connected subsets of T , which are maximal (with respect to inclusion) among all such subsets. One can show that the connected components of T cover T (if this is not known to the students, this can be part of the exercise. See any standard textbook in topology for the solution, which is not difficult).

Note that if $C \subseteq T$ is connected then so is its closure \bar{C} . Indeed, if $\bar{C} = C_1 \cup C_2$ where C_1 and C_2 are disjoint, non empty and open in \bar{C} , then $C = (C_1 \cap C) \cup (C_2 \cap C)$ and again $C_1 \cap C$ and $C_2 \cap C$ are disjoint, non empty and open in C . Hence the connected components of T are closed. This fact is not needed for (2) but will be used in (3).

Now suppose for contradiction that T has infinitely many connected components. In particular, T is not connected. So $T = T_1 \cup T_2$, where T_1 and T_2 are open, non empty and disjoint. In particular T_1 and T_2 are closed. Now either T_1 or T_2 is not connected. Indeed, from the definitions any connected components of T is contained in either T_1 or T_2 . If T_1 and T_2 are connected, then each component is equal to either T_1 or T_2 (by maximality) and so there would be only finitely many. So suppose that T_1 is not connected. Repeating the same reasoning, we obtain a closed and open subset T_{11} in T_1 , which is not equal to T_1 and which is not connected (otherwise all the connected components would be one of T_2 , T_{11} or T_{12}). Continuing in this way, we obtain a decreasing sequence

$$T \supsetneq T_1 \supsetneq T_{11} \supsetneq T_{111} \supsetneq \dots$$

of closed subsets, contradicting the noetherian condition.

(3) Recall that a regular function on a variety V defines a morphism $f : V \rightarrow k$. The image of f is a closed subset of k because V is complete. See Corollary 12.12 for this. Hence $f(V)$ is either finite or it is k (see question 1.1). The second case cannot occur because by (1) the variety k would then be complete, which is not true. To see that k is not complete, let $\pi_2 : k^2 \rightarrow k$ be the second projection. Note that $\pi_2(Z(x_1x_2 - 1)) = k \setminus \{0\}$ and that $k \setminus \{0\}$ is not closed in k . Hence k is not complete (see Definition 12.8).¹ Now if C is a connected component of V , then $f(C)$ is connected and thus $f(C)$ is a point. So f is constant on each connected component of V (use Lemma 12.9). On the other hand, we know by (2) that V only has finitely many connected components so the connected components of V are also open.

Hence we have an isomorphism of k -algebras

$$\mathcal{O}_V(V) \simeq \bigoplus_{C \text{ connected comp. of } V} \mathcal{O}_V(C) \simeq \bigoplus_{C \text{ connected comp. of } V} k$$

Question 3.4. Let V and W be quasi-projective varieties. Suppose that V is irreducible. Let $\text{Mor}(V, W)$ be the set of morphisms from V to W and let $\rho : \text{Mor}(V, W) \rightarrow \text{Rat}(V, W)$ be the natural map (ie ρ sends a morphism to the rational map it represents). Show that ρ is injective.

Solution. We have to show that if $U \subseteq V$ is an open subvariety and $f, g : V \rightarrow W$ are two morphisms such that $f|_U = g|_U$, then $f = g$. Now suppose that there is $v_0 \in V \setminus U$ such that $f(v_0) \neq g(v_0)$. Let $f \times g : V \rightarrow W \times W$ be the morphism of varieties such that $(f \times g)(v) = (f(v), g(v))$ for all $v \in V$. Let $\Delta_W \subseteq W \times W$ be the diagonal, which we know to be closed because W is separated (by Proposition 12.6). In particular, the set $(f \times g)^{-1}(W \times W \setminus \Delta_W)$ is open and contains v_0 . In particular, there is an open set $O \subseteq V$ such that $f(v) \neq g(v)$ for all $v \in O$. But O must meet U , since V is irreducible. This is a contradiction so $f(v) = g(v)$ for all $v \in V$.

Question 3.5. (1) Show that for any $m, n \geq 0$, $k^m \coprod k^n \simeq k^{n+m}$.

(2) Let $V \subseteq k^m$ and $W \subseteq k^n$ be algebraic sets. Show that $V \times W \subseteq k^{n+m}$ is an algebraic set and describe $\mathcal{I}(V \times W)$. Show that the affine variety associated with the algebraic set $V \times W \subseteq k^{n+m}$ is a product of the affine varieties associated with V and W .

¹I am grateful to Dragos Crisan for providing this argument.

Solution. (1) We proceed as in the proof of Theorem 10.2. The projections from $\pi_1 : k^{n+m} \rightarrow k^n$ and $\pi_2 : k^{n+m} \rightarrow k^m$ are clearly morphisms since they are polynomial maps. Now let V be a variety and let $a : V \rightarrow k^n$ and $b : V \rightarrow k^m$ be morphisms. If there is a morphism of varieties $a \amalg b : V \rightarrow k^{n+m}$ such that $\pi_1 \circ a \amalg b = a$ and $\pi_2 \circ a \amalg b = b$ then $a \amalg b = a \times b$, since as a set k^{n+m} is the Cartesian product of k^n and k^m . Hence we only have to show that $a \times b$ is a morphism a varieties. Since a map between varieties is a morphism iff it is a morphism in a neighborhood of all points of the source, we may assume without restriction of generality that V is affine. So suppose that V is an algebraic set in k^t (say). By definition a (resp. b) is then the restriction to V of a polynomial map $A : k^t \rightarrow k^n$ (resp. $B : k^t \rightarrow k^m$). The map $A \times B : k^t \rightarrow k^{n+m}$ is polynomial by definition and $a \times b$ is the restriction to V of $A \times B$. Hence $a \times b$ is a morphism.

(2) The ideal $\mathcal{I}(V \times W)$ is described at the beginning of the proof of Proposition 10.8 and it is also shown there that $V \times W$ is closed in k^{n+m} (because it is a vanishing set of an explicit ideal). It then follows from Corollary 10.6 that $V \times W \subseteq k^{n+m}$ is a product of V and W .

Question 3.6. Let $a : X \rightarrow Y$ be a rational map between two quasi-projective varieties. Suppose that Y is quasi-projective. Show that there is a unique representative $f : O \subseteq X$ of a (where $O \subseteq X$ is an open subvariety of X) such that if $f : U \rightarrow Y$ is a representative of a then $U \subseteq O$. The open set O is called the *open set of definition* of a .

Solution. Let $\{f_i : O_i \rightarrow Y\}$ be the set of all representatives of a . Let $O := \cup_i O_i$. Define the morphism $f : O \rightarrow Y$ in the following way. Let $o \in O$ and let $f_i : O_i \rightarrow Y$ be a representative of a such that $o \in O_i$. Define $f(o) := f_i(o)$. To show that this definition makes sense, we have to show that $f_i(o) = f_j(o)$ if $f_j : O_j \rightarrow Y$ is any other representative of a such that $o \in O_j$. To see this, note that by definition we have $f_i|_{O_i \cap O_j} = f_j|_{O_i \cap O_j}$ by (a small variant of) Question 3.4 so that $f_i(o) = f_j(o)$. To see that f is a morphism note that by construction $f|_{O_i} = f_i$ is a morphism for all i . Since the O_i cover O , f is a morphism because a (ordinary) map between varieties is a morphism iff it is everywhere locally a morphism.

Question 3.7. Let $n \geq 0$ and let $q : k^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n(k)$ be the map such that $q(\bar{v}) = [\bar{v}]$ for all $\bar{v} \in k^{n+1} \setminus \{0\}$. Let $V \subseteq \mathbb{P}^n(k)$ be a closed subset. Endow $k^{n+1} \setminus \{0\}$ with the structure of variety it inherits from k^{n+1} as an open subset.

- (1) Show that q is a morphism of varieties.
- (2) Show that $\mathcal{I}(V)$ is prime iff V is irreducible.
- (3) Show that $q^{-1}(V)$ is irreducible iff V is irreducible.

Solution. (1) Let $i \in \{0, \dots, n\}$. We then have

$$q^{-1}(U_i) = \{\langle X_0, \dots, X_n \rangle \in k^{n+1} \setminus \{0\} \mid X_i \neq 0\}$$

and so $q^{-1}(U_i)$ is an open subset of k^{n+1} . The map $q|_{q^{-1}(U_i)} : q^{-1}(U_i) \rightarrow U_i$ is given by the formula $\langle X_0, \dots, X_n \rangle \mapsto \langle X_0/X_i, \dots, \widehat{X_i/X_i}, \dots, X_n/X_i \rangle$ and so by Proposition 4.5, $q|_{q^{-1}(U_i)} : q^{-1}(U_i) \rightarrow U_i$ is a morphism. Since the $q^{-1}(U_i)$ cover $k^{n+1} \setminus \{0\}$, we conclude that q is a morphism.

(2) We first show that the minimal prime ideals containing $\mathcal{I}(V)$ are homogenous. So let $\{\mathfrak{p}_i\}_{i \in I}$ be the minimal prime ideals containing $\mathcal{I}(V)$. Write $c := \#I$. We proceed as in question 2.4. So let $t \in K \setminus \{0\}$ and let $\rho_t : k[x_0, \dots, x_n] \rightarrow k[x_0, \dots, x_n]$ be the map of k -algebras sending x_i to tx_i . Since $\rho_{1/t} \circ \rho_t = \text{Id}$, the

map ρ_t is a bijection. Note that since $\mathcal{I}(V)$ is homogenous, we have $\rho_t(\mathcal{I}(V)) = \mathcal{I}(V)$. Now

$$\rho_t(\mathcal{I}(V)) = \rho_t(\cap_i \mathfrak{p}_i) = \cap_i \rho_t(\mathfrak{p}_i)$$

and thus by unicity ρ_t permutes the ideals \mathfrak{p}_i (use Theorem 2.4). We conclude that $\rho_{t^{c!}}(\mathfrak{p}_i) = \mathfrak{p}_i$ for all $i \in I$ (since the permutation group on c elements has $c!$ elements). We now reason as in question 2.4. Let $P \in \mathfrak{p}_i$. Let $\delta := \deg(P)$. Since k is infinite, we can find $t_0, \dots, t_\delta \in k$ such that the elements $t_0^{c!}, \dots, t_\delta^{c!} \in k$ are distinct. Then we have

$$\rho_{t^{c!}}(P) = \sum_{j \geq 0} t_l^{j \cdot c!} P_{[j]} \in \mathfrak{p}_i$$

for all $l = 0, \dots, \delta$. This gives a linear system (a Vandermonde matrix) with a unique solution in the $P_{[j]}$ and so we conclude that $P_{[j]} \in \mathfrak{p}_i$. Hence \mathfrak{p}_i is a homogenous ideal.

So now suppose that $\mathcal{I}(V)$ is a prime ideal. Suppose for contradiction that V is not irreducible. By the discussion after Lemma 7.4, we see that there is a $r > 1$ and radical ideals I_1, \dots, I_r such that $\mathcal{I}(V) = \cap_i I_i$ and $I_i \supsetneq \cap_{j \neq i} I_j$ and $I_i \subsetneq \cap_{j \neq i} I_j$ for all i . In particular, there are elements $r_1 \in I_1$ and $r_2 \in \cap_{j \neq 1} I_j$ such that $r_1 \notin \cap_{j \neq 1} I_j$ and $r_2 \notin I_1$. In particular, $r_1, r_2 \notin \mathcal{I}(V) = \cap_i I_i$. However we have $r_1 r_2 \in \mathcal{I}(V)$ so $\mathcal{I}(V)$ is not prime. This is a contradiction.

Conversely, suppose that $\mathcal{I}(V)$ is not a prime ideal. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the minimal prime ideals of $\mathcal{I}(V)$. By assumption, we have $r > 1$ and by the above claim, the \mathfrak{p}_i are also homogenous. By Theorem 2.4, we know that $\mathcal{I}(V) = \cap_i \mathfrak{p}_i$ and that for all i we have $\mathfrak{p}_i \supsetneq \cap_{j \neq i} \mathfrak{p}_j$ and $\mathfrak{p}_i \subsetneq \cap_{j \neq i} \mathfrak{p}_j$. Now note that neither \mathfrak{p}_1 nor $\cap_{j \neq 1} \mathfrak{p}_j$ is the irrelevant ideal. Indeed, the irrelevant ideal contains all the non trivial homogenous ideals and if either \mathfrak{p}_1 or $\cap_{j \neq 1} \mathfrak{p}_j$ were the irrelevant ideal then the previous two equalities would not be satisfied. Thus, applying $Z(\cdot)$ to the same last two last equalities, we conclude that V is the union of two proper closed subsets which are not contained in each other. So V is reducible.

(3) Suppose that V is not empty (otherwise, there is nothing to prove). Then $V = Z(\mathcal{I}(V))$ by Proposition 7.3. In particular, $\mathcal{I}(V)$ is a homogenous ideal which contains no non zero constants. Also, by construction $q^{-1}(V)$ is precisely the zero set of $\mathcal{I}(V)$ in $k^{n+1} \setminus \{0\}$. On the other hand, the zero set of $\mathcal{I}(V)$ in k^{n+1} is $q^{-1}(V) \cup \{0\}$ since any non constant homogenous polynomial vanishes at 0. Since $q^{-1}(V)$ is not empty, it contains the intersection of a line with $k^{n+1} \setminus \{0\}$. By the reasoning in the second part of the proof of Proposition 11.2, the Zariski closure of this line contains 0. Hence the Zariski closure of $q^{-1}(V)$ is $q^{-1}(V) \cup \{0\}$.

Now suppose that $q^{-1}(V)$ is irreducible (in $k^{n+1} \setminus \{0\}$). We conclude from question 2.5 and the last paragraph that $q^{-1}(V) \cup \{0\}$ is closed and irreducible. The last paragraph also implies that the ideal of $q^{-1}(V) \cup \{0\}$ in k^{n+1} is $\mathcal{I}(V)$. Hence $\mathcal{I}(V)$ is prime (by Lemma 2.5). We conclude from (2) that V is irreducible.

Conversely, suppose that V is irreducible. Then $\mathcal{I}(V)$ is prime by (2). By the first paragraph, $q^{-1}(V) \cup \{0\}$ is then closed and irreducible. Hence $q^{-1}(V)$ is irreducible (in $k^{n+1} \setminus \{0\}$), since it is the intersection of an irreducible set and an open set.

Part C

Question 3.8. (1) Let $U \subseteq \mathbb{P}^1(k)$ be an open subset (for the Zariski topology). Let $f : U \rightarrow \mathbb{P}^1(k)$ be a morphism of varieties. Show that there exists a morphism of varieties $g : \mathbb{P}^1(k) \rightarrow \mathbb{P}^1(k)$ such $g|_U = f$.

(2) Show that every automorphism of $\mathbb{P}^1(k)$ is of the form described in question 2.8.

(3) Show that k is not isomorphic to any of its proper open subvarieties (an open subvariety is proper if it is not equal to k).

Solution. (1) First note the following. Let $O \subseteq U$ be an open subset. Let $a : O \rightarrow \mathbb{P}^1(k)$ be a morphism. Then: if a extends to a morphism $U \rightarrow \mathbb{P}^1(k)$, then this extension is unique. This follows from question 3.4. We may thus without restriction of generality replace U by one of its open subsets.

Now since the coordinate charts U_0 and U_1 cover $\mathbb{P}^1(k)$ we know that $f^{-1}(U_i) \neq \emptyset$ for either $i = 0$ or $i = 1$. Supposing that $f^{-1}(U_0) \neq \emptyset$, we may thus replace U by $f^{-1}(U_0)$ and thus suppose that $f(U) \subseteq U_0$. Further, replacing U by $U \cap U_0$ we may also suppose that $U \subseteq U_0$. Finally, by Lemma 4.1, we may without restriction of generality suppose that $u_0^{-1}(U)$ is an open affine subvariety with coordinate ring isomorphic to $k[x_1][h^{-1}]$, where $h \in k[x_1]$. Let now $j : u_0^{-1}(U) \rightarrow k$ be the map such that $u_0 \circ j = f \circ u_0$.

By Theorem 3.7, the map j is induced by a map of k -algebras $j^* : k[x_1] \rightarrow k[x_1][h^{-1}]$. Let $P(x_1)/h^l(x_1) =: j^*(x_1)$, where $l \geq 0$ and where we suppose without restriction of generality that P and h are coprime. If $P = 0$ then j and therefore f is a constant map and then g can be defined on all of $\mathbb{P}^1(k)$ to be the constant map with the same value. So we may suppose that $P \neq 0$.

Now by construction, we have

$$j(X_1) = P(X_1)/h^l(X_1)$$

for all $X_1 \in u_0^{-1}(U) \subseteq k$ (see Corollary 4.4 for more details about this). Let $\delta := \max(\deg(P), l \cdot \deg(h))$. Let $A(x_0, x_1) := x_0^\delta P(x_1/x_0)$ and $B(x_0, x_1) := x_0^\delta h^l(x_1/x_0)$. Note that A and B are homogenous. Note also that either $A(0, 1) \neq 0$ or $B(0, 1) \neq 0$ because we have either $\delta = \deg(P)$ or $\delta = \deg(h^l)$.

Now define a map $\mathbb{P}^1(k) \rightarrow \mathbb{P}^1(k)$ by the formula

$$[X_0, X_1] \mapsto [B(X_0, X_1), A(X_0, X_1)]$$

for all $X_0, X_1 \in k$, not both zero. Note that if $X_0 = 0$ then $\langle A(X_0, X_1), B(X_0, X_1) \rangle \neq 0$ since either $A(0, 1) \neq 0$ or $B(0, 1) \neq 0$. If $X_0 \neq 0$ then $\langle A(X_0, X_1), B(X_0, X_1) \rangle \neq 0$ for all X_1 because P and h are coprime. So this map is well-defined. If $[X_0, X_1] \in U$ then by assumption we have $X_0 \neq 0$ and $h(X_1/X_0) \neq 0$ so that

$$\begin{aligned} [B(X_0, X_1), A(X_0, X_1)] &= [X_0^\delta h^l(X_1/X_0), X_0^\delta P(X_1/X_0)] = [h^l(X_1/X_0), P(X_1/X_0)] \\ &= [1, P(X_1/X_0)/h^l(X_1/X_0)] = [1, j(X_1/X_0)] = u_0(j(X_1/X_0)) = f(u_0(X_1/X_0)) = f([1, X_1/X_0]) = f([X_0, X_1]) \end{aligned}$$

so the map $[X_0, X_1] \mapsto [B(X_0, X_1), A(X_0, X_1)]$ is a morphism $\mathbb{P}^1(k) \rightarrow \mathbb{P}^1(k)$ extending f .

(2) Let $A : \mathbb{P}^1(k) \rightarrow \mathbb{P}^1(k)$ be an automorphism. Write $\infty := [0, 1] \in \mathbb{P}^1(k)$. We saw in the solution of question 2.8 that any point of $\mathbb{P}^1(k)$ can be moved to ∞ (or any other point) by an automorphism of the required type (ie given by an invertible 2×2 -matrix). Composing A with a suitable automorphism of the required type, we may thus suppose that $A(\infty) = \infty$. In that case, the restriction of A to U_0 gives an automorphism $U_0 \simeq U_0$ (since $\mathbb{P}^1(k) = U_0 \cup \{\infty\}$). Now note that by Theorem 3.7, an automorphism of $U_0 \simeq k$ corresponds to a k -algebra automorphism ϕ of $\mathcal{C}(U_0) \simeq k[x_1]$. Note that for any polynomial $P(x_1) \in k[x_1]$, $\deg(P) = \dim_k k[x_1]/(P(x_1))$. Since ϕ induces an isomorphism of k -algebras (and hence k -vector spaces) $k[x_1]/(P(x_1)) \simeq k[x_1]/(\phi(P(x_1)))$ we thus have $\deg(\phi(P(x_1))) = \deg(P(x_1))$. So any automorphism of $k[x_1]$ sends x_1 to $ax_1 + b$ for some $a, b \in k$ with $a \neq 0$. We conclude that there are elements $a, b \in k$ such that $a \neq 0$ and such that

$$A([1, X_1]) = [1, aX_1 + b]$$

for all $X_1 \in k$. Thus, if $X_0 \neq 0$ we have

$$A([X_0, X_1]) = A([1, X_1/X_0]) = [1, a(X_1/X_0) + b] = [X_0, aX_1 + bX_0].$$

Now consider the matrix

$$M := \begin{pmatrix} 1 & 0 \\ b & a \end{pmatrix}$$

This matrix has determinant a and thus lies in $\mathrm{GL}_2(k)$. By construction, the automorphism a_M defined by M restricts to A on U_0 and hence by question 3.4 we have $A = a_M$.

(3) Suppose for contradiction that $U \subsetneq k$ is a proper open subvariety and that $f : k \rightarrow U$ is an isomorphism. Identify k with U_0 , where $U_0 \subseteq \mathbb{P}^1(k)$ is the well-known coordinate chart. By composition, f induces a morphism $\phi : U_0 \rightarrow \mathbb{P}^1(k)$. By (1), ϕ extends to a morphism $g : \mathbb{P}^1(k) \rightarrow \mathbb{P}^1(k)$, such that $g(U_0) = U$. We know that $g(\mathbb{P}^1(k))$ is closed by Corollary 12.12. Since $g(\mathbb{P}^1(k))$ also contains a non-empty subset, we see that $g(\mathbb{P}^1(k)) = \mathbb{P}^1(k)$. In particular, we must have $g([0, 1]) = [0, 1]$, otherwise g is not surjective. But then $g(\mathbb{P}^1(k))$ does not contain $U_0 \setminus U$ and so g is not surjective, which is a contradiction.

Alternatively, note that if $U \neq k$, then there are non constant invertible regular functions on U . Indeed let $a \in k \setminus U$. Then the function $x - a$ never vanishes on U . On the other hand there are no non constant invertible functions on k . So k cannot be isomorphic to U .

Exercise sheet 4. All lectures.

Part A.

Question 4.1. Let $f : X \rightarrow Y$ be a regular map between varieties. Suppose that X is quasi-projective. Let $\sigma : Y \rightarrow X$ be a regular map such that $f \circ \sigma = \text{Id}_Y$ (such a map is called a *section* of f). Show that $\sigma(Y)$ is closed in X .

Solution. Note that we have by construction

$$\sigma(Y) = \{x \in X \mid (\sigma \circ f)(x) = x\}.$$

Let $\delta_X : X \rightarrow X \times X$ (resp. $\Delta_X \subseteq X \times X$) be the diagonal map (resp. the diagonal) of X . Let $\Gamma_{\sigma \circ f} \subseteq X \times X$ be the graph of $\sigma \circ f$ (see Prop.-Def. 12.7). By the above, we have

$$\sigma(Y) = \delta_X^{-1}(\Delta_X \cap \Gamma_{\sigma \circ f})$$

and this set is closed because Δ_X and $\Gamma_{\sigma \circ f}$ are closed (because X is separated).

Part B.

Question 4.2. Suppose in this exercise that $\text{char}(k) = 0$. Find the singularities of the following curves C in k^2 . For each singular point $P \in C$ compute the dimension of $\mathfrak{m}_P/\mathfrak{m}_P^2$ as a k -vector space. Here \mathfrak{m}_P is the maximal ideal of $\mathcal{O}_{C,P}$.

(1) $Z(x^6 + y^6 - xy)$

(2) $Z(y^2 + x^4 + y^4 - x^3)$

You may assume that the polynomials $x^6 + y^6 - xy$ and $y^2 + x^4 + y^4 - x^3$ are irreducible.

Solution. (1) Note that $\dim(C) = 1$ by Krull's theorem and by Theorem 8.7. Thus we need to find the points of C where the gradient of $x^6 + y^6 - xy$ vanishes. The gradient of $x^6 + y^6 - xy$ is $\langle 6x^5 - y, 6y^5 - x \rangle$. Hence we need to solve the equations $x^6 + y^6 - xy = 6x^5 - y = 6y^5 - x = 0$. We have

$$(x/6)(6x^5 - y) - (y/6)(6y^5 - x) + 2y^6 - xy = x^6 + y^6 - xy$$

and thus these equations are equivalent to

$$y(2y^5 - x) = 6x^5 - y = 6y^5 - x = 0.$$

Now if $y = 0$ then $x = 0$. If $y \neq 0$ then $y^5 = x/2 = x/6$ so $y = 0$, which is a contradiction. So we must have $x = y = 0$. So $\langle 0, 0 \rangle$ is the only singular point of C .

For $P = \langle 0, 0 \rangle$ the dimension of $\mathfrak{m}_P/\mathfrak{m}_P^2$ as a k -vector space cannot be 1, since otherwise the ring $\mathcal{O}_{C,P}$ would be regular (apply Proposition 13.3). Since \mathfrak{m}_P is generated as a $k[x, y]$ -module by the elements x and y , we see that $\mathfrak{m}_P/\mathfrak{m}_P^2$ has dimension at most 2. Hence $\mathfrak{m}_P/\mathfrak{m}_P^2$ has dimension 2.

(2) The reasoning is similar. Solve $y^2 + x^4 + y^4 - x^3 = 4x^3 - 3x^2 = 2y + 4y^3 = 0$. Combining, we obtain

$$4(y^2 + x^4 + y^4 - x^3) + (1/4 - x)(4x^3 - 3x^2) - y(2y + 4y^3) = (-3/4)x^2 + 2y^2 = 0.$$

Now if $x \neq 0$ then $x = 3/4$ since $4x^3 - 3x^2 = 0$ and so $y^2 = 27/128$. But then $y(2y + 4y^3) = 6503409/67108864$ which is a contradiction. So we have $x = 0$ and also $y = 0$. We conclude again that the origin is the only singular point of C . By the same reasoning as above, we see that $\mathfrak{m}_P/\mathfrak{m}_P^2$ has dimension 2.

Question 4.3. Let C be the plane curve considered in (1) of question 4.2. Consider the blow-up B of C at each of its singular points in turn. How many irreducible components does the exceptional divisor of B have? Is B nonsingular?

Solution. Consider the curve $Z(x_1x_2 - x_1^6 - x_2^6) \subseteq k^2$ of (1) of question 4.2. Use the terminology of Propositions 14.1 and 14.2, letting $n = 2$ and $X = Z(x_1x_2 - x_1^6 - x_2^6) = Y$ (note that the point to blow-up is the origin by the solution of question 4.2 (1) so we do not have to translate X). We first compute $\phi^{-1}(X)$. Let $\pi : k^n \times \mathbb{P}^1(k) \rightarrow k^n$ be the natural projection. By definition

$$\phi^{-1}(X) = \pi^{-1}(X) \cap Z = Z(x_1y_2 - x_2y_1, x_1x_2 - x_1^6 - x_2^6)$$

Let $U_1 := \{[1, Y_2] \mid Y_2 \in k\} \subseteq \mathbb{P}^1(k)$. In $k^2 \times U_1$, we have

$$\begin{aligned} \phi^{-1}(X) \cap (k^2 \times U_1) &= Z(x_1y_2 - x_2, x_1x_2 - x_1^6 - x_2^6) = Z(x_1y_2 - x_2, x_1^2y_2 - x_1^6 - x_1^6y_2^6) \\ &= Z(x_1y_2 - x_2, x_1^2(y_2 - x_1^4 - x_1^4y_2^6)) = Z(x_1y_2 - x_2, x_1) \cup Z(x_1y_2 - x_2, y_2 - x_1^4 - x_1^4y_2^6) \\ &= \{0\} \times U_1 \cup Z(x_1y_2 - x_2, y_2 - x_1^4 - x_1^4y_2^6) \end{aligned}$$

Now $Z(x_1y_2 - x_2, y_2 - x_1^4 - x_1^4y_2^6)$ does not contain $\{0\} \times U_1$ (since setting $x_1 = x_2 = 0$ implies that $y_2 = 0$) so we have $\text{Bl}(X, 0) \cap (k^2 \times U_1) = Z(x_1y_2 - x_2, y_2 - x_1^4 - x_1^4y_2^6)$ by Proposition 14.2 (2). Finally, note that $Z(x_1y_2 - x_2, y_2 - x_1^4 - x_1^4y_2^6) \cap (\{0\} \times U_1)$ contains only the point $\{0\} \times \{[1, 0]\}$. In other words, the intersection of the exceptional divisor of $\text{Bl}(X, 0)$ with $\{0\} \times U_1$ is the point $\{0\} \times \{[1, 0]\}$.

Let now $U_2 := \{[Y_1, 1] \mid Y_1 \in k\} \subseteq \mathbb{P}^1(k)$. We compute as before

$$\begin{aligned} \phi^{-1}(X) \cap (k^2 \times U_2) &= Z(x_1 - x_2y_1, x_1x_2 - x_1^6 - x_2^6) = Z(x_1 - x_2y_1, y_1x_2^2 - x_2^6y_1^6 - x_2^6) \\ &= Z(x_1 - x_2y_1, x_2) \cup Z(x_1 - x_2y_1, y_1 - x_2^4y_1^6 - x_2^4) = \{0\} \times U_2 \cup Z(x_1 - x_2y_1, y_1 - x_2^4y_1^6 - x_2^4) \end{aligned}$$

We conclude as before that

$$\text{Bl}(X, 0) \cap (k^2 \times U_2) = Z(x_1 - x_2y_1, y_1 - x_2^4y_1^6 - x_2^4)$$

We compute $Z(x_1 - x_2y_1, y_1 - x_2^4y_1^6 - x_2^4) \cap (\{0\} \times U_2) = \{0\} \times \{[0, 1]\}$. So the intersection of the exceptional divisor of $\text{Bl}(X, 0)$ with $\{0\} \times U_2$ is the point $\{0\} \times [0, 1]$.

Putting everything together, we see that the exceptional divisor of $\text{Bl}(X, 0)$ consists of the points $\{0\} \times \{[1, 0]\}$ and $\{0\} \times \{[0, 1]\}$. In particular, the exceptional divisor of $\text{Bl}(X, 0)$ has two irreducible components.

We now check nonsingularity. We only have to check the nonsingularity of $\text{Bl}(X, 0)$ at $\{0\} \times \{[1, 0]\}$ and $\{0\} \times \{[0, 1]\}$ since $\text{Bl}(X, 0) \setminus \{\{0\} \times \{[1, 0]\} \cup \{0\} \times \{[0, 1]\}\}$ is isomorphic to $X \setminus \{0\}$ and $X \setminus \{0\}$ is nonsingular by the solution of question 4.2 (1).

We first check nonsingularity at $\{0\} \times \{[1, 0]\}$. Let $Q_1 := x_1y_2 - x_2$ and $Q_2 := y_2 - x_1^4 - x_1^4y_2^6$. We have

$$\begin{pmatrix} \frac{\partial}{\partial x_1} Q_1 & \frac{\partial}{\partial x_2} Q_1 & \frac{\partial}{\partial y_2} Q_1 \\ \frac{\partial}{\partial x_1} Q_2 & \frac{\partial}{\partial x_2} Q_2 & \frac{\partial}{\partial y_2} Q_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -4x_1^3 & -4x_2^3y_2^2 & 1 - 2x_2^4y_2 \end{pmatrix}$$

and evaluating at 0 we get the matrix

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which has rank 2. Using Lemma 13.6 we see that $\text{Bl}(X, 0)$ is nonsingular at $\{0\} \times \{[1, 0]\}$.

We now check nonsingularity at $\{0\} \times \{[0, 1]\}$. Let $Q_1 := x_1 - x_2 y_1$ and $Q_2 := y_1 - x_2^4 y_1^6 - x_2^4$. We have

$$\begin{pmatrix} \frac{\partial}{\partial x_1} Q_1 & \frac{\partial}{\partial x_2} Q_1 & \frac{\partial}{\partial y_2} Q_1 \\ \frac{\partial}{\partial x_1} Q_2 & \frac{\partial}{\partial x_2} Q_2 & \frac{\partial}{\partial y_2} Q_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4x_2^3 - 4x_2^3 y_1^6 & 1 - 6x_2^4 y_1^5 \end{pmatrix}$$

and evaluating at 0 we get the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which again has rank 2. Again using Lemma 13.6 we see that $\text{Bl}(X, 0)$ is nonsingular at $\{0\} \times \{[0, 1]\}$.

So all in all $\text{Bl}(X, 0)$ is nonsingular and its exceptional divisor has two irreducible components (which are points).

Question 4.4. Let $V \subseteq k^2$ be the algebraic set defined by the equation $x_1 x_2 = 0$. Show that $\text{Bl}(V, 0)$ has two disjoint irreducible components and that each of these components is isomorphic to k .

Solution. Use the terminology of questions Proposition 14.1 and 14.2, letting $n = 2$ and $X = Z(x_1 x_2) = Y$ (note that the point to blow-up is the origin so we do not have to translate X). We first compute $\phi^{-1}(X)$. Let $\pi : k^n \times \mathbb{P}^1(k) \rightarrow k^n$ be the natural projection. By definition

$$\phi^{-1}(X) = \pi^{-1}(X) \cap Z = Z(x_1 y_2 - x_2 y_1, x_1 x_2)$$

Let $U_1 := \{[1, Y_2] \mid Y_2 \in k\} \subseteq \mathbb{P}^1(k)$. In $k^2 \times U_1$, we have

$$\begin{aligned} \phi^{-1}(X) \cap (k^2 \times U_1) &= Z(x_1 y_2 - x_2, x_1 x_2) = Z(x_1 y_2 - x_2, x_1) \cup Z(x_1 y_2 - x_2, x_2) \\ &= \{0\} \times U_1 \cup Z(x_1 y_2, x_2) = \{0\} \times U_1 \cup Z(x_1, x_2) \cup Z(y_2, x_2) = \{0\} \times U_1 \cup Z(y_2, x_2) \end{aligned}$$

Now note that by definition $\text{Bl}(X, 0)$ is the closure of $\phi^{-1}(X \setminus 0)$. In particular, $\text{Bl}(X, 0)$ is the union of the closures of $\phi^{-1}(Z(x_1) \setminus 0)$ and $\phi^{-1}(Z(x_2) \setminus 0)$, ie the blow-ups of $Z(x_1)$ and of $Z(x_2)$, respectively. Now note that $\phi^{-1}(Z(x_1) \setminus 0) \cap (k^2 \times U_1) = \emptyset$. Noting also that $Z(y_2, x_2)$ is irreducible, we see that $\text{Bl}(X, 0) \cap (k^2 \times U_1) = Z(y_2, x_2)$.

A completely similar reasoning with U_2 in place of U_1 shows that $\text{Bl}(X, 0) \cap (k^2 \times U_2) = Z(y_1, x_1)$. Hence $\text{Bl}(X, 0) \subseteq Z(y_2, x_2) \cup Z(y_1, x_1) \subseteq k^2 \times \mathbb{P}^1(k)$, where we view the polynomials x_1, x_2, y_1, y_2 as homogenous polynomials in the y -variables. On the other hand we have $Z(y_2, x_2) \cap Z(y_1, x_1) = Z(x_1, x_2, y_1, y_2) = \emptyset$ and $Z(y_2, x_2) \simeq Z(y_1, x_1) \simeq k$. Since $\text{Bl}(X, 0)$ has two irreducible components of dimension 1 by the above, we thus have $\text{Bl}(X, 0) = Z(y_2, x_2) \cup Z(y_1, x_1)$.

Question 4.5. Let $C \subseteq k^2$ be defined by the equation $P(x_1, x_2) = 0$, where $P(x_1, x_2)$ is an irreducible polynomial. Suppose that C goes through the origin 0 of k^2 and is non singular there. Show that the natural morphism $\text{Bl}(C, 0) \rightarrow C$ is an isomorphism. [Hint: construct an inverse map directly, without looking at coordinate charts]

Solution. Use the terminology of Proposition 14.1 and 14.2, letting $n = 2$ and

$$X = Z(P(x_1, x_2)) = Y$$

(note that the point to blow up is the origin so we do not have to translate X). We first compute $\phi^{-1}(X)$. Let $\pi : k^n \times \mathbb{P}^1(k) \rightarrow k^n$ be the natural projection. By definition

$$\phi^{-1}(X) = \pi^{-1}(X) \cap Z = Z(x_1 y_2 - x_2 y_1, P(x_1, x_2)).$$

By assumption, we have either have $P_{x_1}(0,0) \neq 0$ or $P_{x_2}(0,0) \neq 0$. We suppose that $P_{x_1}(0,0) \neq 0$, the reasoning being similar if $P_{x_2}(0,0) \neq 0$. Also, note that by assumption, we have $P(0,0) = 0$. This implies that we can write

$$P(x_1, x_2) = A(x_2) + x_1 Q(x_1, x_2)$$

where $Q(0,0) \neq 0$ and $A(0) = 0$. In particular, $x_2 | A(x_2)$. Write $B(x_2) = A(x_2)/x_2$. We define a map $\sigma : X \rightarrow k^n \times \mathbb{P}^1(k)$ by the formulae

$$\sigma(X_1, X_2) = \langle X_1, X_2 \rangle \times [B(X_2), -Q(X_1, X_2)] \quad (*)$$

if $Q(X_1, X_2) \neq 0$ and

$$\sigma(X_1, X_2) := \langle X_1, X_2 \rangle \times [X_1, X_2] \quad (**)$$

otherwise. We have

$$B(X_2)X_2 - (-Q(X_1, X_2))X_1 = 0$$

so that whenever $\langle X_1, X_2 \rangle \neq \langle 0, 0 \rangle$ and $\langle X_1, X_2 \rangle \in X$, the formulae $(*)$ and $(**)$ give the same point in $k^n \times \mathbb{P}^1(k)$. Also, by construction, $\sigma(X) \subseteq \phi^{-1}(X)$ and $\phi \circ \sigma = \text{Id}_X$. Now by Question 4.1, the set $\sigma(X)$ is closed. The set $\sigma(X)$ is thus closed and irreducible and it is isomorphic to X as a closed subvariety of $\phi^{-1}(X)$ (since $\phi|_{\sigma(X)}$ provides an inverse to $\sigma : X \rightarrow \sigma(X)$). Finally, $\sigma(X)$ is not contained in $\{0\} \times \mathbb{P}^1(k)$. Hence it is contained in $\text{Bl}(X, 0)$ (see Proposition 14.2) and it thus coincides with it since $\text{Bl}(X, 0)$ is birational to X and thus has the same dimension as X . The map σ thus provides an isomorphism between $\text{Bl}(X, 0)$ and X .

Part C.

Question 4.6. (1) Let $f : X \rightarrow Y$ be a dominant morphism of varieties. Suppose that Y is irreducible. Show that $\dim(X) \geq \dim(Y)$.

(2) Let $f : X \rightarrow Y$ be a dominant morphism of irreducible varieties. Suppose that the field extension $\kappa(X)|\kappa(Y)$ is algebraic. Show that there are affine open subvarieties $U \subseteq X$ and $W \subseteq Y$ such that $f(U) = W$ and such that the map of rings $\mathcal{O}_X(U) \rightarrow \mathcal{O}_Y(W)$ is injective and finite.

(3) Let $f : X \rightarrow Y$ be a dominant morphism of irreducible quasi-projective varieties. Show that there is a $y \in Y$ such that we have $\dim(f^{-1}(\{y\})) \geq \dim(X) - \dim(Y)$. [Hint. Reduce to the situation where Y is affine and apply Noether's normalisation lemma to show that you may assume wlog that $Y = k^n$ for some n . Now use the existence of transcendence bases and (2) to show that there is an open subvariety $U \subseteq X$ and an open subvariety W of $k^{\dim(X)-\dim(Y)} \times k^n$ such that $f|_U$ factors as a finite and surjective morphism $U \rightarrow W$, followed by the projection to k^n . Now deduce the result from (1) and a computation of the dimension of the fibres of the projection $k^{\dim(X)-\dim(Y)} \times k^n \rightarrow k^n$.]

(4) Deduce that in the situation of (3), the set of $y \in Y$ such that we have $\dim(f^{-1}(\{y\})) \geq \dim(X) - \dim(Y)$ is dense in Y .

Solution. (1) Let $\{X_i\}$ be the irreducible components of X . Then $f(X_i)$ is irreducible for all i and hence the closure $\overline{f(X_i)}$ is also irreducible for all i (by question 2.5 (1)). Hence we must have $\cup_i \overline{f(X_i)} = Y$, otherwise f is not dominant. Now if $\overline{f(X_i)} \neq Y$ for all i then Y is not irreducible, which is impossible. So there is an index i_0 such that $\overline{f(X_{i_0})} = Y$. In that case we have a field extension $\kappa(X_{i_0})|\kappa(Y)$ and thus $\dim(X_{i_0}) \geq \dim(Y)$ by Proposition 9.4. It now follows from the definition of dimension that $\dim(X) \geq \dim(Y)$.

(2) We first prove the following statement of commutative algebra. Let $\phi : A \rightarrow B$ be a homomorphism of finitely generated integral k -algebras. Suppose that $\text{Spm}(\phi)(\text{Spm}(B))$ is dense in $\text{Spm}(A)$ and suppose that the induced map $\text{Frac}(\phi) : \text{Frac}(A) \rightarrow \text{Frac}(B)$ is an algebraic extension of fields. Then there is an element $f \in A$ such that the induced map $A[f^{-1}] \rightarrow B[\phi(f)^{-1}]$ is injective and finite.

To prove this assertion, note that by question 1.5 we already know that under the given assumptions, ϕ must be injective. Note also that since we have a commutative diagram

$$\begin{array}{ccc} \text{Frac}(A) & \xrightarrow{\text{Frac}(\phi)} & \text{Frac}(B) \\ \uparrow & & \uparrow \\ A & \xrightarrow{\phi} & B \end{array}$$

all whose maps are injective, the induced map $A[f^{-1}] \rightarrow B[\phi(f)^{-1}]$ is injective for any choice of $f \in A \setminus \{0\}$ (remember that A and B are integral domains). Thus we only have to show that there is $f \in A \setminus \{0\}$ such that the induced map $A[f^{-1}] \rightarrow B[\phi(f)^{-1}]$ is finite. Now let b_1, \dots, b_l be generators of B as a k -algebra. By assumption, each $b_i/1 \in \text{Frac}(B)$ satisfies a monic polynomial equation with coefficients in $\text{Frac}(A)$. Let $f \in A$ be the product of the denominators of all the coefficients of all these equations. Note that $B[\phi(f)^{-1}]$ is generated as a k -algebra by $1/\phi(f)$ and by the elements $b_i/1$ (use Lemma 5.3 in CA). In particular, $B[\phi(f)^{-1}]$ is generated by the $b_i/1$ as a $A[f^{-1}]$ -algebra. On the other hand, by construction, the elements $b_i/1$ all satisfy integral equations over $A[f^{-1}]$. Hence $A[f^{-1}] \rightarrow B[\phi(f)^{-1}]$ is a finite map of rings (see section 8 in CA).

Note that the fact that $A[f^{-1}] \rightarrow B[\phi(f)^{-1}]$ is injective and finite implies that the induced map

$$\text{Spm}(B[\phi(f)^{-1}]) \rightarrow \text{Spm}(A[f^{-1}])$$

is surjective (use Th. 8.8 and Cor. 8.10 in CA).

Returning to the problem at hand, note that we may wlog assume that X and Y are affine (take an affine open Y' in Y and an affine open X' in $f^{-1}(Y')$ and replace X by X' (resp. Y by Y'). Applying the result of commutative algebra that we just proved to $A = \mathcal{O}_X(X)$ and $B = \mathcal{O}_Y(Y)$ we obtain the desired result.

(3) Note that Th. 9.1 (Noether's normalisation lemma), Prop. 8.12, Th. 8.8 and Cor. 8.10 in CA imply that for some $n \geq 0$ there is a surjective morphism $h : Y \rightarrow k^{\dim(Y)}$, such that the fibre $h^{-1}(\bar{v})$ of h over \bar{v} is finite for all $\bar{v} \in k^n$. Since the fibres of the composed morphism $h \circ f$ are finite disjoint unions of fibres of f , we may thus replace f by $h \circ f$ and suppose that $Y = k^n$ for some $n \geq 0$.

Now consider the field extension $\kappa(X)|\kappa(Y)$. Choose a transcendence basis $b_1, \dots, b_\delta \in \kappa(X)$ of $\kappa(X)$ over $\kappa(Y)$. Write $\kappa(Y) = \kappa(k^n) = k(x_1, \dots, x_n)$. The set $x_1, \dots, x_n, b_1, \dots, b_\delta$ is then by construction a transcendence basis for $\kappa(X)$ over k . Since we know that $\dim(k^n) = n$ (see Theorem 8.4), we deduce from Proposition 9.4 that $\delta = \dim(X) - \dim(Y)$. Now the subfield $\kappa(Y)(b_1, \dots, b_\delta)$ of $\kappa(X)$ is isomorphic as a k -algebra to $k(x_1, \dots, x_n, y_1, \dots, y_\delta)$, which is the function field of $k^{n+\delta}$. The inclusion $k(x_1, \dots, x_n) \hookrightarrow k(x_1, \dots, x_n, y_1, \dots, y_\delta)$ is induced by the natural projection morphism $\pi : k^{n+\delta} \rightarrow k^n$ (unroll the definitions). Hence we have a rational dominant map $a : X \rightarrow k^{n+\delta}$ such that the rational dominant map associated with the morphism $f : X \rightarrow Y$ is the composition of a with the rational dominant map associated with π (apply Proposition 9.6 and question 3.4). Applying (2) we obtain open affine subvarieties $U \subseteq X$ and $W \subseteq k^{n+\delta}$ and a surjective morphism $g : U \rightarrow W$, which represents a . Let now $f' = \pi \circ g$.

Note that by question 3.4 again, we have $f' = f|_U$. Let $y \in \pi(W) = f'(U) = f(U)$. We compute

$$\begin{aligned} \dim(f^{-1}(y)) &\geq \dim(f^{-1}(y) \cap U) = \dim((f')^{-1}(y)) \\ &= \dim(g^{-1}(\pi^{-1}(y) \cap W)) \geq \dim(\pi^{-1}(y) \cap W) = \dim(\pi^{-1}(y)) = \delta = \dim(X) - \dim(Y) \end{aligned}$$

Here we used question 2.7 for the first inequality and we used (1) for the inequality

$$\dim(g^{-1}(\pi^{-1}(y) \cap W)) \geq \dim(\pi^{-1}(y) \cap W)$$

(remember that g is surjective). To justify the equality

$$\dim(\pi^{-1}(y) \cap W) = \dim(\pi^{-1}(y)) = \delta$$

note that $\pi^{-1}(y) \simeq k^\delta$. We thus have $\dim(\pi^{-1}(y) \cap W) = \dim(\pi^{-1}(y))$ by Proposition 9.4 and we have $\dim(\pi^{-1}(y)) = \delta$ by Theorem 8.4.

(4) Let $U \subseteq Y$ be an open subvariety. Applying (3) to the morphism $f^{-1}(U) \rightarrow U$, we see that there is a point $y \in U$ such that $\dim(f^{-1}(y)) \geq \dim(f^{-1}(U)) - \dim(U) = \dim(X) - \dim(Y)$. Since U was arbitrary, this shows what we want.

Question 4.7. (1) Show that all the morphisms from $\mathbb{P}^2(k)$ to $\mathbb{P}^1(k)$ are constant. [Hint: Use question 4.6 and the projective dimension theorem.]

(2) Using (1) or using another method, show that the morphisms from $\mathbb{P}^n(k)$ to $\mathbb{P}^1(k)$ are constant.

Solution. (1) Let $f : \mathbb{P}^2(k) \rightarrow \mathbb{P}^1(k)$ is a morphism. Suppose for contradiction that f is not constant. By Corollary 12.12, the image $f(\mathbb{P}^2(k))$ is closed, and it is also irreducible, since $\mathbb{P}^2(k)$ is irreducible. Hence $f(\mathbb{P}^2(k)) = \mathbb{P}^1(k)$ (because $\dim(\mathbb{P}^1(k)) = 1$). Now let $y_1, y_2 \in \mathbb{P}^1(k)$ be such that $y_1 \neq y_2$ and $\dim(f^{-1}(y_1)), \dim(f^{-1}(y_2)) \geq \dim(\mathbb{P}^2(k)) - \dim(\mathbb{P}^1(k)) = 1$. This exists by question 4.6. Since $\dim(\mathbb{P}^2(k)) = 2$ we then actually have $\dim(f^{-1}(y_1)) = \dim(f^{-1}(y_2)) = 1$. Let C_1 (resp. C_2) be an irreducible component of $\dim(f^{-1}(y_1))$ (resp. $\dim(f^{-1}(y_2))$) such that $\dim(C_1) = \dim(C_2) = 1$. We have $\dim(C_1) + \dim(C_2) - 2 = 0$ and so by Proposition 11.2 we have $C_1 \cap C_2 \neq \emptyset$. This is a contradiction.

(2) Let $n \geq 2$. First note that $\mathbb{P}^2(k)$ is isomorphic to the closed subvariety $Z(x_3, x_4, \dots, x_n)$ of $\mathbb{P}^n(k)$. To see this note that the image of the morphism $\iota : \mathbb{P}^2(k) \rightarrow \mathbb{P}^n(k)$ given by the formula

$$[X_0, X_1, X_2] \mapsto [X_0, X_1, X_2, 0 \dots ((n-2)\text{-times}) \dots, 0]$$

is $Z(x_3, x_4, \dots, x_n)$. This morphism is an isomorphism onto $Z(x_3, x_4, \dots, x_n)$ because the morphism

$$\mathbb{P}^n(k) \setminus Z(x_0, x_1, x_2) \rightarrow \mathbb{P}^2(k)$$

given by the formula

$$[X_0, X_1, X_2, \dots, X_n] \mapsto [X_0, X_1, X_2]$$

gives an inverse to ι when restricted to $Z(x_3, x_4, \dots, x_n)$.

Let now $f : \mathbb{P}^n(k) \rightarrow \mathbb{P}^1(k)$ be a morphism. Suppose for contradiction that f is not constant. Let $\bar{v}_1, \bar{v}_2 \in \mathbb{P}^n(k)$ be two points such that $f(\bar{v}_1) \neq f(\bar{v}_2)$. Let M be an invertible $(n+1) \times (n+1)$ -matrix such that $M([1, 0, 0, \dots, 0]) = \bar{v}_1$ and $M([0, 1, 0, 0, \dots, 0]) = \bar{v}_2$. Let $\phi_M : \mathbb{P}^n(k) \rightarrow \mathbb{P}^n(k)$ be the automorphism defined by M (see question 2.8). The morphism $f \circ \phi_M \circ \iota : \mathbb{P}^2(k) \rightarrow \mathbb{P}^1(k)$ is then not constant, which is a contradiction by (1).