C8.4 Probabilistic Combinatorics

Sheet 0 — HT26

Not for classes

This is a preliminary problem sheet, to get the ball rolling. There is a 'bonus' problem for MFoCS students overleaf. Solutions will be put on the website near the end of week 1. Problem sheet 1 (based on the first two weeks' lectures) will be for the first class.

Estimates and asymptotics, union bound and first-moment method

1. Prove the following inequalities:

   (a) $1 + x \leqslant e^x$ for all real x.

   (b) $e^{nx/(1+x)} \leqslant (1+x)^n \leqslant e^{nx}$ for $x > -1$, $n \geqslant 0$.

   (c) $k! \geqslant (k/e)^k$ for $k \geqslant 1$.

   (d) $\left(\frac{n}{k}\right)^k \leqslant \binom{n}{k} \leqslant \frac{n^k}{k!} \leqslant \left(\frac{en}{k}\right)^k$ for $1 \leqslant k \leqslant n$.

   Solution: (a) $f(x) = e^x - 1 - x$ is increasing for $x \geqslant 0$ as $f'(x) = e^x - 1 \geqslant 0$, decreasing for $x \leqslant 0$ as $f'(x) \leqslant 0$, and $f(0) = 0$. Hence $f(x) \geqslant 0$ for all x.

   (b) $0 < 1 + x \leqslant e^x$ implies $(1+x)^n \leqslant e^{nx}$ and $(1+x)^n = 1/(1 - x/(1+x))^n \geqslant 1/e^{-nx/(1+x)} = e^{nx/(1+x)}$.

   (c) Expand $e^k = 1 + k + \frac{k^2}{2!} + \cdots + \frac{k^k}{k!} + \cdots \geqslant \frac{k^k}{k!}$. Hence $k! \geqslant k^k/e^k = (k/e)^k$.

   (d) We have $\binom{n}{k} = \frac{n}{k} \cdot \frac{n-1}{k-1} \cdots \frac{n-k+1}{1} \geqslant \left(\frac{n}{k}\right)^k$ as $\frac{n-i}{k-i} \geqslant \frac{n}{k}$ for $0 \leqslant i < k \leqslant n$. Also $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \leqslant \frac{n^k}{k!}$ and $\frac{n^k}{k!} \leqslant \frac{n^k}{(k/e)^k} = \left(\frac{en}{k}\right)^k$ by (c).

2. For the following functions $f(n)$ and $g(n)$, decide whether $f = o(g)$ or $g = o(f)$ or $f = \Theta(g)$ as $n \to \infty$:

   (a) $f(n) = \binom{n}{k}$, $g(n) = n^k$, first for k fixed and then for the case where $k = k(n) \to \infty$ as $n \to \infty$;

   (b) $f(n) = (\log n)^{1000}$, $g(n) = n^{1/1000}$.

   Solution:
   (a) For k fixed, $f(n) = \Theta(g(n))$ (as $f(n) \leqslant g(n)$ and $f(n) \geqslant g(n)/k^k$ for $n \geqslant k$ by 1(d)). For $k = k(n) \to \infty$, $f(n) = o(g(n))$ (as $f(n) \leqslant g(n)/k!$).

   (b) $f(n) = o(g(n))$. This is just a version of 'exponentials beat powers'.

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3. Find the simplest function $f(n)$ you can such that $(n-2)^{n+2}/n^n \sim f(n)$ as $n \to \infty$.

   Solution: Write $(n-2)^{n+2}/n^n = (1-\frac{2}{n})^n(n-2)^2 \sim e^{-2}n^2$, so $f(n) = (n/e)^2$.

4. Show that if $n, k, \ell \geqslant 1$ are integers and $0 < p < 1$, then

$$R(k,\ell) > n - \binom{n}{k}p^{\binom{k}{2}} - \binom{n}{\ell}(1-p)^{\binom{\ell}{2}}.$$

   Solution: Colour the edges of K_n independently, each red with probability p and blue otherwise. Let X be the number of red K_ks and Y the number of blue K_ℓs. Then

$$\mathbb{E}[X+Y] = \mu := \binom{n}{k}p^{\binom{k}{2}} + \binom{n}{\ell}(1-p)^{\binom{\ell}{2}}.$$

   Now use the fact that $\mathbb{P}(X+Y \leqslant \mathbb{E}[X+Y]) > 0$ to show that there is a colouring with $X+Y \leqslant \mu$. Deleting one vertex from each monochromatic clique gives a colouring of $K_{n'}$ with no red K_k or blue K_ℓ and $n' \geqslant n-\mu$. Thus $R(k,\ell) > n' \geqslant n-\mu$ as required. [Note that it's not enough to argue that the events $A = \{X \leqslant \mathbb{E}[X]\}$ and $B = \{Y \leqslant \mathbb{E}[Y]\}$ both have positive probability.]

5. Let H be an r-uniform hypergraph with fewer than $\frac{3^{r-1}}{2^r}$ edges. Prove that the vertices of H can be coloured using three colours in such a way that in each edge, all three colours are represented.

   Solution: Pick a 3-colouring of the vertices uniformly at random. Call an edge e *bad* if e gets at most 2 colours. Then $\mathbb{P}(e \text{ is bad}) \leqslant 3\,(\frac{2}{3})^r$, and so the expected number of bad edges is < 1. (The result is true even if r is 1 or 2, since then H has no edges.)

6. Let F be a collection of binary strings ("codewords") of finite length, where the ith codeword has length c_i. Suppose that no member of F is an initial segment of another member (so you can decode any string made up by concatenating codewords as you go along, without looking ahead). Show that $\sum_i 2^{-c_i} \leqslant 1$ (the *Kraft inequality* for prefix-free codes).

   Solution: Consider an infinite binary string in which each bit is 0 or 1 independently with probability 1/2. Let A_i be the event that the initial c_i bits form the ith codeword. Then $\mathbb{P}(A_i) = 2^{-c_i}$. But the events are disjoint as no codeword is an initial segment of another. Thus $\sum_i 2^{-c_i} = \mathbb{P}(\text{string starts with some codeword}) \leqslant 1$.

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Bonus question (for MFoCS students, optional for others):

A (finite, or infinite and convergent) sum $S = \sum_{i\geqslant 0} a_i$ is said to *satisfy the alternating inequalities* if the partial sum $\sum_{i=0}^{t} a_i$ is at least S for all even t and at most S for all odd t; that is, the partial sums alternately over- and under-estimate the final result.

7. Let $\mathbb{1}_1, \dots, \mathbb{1}_n$ be the indicator functions of n events $E_1, \dots, E_n$. For $0 \leqslant r \leqslant n$ let $S_r = \sum_{A\subseteq[n],\,|A|=r} \prod_{i\in A} \mathbb{1}_i$, where $[n] = \{1, 2, \dots, n\}$. Show that

$$\prod_{i=1}^{n}(1 - \mathbb{1}_i) = \sum_{r=0}^{n}(-1)^r S_r, \tag{1}$$

and that the sum satisfies the alternating inequalities. [Both sides are random; the statement is that the relevant inequalities *always* hold. You may want to consider different cases according to how many of the events E_i hold.] Deduce that

$$\mathbb{P}(\text{no } E_i \text{ holds}) = \sum_{r=0}^{n}(-1)^r \sum_{A\subseteq[n],\,|A|=r} \mathbb{P}\Big(\bigcap_{i\in A} E_i\Big), \tag{2}$$

and that the sum satisfies the alternating inequalities. [This is a form of the inclusion–exclusion formula.]

Solution: Let K be the set of i such that A_i holds, and let $k = |K|$. Then the RHS of (1) is

$$\sum_{r=0}^{k}(-1)^r S_r = \sum_{r=0}^{k}(-1)^r \sum_{A\subseteq K,\,|A|=r} 1 = \sum_{r=0}^{k}(-1)^r \binom{k}{r} = (1-1)^k,$$

which is zero unless $k = 0$, when it is 1. Thus (1) holds. Taking expectations of both sides gives (2).

For the alternating inequalities, again consider the RHS in (1). Arguing as before, it suffices to check alternating inequalities for $\sum_{r\geqslant 0}(-1)^r\binom{k}{r}$. If $k = 0$, the LHS is 1 and $\sum_{r=0}^{m}$ is 1 for each $m \geqslant 0$. Suppose that $k \geqslant 1$, so the LHS is 0. If $m \geqslant k$ then $\sum_{r=0}^{m}(-1)^r\binom{k}{r} = 0$. **Method 1.** Let $0 \leqslant m \leqslant (k+1)/2$. For $r \leqslant (k+1)/2$, $\binom{k}{r}$ increases, and so $\sum_{r=0}^{m}(-1)^r\binom{k}{r}$ is $\geqslant 0$ for m even and $\leqslant 0$ for m odd, as required.

Let $(k+1)/2 < m < k$. We may use

$$\sum_{r=0}^{m}(-1)^r\binom{k}{r} = -\sum_{r=m+1}^{k}(-1)^r\binom{k}{r} = -(-1)^k\sum_{s=0}^{k-m-1}(-1)^s\binom{k}{s}$$

(setting $s = k - r$) to see from the previous case that the alternating inequalities hold for such m.

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Method 2. (The slick way.) Notice that

$$\sum_{r=0}^{m}(-1)^r\binom{k}{r} = (-1)^m\binom{k-1}{m},$$

which easily follows from $\binom{k}{r} = \binom{k-1}{r-1} + \binom{k-1}{r}$.

Either way, we have the alternating inequalities for $\sum_{r\geqslant 0}(-1)^r S_r$ in (1), and taking expectations gives the corresponding result for (2).

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