

# Additional Lecture Notes for W7

**Theorem 1.** Let  $X$  be a reflexive separable Banach space,  $X^*$  its dual, and  $M$  a non-empty, convex closed subset of  $X$ . Let  $F : M \rightarrow X^*$  satisfies

(H1)  $F$  maps bounded sets in  $X$  into bounded sets in  $X^*$ .

(H2) (Coercivity) There exists  $u_0 \in M$  such that

$$\lim_{\|u\| \rightarrow \infty} \frac{\langle F(u), u - u_0 \rangle}{\|u\|} = \infty.$$

(H3) Weak (sequential) lower semi-continuity relative to  $M$ : If  $u_n \rightharpoonup u$  and  $F(u_n) \rightharpoonup \xi$ , then

$$\langle \xi, u \rangle \leq \liminf_{n \rightarrow \infty} \langle F(u_n), u_n \rangle,$$

and if equality holds then

$$\langle F(u) - \xi, u - v \rangle \leq 0 \text{ for all } v \in M.$$

Then there exists  $u \in M$  such that

$$\langle F(u), u - v \rangle \leq 0 \text{ for all } v \in M.$$

We will only present the proof in the case  $M$  is a closed linear subspace of  $X$ . See the lecture notes for the proof in the full generality.

Denote by  $M^\circ \subset X^*$  the annihilator of  $M$ , i.e.,

$$M^\circ = \{f \in X^* : \langle f, v \rangle = 0, \text{ for all } v \in M\}.$$

**Theorem 2.** Let  $X$  be a reflexive separable Banach space,  $X^*$  its dual, and  $M \subset X$  a closed linear subspace of  $X$ . Let  $F : M \rightarrow X^*$  satisfies

(H1)  $F$  maps bounded sets into bounded set.

(H2) (Star-like at infinity): There exists  $u_0 \in M$  and  $K > 0$  such that

$$\langle F(u), u - u_0 \rangle > 0 \text{ if } \|u\| \geq K.$$

(H3) Weak (sequential) lower semi-continuity relative to  $M$ : If  $u_n \rightharpoonup u$  and  $F(u_n) \rightharpoonup \xi$ , then

$$\langle \xi, u \rangle \leq \liminf_{n \rightarrow \infty} \langle F(u_n), u_n \rangle,$$

and if equality holds then

$$F(u) - \xi \in M^\circ.$$

Then there exists  $u \in M$  such that

$$F(u) \in M^\circ \text{ and } \|u\| \leq K.$$

*Proof.* Step 1: Finite dimensional reduction. Suppose that the theorem is established when  $\dim X < \infty$ . We show that the theorem remains true in the general case.

Since  $X$  is separable, there exist linearly independent vectors  $e_1, e_2, \dots$  such that  $X$  is the closed linear span of  $\{e_1, e_2, \dots\}$ . Let  $X_n = \text{Span}(\{e_1, \dots, e_n\})$  and  $M_n = M \cap X_n$ .

For technical reason (see below), we assume that the vector  $u_0$  in (H2) belongs to  $\text{Span}(e_1) = X_1$ .

Noting that, for every  $u \in M$ ,  $F(u)$  is a bounded linear functional on  $X$  and so is also a bounded linear function on  $X_n$ . Hence  $F_n := F|_{M_n} : M_n \rightarrow X_n^*$ . It's easy to see that  $F_n$  satisfies (H1) and (H3). As  $u_0 \in X_1 \subset X_n$ ,  $F_n$  satisfies (H2). We may thus apply the theorem in the finite dimensional case to obtain  $u_n \in M_n$  with  $\|u_n\| \leq K$  such that

$$F(u_n) \in (M_n)^\circ \quad \text{that is } \langle F(u_n), v \rangle = 0 \text{ for all } v \in M_n. \quad (*)$$

Since  $(u_n)$  is bounded, so is  $F(u_n)$  by (H1). Thus, by reflexivity and after passing to a subsequence, we have  $u_n \rightharpoonup u$  (with  $\|u\| \leq K$ ) and  $F(u_n) \rightharpoonup \xi$ . We would like to show that  $F(u) \in M^\circ$ .

By (\*), we have that

$$0 = \lim_{n \rightarrow \infty} \langle F(u_n), v \rangle = \langle \xi, v \rangle \text{ for all } v \in \bigcup_n M_n \text{ and so all } v \in \overline{\bigcup_n M_n} = M,$$

that is  $\xi \in M^\circ$ . In particular, we have  $\langle \xi, u \rangle = 0$ . On the other hand, we have  $\langle F(u_n), u_n \rangle = 0$  by (\*), and so by (H3),

$$\langle \xi, u \rangle \leq \liminf \langle F(u_n), u_n \rangle = 0.$$

Hence

$$\langle \xi, u \rangle = \liminf \langle F(u_n), u_n \rangle = 0,$$

which together with (H3) implies  $F(u) - \xi \in M^\circ$ . Hence  $F(u) = \xi + (F(u) - \xi) \in M^\circ$ .

Step 2: Reduction to Hilbert setting when  $\dim X < \infty$ .

Note that  $F : M \rightarrow X^* \hookrightarrow M^*$ . Thus, we may replace  $X$  by  $M$ , i.e. we may assume  $M = X$ . Notice that  $X^\circ = \{0\}$ .

Note that when  $\dim X$  is finite, weak and strong convergence are equivalent. Hence (H3) reads as follows: If  $u_n \rightarrow u$  and  $F(u_n) \rightarrow \xi$ , then  $F(u) = \xi$ . Note also that, by (H1), for any bounded sequence  $u_n$ ,  $F(u_n)$  is bounded and hence every subsequence of  $(F(u_n))$  has a subsequence which converges to  $F(u)$ . This implies that  $F$  is continuous.

Since  $\dim X$  is finite, we can equip  $X$  with an inner product denoted by a dot, whose norm is denoted  $|\cdot|$ . By Riesz representation theorem, every linear function  $T \in X^*$  can be identified with an element  $t = J(T) \in X$  such that

$$\langle T, x \rangle = t \cdot x.$$

Let  $\tilde{F} = J \circ F : X \rightarrow X$ . Then

(H1)  $\tilde{F}$  maps bounded sets into bounded set.

(H2) (Star-like at infinity): There exists  $u_0 \in X$  and  $K > 0$  such that

$$\tilde{F}(u) \cdot (u - u_0) > 0 \text{ if } \|u\| \geq K.$$

(H3) Continuity:  $F$  is continuous.

We aim to show that there exists  $u \in X$  such that

$$\tilde{F}(u) = 0 \text{ and } \|u\| \leq K.$$

Note that if  $\tilde{F}(u) = 0$ , then by (H2),  $\|u\| \leq K$ . Thus, we only need to find  $u \in X$  such that  $\tilde{F}(u) = 0$ .

Step 3: Proof of the theorem when  $X$  is a finite dimensional Hilbert space.

Consider  $T = Id - \tilde{F}$  and we aim to show that  $T$  has a fixed point. Fix some  $R > 0$  for the moment and let  $P_R$  denote the closest point projection onto  $\bar{B}_R(0)$ , that is

$$P_R(x) = \frac{Rx}{|x|} \text{ if } |x| > R, \quad \text{and} \quad P_R(x) = x, \quad \text{if } |x| \leq R.$$

Then  $P_R \circ T$  is a continuous map from  $\bar{B}_R$  into  $\bar{B}_R$ . By Brouwer's fixed point theorem, there exists  $x_R \in \bar{B}_R$  which is fixed by  $P_R \circ T$ .

Notice that if  $|x_R| < R$  or  $|P_R(T(x_R))| \leq R$ , then  $x_R$  is a fixed point for  $T$  and the proof is complete. d

Suppose for a contradiction that  $|x_R| = R$  for some  $R \geq \max\{|u_0|, K\}$  with

$$x_R = P_R(x_R - \tilde{F}(x_R)) \quad \text{and} \quad |x_R - \tilde{F}(x_R)| > R.$$

It follows that, for any  $v \in \bar{B}_R$  (for the first inequality draw a picture, notice that the angle between  $(x_R - \tilde{F}(x_R)) - x_R$  and  $v - x_R$  is larger than  $\pi/2$ , since  $v$  is an interior point and  $(x_R - \tilde{F}(x_R)) - x_R$  is normal to  $\partial B_R$  outward pointing. See picture in the lecture notes.)

$$0 \geq [(x_R - \tilde{F}(x_R)) - x_R] \cdot (v - x_R) = -\tilde{F}(x_R) \cdot (v - x_R) = \tilde{F}(x_R) \cdot (x_R - v).$$

As  $R \geq |u_0|$ , we may take  $v = u_0$  in the above inequality and obtain a contradiction to (H2), since  $|x_R| \geq K$ .  $\square$