Problem Sheet 2

Section B

QUESTION 3. **Leray-Schauder/Schaefer Theorem.**

Let X be a Banach space.

(a) Prove the following result.
Assume that $T\colon X \to X$ is a compact map with the following property: there exists $R > 0$ such that the statement ($x = \tau Tx$ with $\tau \in [0, 1)$) implies $\|x\|_X < R$. Then T has a fixed point x^* such that $\|x^*\|_X \leq R$.
Hint: Consider the operator

$$\tilde{T}(x) := \begin{cases} Tx & \text{if } \|Tx\|_X \leq R, \\ \frac{R}{\|Tx\|_X} Tx & \text{else} \end{cases}$$

on a suitable domain and prove that it is compact.

(b) Let $T : X \to X$ be a compact map with the following property: there exists $R > 0$ such that $\|Tx - x\|_X^2 \geq \|Tx\|_X^2 - \|x\|_X^2$ for all $\|x\|_X \geq R$. Show that T admits a fixed point.

Solution. Part (a). Following the hint, consider the operator

$$\tilde{T}(x) := \begin{cases} Tx & \text{if } \|Tx\|_X \leq R, \\ \frac{R}{\|Tx\|_X} Tx & \text{else.} \end{cases}$$

By construction, we have that $\tilde{T} : X \to \overline{B_R(0)}$. In particular, $\tilde{T} : \overline{B_R(0)} \to \overline{B_R(0)}$.

Claim: $\tilde{T} : \overline{B_R(0)} \to X$ is a compact operator.

We first observe that $\tilde{T} : \overline{B_R(0)} \to X$ is continuous, as T is continuous, $\frac{R}{\|Tx\|_X} Tx$ is continuous for $Tx \in X \setminus \overline{B_R(0)}$, and the two maps agree for $\|Tx\|_X = R$.

Let $(x_j)_j \subset \overline{B_R(0)}$. Since by assumption T is a compact operator, the sequence $(Tx_j)_j$ has a converging sub-sequence $(Tx_{j_k})_k$. It easily seen that $(\tilde{T}x_{j_k})_k$ converges as well. The claim is proved.

Since $\tilde{T} : \overline{B_R(0)} \to X$ is a compact operator, from the 3rd formulation of Schauder's fixed point Theorem we infer that there exists a fixed point

$$x^* \in \overline{B_R(0)}, \quad \tilde{T}x^* = x^*.$$

Claim: $Tx^* = x^*$.

By the explicit expression of $\tilde{T}$, if it is not true that $x^* = \tilde{T}x^* = Tx^*$, then $x^* = \tilde{T}x^* = \tau Tx^*$ for some $\tau \in (0, 1]$ and $\|\tilde{T}x^*\|_X = R$. Using the assumption on T, we infer that if the latter holds, then $\|x^*\|_X < R$. We thus get the contradiction:

$$R = \|\tilde{T}x^*\|_X = \|x^*\|_X < R,$$

proving the claim.

Part (b). In order to apply part (a) we claim that:

Claim: if $x = \tau Tx$ with $\tau \in [0, 1)$, then $\|x\|_X < R$.

If $x = \tau Tx$ then either $\tau = 0$ and thus $\|x\|_X = 0 < R$, or $\tau \in (0, 1)$ and thus

$$\begin{aligned} \|Tx - x\|_X^2 &= \|Tx - \tau Tx\|_X^2 = (1 - \tau)^2 \|Tx\|_X^2 = (1 - 2\tau + \tau^2) \|Tx\|_X^2 \\ &< (1 - \tau^2) \|Tx\|_X^2 = \|Tx\|_X^2 - \|\tau Tx\|_X^2 \\ &= \|Tx\|_X^2 - \|x\|_X^2 . \end{aligned}$$

Using the assumption on T we infer that $\|x\|_X < R$, proving the claim.

We conclude by applying part (a). □

QUESTION 4. **Integral operators on $L^2(\Omega)$ vs. $C(\bar{\Omega})$** As always, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain.

(a) Let $a : \bar{\Omega} \times \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a continuous map, and let

$$A(u)(x) = \int_\Omega a(x, y, u(y))dy.$$

show that $A : C(\bar{\Omega}) \to C(\bar{\Omega})$ is well defined and compact. *Hint: use Arzela-Ascoli Theorem.*

(b) Let $k \in L^2(\Omega \times \Omega)$ and define

$$(Ku)(x) = \int_\Omega k(x, y)u(y)dy.$$

Show that $K : L^2(\Omega) \to L^2(\Omega)$ is well defined and compact. You can use for example that $C_0^\infty(\Omega \times \Omega)$ is dense in $L^2(\Omega \times \Omega)$, and therefore there is a sequence $k_m \in C_0^\infty(\Omega \times \Omega)$ such that $k_m \to k$ in $L^2(\Omega)$.

(c) Give an example of continuous a such that A (defined as above) is not well defined as an operator from $L^2(\Omega) \to L^2(\Omega)$.

Solution of (a).

Claim. $x \mapsto A(u)(x)$ is continuous from $\bar{\Omega}$ to $\mathbb{R}$.

By assumption $a : \bar{\Omega} \times \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous. If $u : \bar{\Omega} \to \mathbb{R}$ is continuous, then also $(x, y) \mapsto a(x, y, u(y))$ is continuous from $\bar{\Omega} \times \bar{\Omega} \to \mathbb{R}$. Since $\bar{\Omega} \times \bar{\Omega}$ is compact, we infer that $(x, y) \mapsto a(x, y, u(y))$ is uniforrmly continuous, impliying that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} |A(u)(x_1) - A(u)(x_2)| &\leq \int_{\bar{\Omega}} |a(x_1, y, u(y)) - a(x_2, y, u(y))|dy \\ &\leq |\Omega| \, \varepsilon, \quad \text{for all } x_1, x_2 \in \bar{\Omega} \text{ with } |x_1 - x_2| \leq \delta. \end{aligned} \tag{1}$$

This shows the first claim and thus the fact that $A : C(\bar{\Omega}) \to C(\bar{\Omega})$ is well defined.

Claim. For every bounded subset $\mathcal{M} \subset C(\bar{\Omega})$, the image $A(\mathcal{M})$ is precompact.

In order to show the claim, it is enough to show that $A(\mathcal{M})$ is bounded and equi-continuous, as the pre-compactness would then follow from Arzelá-Ascoli's Theorem.

We first show that $A(\mathcal{M})$ is bounded. Let $C = \sup_{u \in \mathcal{M}} \|u\|_{C^0}$. We have

$$\begin{aligned} \|A(u)\|_{C^0} &= \sup_{x \in \bar{\Omega}} |A(u)(x)| \leq \sup_{x \in \bar{\Omega}} \int_\Omega |a(x, y, u(y))|dy \\ &\leq |\Omega| \sup_{(x,y,z) \in \bar{\Omega} \times \bar{\Omega} \times [-C,C]} |a(x, y, z)| =: \bar{C} < \infty, \quad \forall u \in \mathcal{M}. \end{aligned}$$

We next show that $A(\mathcal{M})$ is equi-continuous. Since $(x, y, z) \mapsto a(x, y, z)$ is uniformly continuous on the compact set $\bar{\Omega} \times \bar{\Omega} \times [-C, C]$, the same estimates as in (1) give that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|A(u)(x_1) - A(u)(x_2)| \leq |\Omega| \, \varepsilon, \quad \text{for all } u \in \mathcal{M} \text{ and all } x_1, x_2 \in \bar{\Omega} \text{ with } |x_1 - x_2| \leq \delta.$$

Solution of (b). This can be either approached directly, working with $k \in L^2(\Omega \times \Omega)$ and arguing with Fubini-Tonelli's theorem (for the good definition and continuity of K, and using Kolmogorov-Riesz for the compactness of the map K), or arguing by approximation as suggested in the hint. We take the second approach (the advantage is that, for proving the compactness, we will only need Arzelá-Ascoli and not Kolmogorov-Riesz).

Let $k_m \in C_0^\infty(\Omega \times \Omega)$ such that $k_m \to k$ in $L^2(\Omega)$. For all $u \in L^2(\Omega)$, define

$$(K_m u)(x) = \int_\Omega k_m(x, y)u(y)dy.$$

Since $k_m(x, \cdot) \in L^2(\Omega)$, we have that $(K_m u)(x)$ is well defined for all $x \in \Omega$.

Claim 1. $K_m u \in L^2(\Omega)$ for all $u \in L^2(\Omega)$ and $K_m : L^2(\Omega) \to L^2(\Omega)$ is continuous. Using Cauchy-Schwarz and Fubini-Tonelli, we get

$$\begin{aligned} \|K_m u\|_{L^2}^2 &= \int_\Omega \left(\int_\Omega k_m(x,y)u(y)dy \right)^2 dx \le \int_\Omega \|k_m(x,\cdot)\|_{L^2}^2 \|u\|_{L^2}^2 dx \\ &= \|u\|_{L^2}^2 \int_\Omega \left(\int_\Omega k_m(x,y)^2 dy \right) dx = \|u\|_{L^2}^2 \int_{\Omega\times\Omega} k_m(x,y)^2 dxdy \\ &= \|u\|_{L^2}^2 \|k_m\|_{L^2}^2. \end{aligned} \tag{2}$$

This shows that $K_m u \in L^2$. The operator $K_m : L^2(\Omega) \to L^2(\Omega)$ is clearly linear, thus the above estimate also shows the continuity of K_m as endomorphism of L^2. The proof of the claim is complete.

From (2), it also follows that

$$\|K_m u - K_n u\|_{L^2}^2 \le \|u\|_{L^2}^2 \|k_m - k_n\|_{L^2}^2 \to 0 \quad \text{as } m, n \to \infty,$$

i.e. the sequence $(K_m u)_m$ is Cauchy in L^2 and thus converges. Define

$$Ku := \lim_{m\to\infty} K_m u.$$

We next establish compactness of K by first proving the compactness of K_m and then applying a diagonal argument.

Claim 2. Fix $m \in \mathbb{N}$. For any bounded sequence $(u_n)_n \subset L^2(\Omega)$ there exists a subsequence u_{n_j} such that $(K_m u_{n_j})_j \subset L^2$ converges.
Since $k_m \in C_0^\infty(\Omega \times \Omega)$, we get that $(K_m u_n)_n$ are uniformly bounded:

$$|K_m u_n(x)| \le \|k_m\|_{L^\infty} \|u_n\|_{L^1} \le \|k_m\|_{L^\infty} |\Omega|^{1/2} \|u_n\|_{L^2}, \quad \text{for all } x \in \Omega$$

and equicontinuous, as k_m is uniformly continuous:

$$\begin{aligned} \sup_n \sup_{|x_1-x_2|<\delta} |K_m u_n(x_1) - K_m u_n(x_2)| &\le |\Omega|^{1/2} \sup_n \|u_n\|_{L^2} \sup_{y\in\Omega, |x_1-x_2|<\delta} |k_m(x_1,y) - k_m(x_2,y)| \\ &\to 0 \quad \text{as } \delta \to 0, \quad \text{for fixed } m. \end{aligned}$$

Thus the sequence $(K_m u_n)_n$ is pre-compact in $C^0(\Omega)$ by Arzelá-Ascoli Theorem, i.e. it admits a subsequence which converges uniformly, hence in particular in L^2, showing the claim.

By a diagonal argument, from claim 2, it follows that for any bounded sequence $(u_n)_n \subset L^2(\Omega)$ there exists a subsequence u_{n_j} such that $(K_m u_{n_j})_j \subset L^2$ converges for every $m \in \mathbb{N}$.
We now show the compactness of the map K:

Claim 3. For any bounded sequence $(u_n)_n \subset L^2(\Omega)$ there exists a subsequence u_{n_j} such that $(Ku_{n_j})_j \subset L^2$ converges.
Let $(u_{n_j})_j$ be the subsequence such that $(K_m u_{n_j})_j \subset L^2$ converges for every $m \in \mathbb{N}$. Then

$$\begin{aligned} \|Ku_{n_{j_1}} - Ku_{n_{j_2}}\|_{L^2} &\le \|Ku_{n_{j_1}} - K_m u_{n_{j_1}}\|_{L^2} + \|K_m u_{n_{j_1}} - K_m u_{n_{j_2}}\|_{L^2} + \|K_m u_{n_{j_2}} - Ku_{n_{j_2}}\|_{L^2} \\ &\le 2 \sup_n \|u_n\|_{L^2} \|k - k_m\|_{L^2(\Omega\times\Omega)} + \|K_m(u_{n_{j_1}} - u_{n_{j_2}})\|_{L^2}. \end{aligned}$$

For every $\varepsilon > 0$ let m be such that $\|k - k_m\|_{L^2(\Omega\times\Omega)} \le \varepsilon/(2\sup_n \|u_n\|_{L^2})$. Let also $J > 0$ be such that $\|K_m(u_{n_{j_1}} - u_{n_{j_2}})\|_{L^2} < \varepsilon/2$ for all $j_1, j_2 \ge J$. Then $\|Ku_{n_{j_1}} - Ku_{n_{j_2}}\|_{L^2} < \varepsilon$ for all $j_1, j_2 \ge J$, i.e. it is a Cauchy sequence. The claim follows.

Note. The argument above is very similar to an argument in the lectures, where compact operators are approximated by "finite dimensional operators".

Solution of (c). Let $a(x,y,z) = z^4$. If $u \in L^2$, it is in general not true that $\int_\Omega u^4$ exists finite and thus the corresponding Au may fail to be well defined as an L^2 function.

□

QUESTION 5. **Continuous maps.** Let $g \in C(\mathbb{R} \times \mathbb{R}^n)$ be such that $g(z,p) \leq a + b|z|^\alpha + c|p|$, where a,b and c are non negative constants, and $2\alpha < 2^*$, where $2^* = 2n/(n-2)$ if $n \geq 3$, and $2^* = \infty$ if $n = 1, 2$. Then the map $u \mapsto g(u, \nabla u)$ is continuous from $H_0^1(\Omega)$ to $L^2(\Omega)$ and maps bounded subsets of $H_0^1(\Omega)$ to bounded subsets of $L^2(\Omega)$.

Hint: rewrite $g(u, \nabla u) = \tilde{g}\big(u, \frac{\nabla u}{|\nabla u|^\nu}\big)$ *for a suitable function* $\tilde{g}$ *and a suitable exponent* $0 < \nu < 1$, *and apply Lemma 2.6 from the lecture notes.*

Solution. From Lemma 2.6 in the lecture notes, we know that if $f \in C(\mathbb{R})$ satisfies

$$|f(x)| \leq M_1 + M_2|x|^r, \quad \forall x \in \mathbb{R}^n$$

then the map $u \mapsto f(u)$ is well defined and continuous from L^p to $L^{p/r}$, and maps bounded sets to bounded sets.

If $\alpha \leq 1$ and $g \in C(\mathbb{R} \times \mathbb{R}^n)$ satisfies $g(z,p) \leq a + b|z|^\alpha + c|p|$, then the claim follows immediately, as $b|z|^\alpha \leq b(|z|+1)$ and as the map $u \mapsto \nabla u$ is continuous from H_0^1 to L^2.

If $\alpha > 1$ then, using that by assumption $2\alpha < 2^*$ and Sobolev embedding theorem, we get that $u \mapsto u$ is continuous as a map from H_0^1 to $L^{2\alpha}$.
At the same time, consider the map

$$v \mapsto \frac{v}{|v|^\nu}, \quad 0 < \nu < 1, \tag{3}$$

which is well defined and continuous from $L^2(\Omega, \mathbb{R}^n)$ to $L^{2/(1-\nu)}(\Omega, \mathbb{R}^n)$ and maps bounded sets to bounded sets, by Lemma 2.6. Hence, choosing ν so that $\frac{2}{1-\nu} = 2\alpha$, we get that (3) is continuous from $L^2(\Omega, \mathbb{R}^n)$ to $L^{2\alpha}(\Omega, \mathbb{R}^n)$.

Combining the above, we obtain that

$$h : \ u \mapsto \left(u, \frac{\nabla u}{|\nabla u|^\nu}\right) \tag{4}$$

is well defined and continuous from $H_0^1(\Omega)$ to $L^{2\alpha}(\Omega, \mathbb{R} \times \mathbb{R}^n)$. Following the hint, consider the function $\tilde{g} \in C(\mathbb{R} \times \mathbb{R}^n)$ such that

$$g(u, \nabla u) = \tilde{g}\left(u, \frac{\nabla u}{|\nabla u|^\nu}\right),$$

(i.e., $\tilde{g}(v,w) := g(v, |w|^\beta w)$, with $\beta = \nu/(1-\nu)$). Then

$$\begin{aligned} |\tilde{g}(v,w)| &\leq a + b|v|^\alpha + c|w|^{1/(1-\nu)} \\ &= a + b|v|^\alpha + c|w|^\alpha. \end{aligned}$$

We conclude that

$$H_0^1(\Omega) \xrightarrow{h} L^{2\alpha}(\Omega, \mathbb{R} \times \mathbb{R}^n) \xrightarrow{\tilde{g}} L^2(\Omega)$$

is well defined, continuous, and maps bounded sets to bounded sets, as composition of maps with such properties.

□