Geometric Group Theory

Cornelia Druţu

University of Oxford

Part C course HT 2025

Graphs of groups and actions on trees

Theorem

$H = \pi_1(G, Y, a_0)$ acts on a tree T without inversions and such that

1. *The quotient graph $H\backslash T$ can be identified with Y;*
2. *Let $q : T \to Y$ be the quotient map:*
   - a. *For all $v \in V(T)$, $\mathrm{Stab}_H(v)$ is a conjugate in H of $G_{q(v)}$;*
   - b. *For all $e \in E(T)$, $\mathrm{Stab}_H(e)$ is a conjugate in H of $G_{q(e)}$.*

Conversely, if a group Γ acts on a tree T with quotient Y then there exists a graph of groups (G, Y) such that $\Gamma \simeq \pi_1(G, Y, a_0)$.

Graphs of groups and actions on trees

Conversely, if a group Γ acts on a tree T with quotient Y then there exists a graph of groups (G, Y) such that $\Gamma \simeq \pi_1(G, Y, a_0)$.

Indeed, suppose $\Gamma \curvearrowright T$, $Y = \Gamma\backslash T$ and $p : T \to Y$.

Let $X \subset S \subset T$ be such that $p(X)$ is a maximal tree of Y, $p(S) = Y$ and $p\big|_{\text{edges of } S}$ is 1-to-1.

Notation: If v is a vertex of Y and e is an edge of Y, let

- v^X be the vertex of X such that $p(v^X) = v$;
- e^S be the edge of S such that $p(e^S) = e$.

We define a graph of groups with graph Y:

1. The map G:
    - Let $G_v = \mathrm{Stab}_\Gamma(v^X)$;
    - Let $G_e = \mathrm{Stab}_\Gamma(e^S)$.

Graphs of groups and actions on trees

1. The map G:
   - Let $G_v = \mathrm{Stab}_\Gamma(v^X)$;
   - Let $G_e = \mathrm{Stab}_\Gamma(e^S)$.
2. For each edge e, we define $\alpha_e : G_e \to G_{t(e)}$: For all $x \in V(S)$, define

$$g_x = \begin{cases} 1 & \text{if } x \in V(X) \\ \text{some } g_x \text{ such that } g_x x \in V(X) & \textit{otherwise}. \end{cases}$$

   Define $\alpha_e : G_e \to G_{t(e)}, \alpha_e(g) = g_{t(e)} g g_{t(e)}^{-1}$.

We can define a homomorphism $\varphi : F(G, Y) \to \Gamma$ by:

- $\forall a \in V(Y)$, $\varphi\big|_{G_a} = \mathrm{incl}_{G_a}$;
- $\forall e \in E(Y)$, $e = [y, x]$, $\varphi(e) = g_y g_x^{-1}$.

We can define a homomorphism $\varphi : F(G,Y) \to \Gamma$ by:

- $\forall a \in V(Y)$, $\varphi\big|_{G_a} = \mathrm{incl}_{G_a}$;
- $\forall e \in E(Y)$, $e = [y,x]$, $\varphi(e) = g_y g_x^{-1}$.

It satisfies the relations:

$$\varphi(\bar{e}) = g_x g_y^{-1} = (g_y g_x^{-1})^{-1} = \varphi(e)^{-1}$$

$$\varphi(e\alpha_e(g)e^{-1}) = (g_y g_x^{-1})(g_x g g_x^{-1})(g_x g_y^{-1}) = g_y g g_y^{-1} = \varphi(\alpha_{\bar{e}}(g))$$

Also, $\forall e \in p(X)$, $\varphi(e) = 1$. Hence, φ defines a homomorphism

$$\bar{\varphi} : \pi_1(G,Y,p(X)) \simeq \pi_1(G,Y,a_0) \to \Gamma$$

Graphs of groups and actions on trees

Hence, φ defines a homomorphism

$$\bar{\varphi} : \pi_1(G, Y, p(X)) \simeq \pi_1(G, Y, a_0) \to \Gamma$$

Theorem

The homomorphism $\bar{\varphi}$ is an isomorphism. If $\widetilde{\mathcal{T}} = \mathcal{T}(G, Y, a_0)$ is the universal covering tree of (G, Y) then there exists a graph isomorphism $f : \widetilde{\mathcal{T}} \to T$ such that $\forall g \in \pi_1(G, Y, a_0)$, $\forall v \in V(\widetilde{\mathcal{T}})$,

$$f(g \cdot v) = \bar{\varphi}(g) \cdot f(v).$$

Proof: Not provided and non-examinable.

Subgroups

Theorem

Let $\Gamma = \pi_1(G, Y, a_0)$. If $B \leq \Gamma$ then there exists (H, Z) a graph of groups such that $B = \pi_1(H, Z, b_0)$ and

- *for all $v \in V(Z)$, $H_v \leq gG_a g^{-1}$ for some $a \in V(Y)$, $g \in \Gamma$;*
- *for all $e \in E(Z)$, $H_e \leq \gamma G_y \gamma^{-1}$, for some $y \in E(Y)$, $\gamma \in \Gamma$.*

Proof.

Γ acts on a tree T with quotient a graph of groups (G, Y). The subgroup B acts on T, $\mathrm{Stab}_B(v) \leq \mathrm{Stab}_\Gamma(v)$ for all $v \in V(T)$ and $\mathrm{Stab}_B(e) \leq \mathrm{Stab}_\Gamma(e)$ for all $e \in E(T)$. □

NB It may be that, while Y is finite, Z is infinite.

Subgroups

Theorem (Kurosh)

Suppose $G = G_1 * \dots * G_n$. *If* $H \leq G$ *then*

$$H = (*_{i \in I} H_i) * F$$

where I *is finite or countable,* F *is a free group and the* H_i *are subgroups of conjugates of* G_j.

Unique decomposition I

We say that G is indecomposable if $G \neq A * B$.

Theorem (Grushko)

Suppose G is finitely generated. There exists indecomposable $G_1, ..., G_k$ such that

$$G = G_1 * ... * G_k * F_n$$

Moreover, if there exist other indecomposable $H_1, ..., H_m$ such that

$$G = H_1 * ... * H_m * F_r$$

then $m = k$, $r = n$ and, after reordering, H_i is conjugate to G_i for all i.

Unique decomposition II

Theorem (Dunwoody)

Suppose Γ is finitely presented. Then Γ can be written as $\pi_1(G, Y, a_0)$ where (G, Y) is a finite graph of groups such that all edge groups are finite and all the G_v do not split over finite groups.

Theorem (Stallings)

A group Γ does not split over finite groups if and only if it is one-ended.

A group Γ is one-ended if any (every) Cayley graph cannot be disconnected by removing a compact subset.

Quasi-isometry

Definition

Let $f : X \to Y$ be a map between metric spaces.

1. We say that f is an (L, A)-quasi-isometric embedding if for some constants $L \geq 1$, $A \geq 0$ and for all $x_1, x_2 \in X$ we have

$$\frac{1}{L}d(x_1, x_2) - A \leq d(f(x_1), f(x_2)) \leq Ld(x_1, x_2) + A$$

   It is called a quasi-isometry if moreover we have that for all $y \in Y$, there exists some $x \in X$ such that $d(y, f(x)) \leq A$.

2. If $I \subseteq \mathbb{R}$ is an interval, then an (L, A)-quasi-isometric embedding $\gamma : I \to X$ is called an (L, A)-quasi-geodesic.
3. If there exists a quasi-isometry $f : X \to Y$ between two metric spaces then we say that X and Y are quasi-isometric.

Quasi-isometry

Examples

1. *$\mathbb{Z}^2$ and $\mathbb{R}^2$ are quasi-isometric.*
2. *If G is a finitely generated group with finite generating sets S, S' then the Cayley graphs $\Gamma(S, G)$, $\Gamma(S', G)$ are quasi-isometric.*
3. *If T_n is the n-valent tree, then $T_n \sim T_3$ for all $n \in \mathbb{N}$.*

The following theorem implies the first example above and is our main source of quasi-isometries.

Quasi-isometry

The following theorem implies the first example above and is our main source of quasi-isometries.

Theorem (Milnor–Švarc)

Suppose G acts by isometries on a metric space X such that

1. a. *X is geodesic;*
   b. *X is proper (closed balls are compact);*
2. *the action is*
   a. *properly discontinuous: i.e. given a compact $K \subseteq X$, the set $\{g \in G : g(K) \cap K \neq \emptyset\}$ is finite;*
   b. *cocompact: i.e. there exists a compact $K' \subseteq X$ such that $GK' = X$;*

then G is finitely generated and every orbit map $G \to X$, $g \mapsto g \cdot x_0$ is a quasi-isometry when G is endowed with a word metric.

Proof is non-examinable.