Geometric Group Theory

Problem Sheet 4

Section A

1. i) Show that the relation of quasi-isometry of metric spaces $\sim$ is an equivalence relation.

ii) Let S_1, S_2 be finite generating sets of a group G. Show that $\Gamma(S_1, G) \sim \Gamma(S_2, G)$.

Solution. i) Let $f : X \to Y$ a (K, A)-quasi-isometry. Define a 'quasi-inverse' $g : Y \to X$ as follows: Given $y \in Y$ pick $x \in X$ such that $d(y, f(x)) \leq A$. Define $g(y) = x$. Then g is also a quasi-isometry: Let $x \in X$ and $y = f(x)$ then $g(y) = x_1$ for some x_1 for which $d(f(x), f(x_1)) \leq A$. So $d(x, x_1) \leq KA + A$.

It is clear that $X \sim Y, Y \sim Z$ implies $X \sim Z$ as the composition of quasi-isometries is a quasi-isometry.

ii) We consider the identity map on the vertices $f : \Gamma(G, S_1) \to \Gamma(G, S_2)$. We can write each element of S_1 as a word on S_2 and each element of S_2 as a word on S_1. The maximum length of all these words controls the quasi-isometry constants.

2. Show that the groups $\mathbb{Z}^2$ and $\mathbb{Z}^3$ are not quasi-isometric (*hint*: growth)

Solution. Let Γ_1, Γ_2 be respectively the Cayley graphs of $\mathbb{Z}^2$, $\mathbb{Z}^3$ with respect to the standard generators. Then the ball or radius n, $B^1(n)$ of Γ_1 has less than $4n^2$ vertices while the ball of radius n, $B^2(n)$ of Γ_2 has more than n^3 vertices. Let $f : \Gamma_2 \to \Gamma_1$ be a quasi-isometry. Without loss of generality we may assume that f maps vertices to vertices.

There is some constant $C > 0$ such that $f(B^2(n))$ is contained in a ball $B^1(Cn)$. Also there is a contant D so that at most D distinct vertices can have the same image under f. It follows that $f(B^2(n))$ has at least n^3/D vertices. On the other hand $B^1(Cn)$ has less than $4Cn^2$ vertices. So we get a contradiction for large n.

Section B

3. Show that the Cayley graph Γ of an infinite finitely generated group G contains a bi-infinite geodesic.

Solution. Let x be a fixed vertex. Consider a sequence of vertices x_n such that $d(x_n, x) \to \infty$. We pick geodesic paths p_n joining x_n to x. Since Γ is locally finite by passing to a subsequence we may assume that the paths p_n converge to an infinite path p. Let y_n be a sequence of distinct vertices on p. For each n we pick g_n such that $g_n y_n = x$. Since the Γ is locally finite, after passing to a subsequence, the paths $g_n p$ converge to a bi-infinite geodesic.

4. i) Show that any metric space X has a $(1,1)$-net.

ii) Show that if $N \subset X$ is a net then $X \sim N$.

iii) Show that $X \sim Y$ if and only if there are nets $N_1 \subset X, N_2 \subset Y$ and a bilipschitz map $f : N_1 \to N_2$.

iv) Let G be a f.g. group. Show that $H < G$ is a net in G if and only if H is a finite index subgroup of G.

Solution. i) Let N be a maximal subset of X such that for any $a, b \in N$ $d(a,b) \geq 1$. Such an N exists by Zorn's lemma. Now if $x \in X$ and $d(x,a) \geq 1$ for any $a \in N$ then N is not maximal. So there is some $a \in N$ such that $d(a,x) \leq 1$.

ii) If N is a (m,n)-net define $f : X \to N$ so that $d(f(x), x) \leq m$ for all x. Clearly this is possible. One sees easily that f is a quasi-isometry.

iii) Let $f : X \to Y$ be a (K,A)-quasi-isometry. Pick N_1 an (n,n)-net in X with $n = 2K(A+1)+A$ (sufficiently large). Then $d(f(x), f(y)) \geq 1$ for $x \neq y$ so f is injective on N_1. Also

$$d(f(x), f(y)) \leq K d(x,y) + A \leq KA d(x,y)$$

and

$$d(f(x), f(y)) \geq \frac{d(x,y)}{K} - A \geq \frac{d(x,y)}{K} - \frac{d(x,y)}{2K} \geq \frac{d(x,y)}{2K}$$

Finally since for any $y \in Y$ there is an $x \in X$ such that $d(y, f(x)) \leq A$ and there is an $a \in N_1$ with $d(a,x) \leq n$ we have that

$$d(f(n), y) \leq A + Kn + A.$$

so $f(N_1) = N_2$ is a net in Y.

iv) Clearly if H is of index n then H is an $(n,1)$ net in G. Assume that H is an $(n,1)$ net in G. Let's say that there are M words on the generating set of G of length $\leq n$. For every $g \in G$ $gw \in H$ for some word of length $\leq n$. So $g \in Hw^{-1}$. It follows that the index of H in G is bounded by M.

5. Show that $\mathbb{F}_2 \times \mathbb{Z}$ has one end (where $\mathbb{F}_2$ is the free group of rank 2).

Solution.

Let a, b be a generating set of $\mathbb{F}_2$. Then $S = (a,0), (b,0), (0,1)$ is a generating set of $G = \mathbb{F}_2 \times \mathbb{Z}$. If $X = \Gamma(S, G)$ has more than one end then there is a finite set of vertices, K, of X such that $X - K$ has more than 1 unbounded component. If $p_2 : \mathbb{F}_2 \times \mathbb{Z} \to \mathbb{Z}$ is the projection map then there is some n such that $p_2(K) \subset [-n, n]$. Let $p_1 : \mathbb{F}_2 \times \mathbb{Z} \to \mathbb{F}_2$ be the projection to $\mathbb{F}_2$ and let $K' = p_1(X) \times [-n, n]$. Then K' is finite and contains K. We claim that any two vertices of $X - K'$ can be connected by a path, hence X is 1 ended. Indeed let (v_1, n_1) be a vertex of $X - K'$ and let $v \notin p_1(K)$. If $N > n$ then we join (v_1, n_1) to (v_1, N) or to $(v_1, -N)$ by a path in $v_1 \times \mathbb{Z}$ that does not meet K'. It is clear that this is possible. Then we join $(v, 0)$ to (v, N) and to $(v, -N)$ by a path in $v \times \mathbb{Z}$. Finally if q is a path joing v_1, v is the Cayley graph of $\mathbb{F}_2$ we may join (v_1, N) and $(v_1, -N)$ to (v, N) resp. $(v, -N)$ by $q \times \{N\}$, resp $q \times \{-N\}$. So (v_1, n_1) can be joined to $(v, 0)$. It follows that $X - K'$ is connected and $X - K$ is one ended.

6. Let X be a δ-hyperbolic geodesic metric space. If L is a geodesic in X and $a \in X$ we say that $b \in L$ is a projection of a to L if

$$d(a, b) = \inf\{d(a, x) : x \in L\}.$$

Show that if b_1, b_2 are projections of a to L then $d(b_1, b_2) \leq 2\delta$.

Solution. This follows easily considering the geodesic triangle $[a, b_1, b_2]$.

7. Let $G =< S|R >$ be a torsion free δ-hyperbolic group. Show that if $g^3 = h^3$ then $g = h$.

Solution. Clearly g, h lie in the centralizer C of g^3. But the centralizer is virtually cyclic so by Stallings theorem it splits over a finite group. So either $C = A * B$ or $C = A*_e$. In the second case $C = A * \mathbb{Z}$. In the first case A, B are infinite so C has exponential growth by normal forms-which is impossible. In the second case if A is non-trivial then C has again exponential growth so in fact $A = \{e\}$ and $C = \mathbb{Z}$. Since $g, h \in C$ we have $g = h$.

8. Let $G =< S|R >$ be δ-hyperbolic group. Show that G has no subgroup isomorphic to $< x, t|txt^{-1} = x^2 >$.

Solution. $t^n x t^{-n} = x^{2^n}$ which contradicts the fact that x^n is a quasi-geodesic.

9. Let $G =< S|R >$ be a be a Dehn presentation of a of a δ-hyperbolic group. Show that we can decide whether a word w on S represents an infinite order element.

Solution. To clarify, our input for the algorithm is the finite presentation $< S|R >$ and δ.

1st solution: We use a Dehn presentation and using the solution to the conjugacy problem we check successively for the powers of w, w^k, whether they are conjugate to an element of length $\leq \max\{|r|+2\}$ where r ranges over all relations of the Dehn presentation. Eventually we will either find that $w^k = 1$ or we will find two powers w^k, w^m which are conjugate to the same element a. It follows that these are conjugate so there is some t such that $tw^kt^{-1} = w^m$. However this contradicts the fact that $< w >$ is a quasi-geodesic as in exercise 8. So either some power is equal to 1 or some power is not conjugate to any element of length $\leq \max\{|r|+2\}$ (and hence w is of infinite order).

2nd solution: Enumerate powers w^n and check if they are equal to 1. In parallel try to find a vertex m of the Cayley graph and a power w^k such that $d(w^{2k}m, w^km) > 2d(m, w^km) - 12\delta$ and $d(e, w^k) > 100\delta$. If w is of finite order the first procedure will terminate. If w is of infinite order then by the proof of the proposition 6.4 in the notes showing that $< w >$ is a quasi-geodesic w^k and m with the above properties exist and we can detect them since the word problem is solvable in G.

Section C

10. Let $G =< S|R >$ be a Dehn presentation of a δ-hyperbolic group. Show that we can decide whether a word w on S lies in the subgroup $< v >$.

Solution. To clarify, our input for the algorithm is the finite presentation $< S|R >$, δ and the words v, w.

The proof shows of proposition 6.4 that there is some vertex m in the Cayley graph and some power v^k such that $d(v^{2k}m, v^km) \geq 2d(v^km, m) - 12\delta$. However since we can solve the word problem we can find v^k, m just by calculating multiplication tables for larger and larger balls and powers of v. Once those are found we get an estimate, as in proposition 6.4, of the form $d(v^n, e) \geq cn - d$ for some $c, d > 0$. So it is enough to check whether $c^n = w$ for all n for which $cn - d \leq |w|$.

11. (Ends of Groups.) i. Show that if a finitely generated group G splits over a finite group then G has more than 1 end.

(*Hint*: This can be done either by constructing the Cayley graph Γ or by normal forms. If e.g. $G = A *_C B$ note that words of the form $(ab)^n$ and $(ba)^n$ lie in different components of $\Gamma \setminus C$.)

ii. Show that two quasi-isometric locally finite graphs have the same number of ends. Deduce that the number of ends of a finitely generated group is well defined (ie it does not depend on the Cayley graph that we pick).

iii. Show that a finitely generated group has 0,1,2 or ∞ ends.

Solution. Let's say that $G = A *_C B$ with C finite. Pick generating sets $S_A = \{s_1, ..., s_n\}, S_B = \{t_1, ..., t_k\}$ of A, B respectively. If $S = S_A \cup S_B$

consider the Cayley graph X of G with respect to S. We claim that $X - C$ has 2 unbounded components. Let $a \in S_A - C, b \in S_B - C$ Consider reduced words of the form $w_n = (ab)^n$ with and $v_n = (ba)^n$. w_n, v_n are vertices of X and any path joining w_n, v_n has to go through C. Indeed a path from v_n to w_n corresponds to a word $p = x_1 ... x_s$ with $x_i \in S_A \cup S_B$. Let k be maximal such that the reduced word corresponding to $w_n x_1 ... x_k$ starts with an element of A. Say $w_n x_1 ... x_k = a' y_1 ... y_r$ where $a' y_1 ... y_r$ is reduced word. Then $w_n x_1 ... x_k x_{k+1} = a' y_1 ... y_r x_{k+1}$. If $r \geq 1$ then after reducing still we get a word starting with an element of A. It follows that $w_n x_1 ... x_k = a'$ and in fact $a' \in C$, so the path p goes through C. Also $d(w_n, C), d(v_n, C) \geq 2n$. A single connected component Y_1 of $X - C$ contains all w_n and a different connected component Y_2 of $X - C$ contains all v_n so Y_1, Y_2 are unbounded. One can argue similarly in the HNN-extension case using again normal forms. Then C separates t^n from t^{-n} where t is the stable letter of the HNN-extension.

A more geometric argument runs as follows:

If $G = A *_H B$ or $G = A *_H$ we may pick a finite generating set S of G so that, in the first case, all generators lie in $A \cup B$ or, in the second case, the generators are given by the stable letter t and a finite set of elements of A. To see this take any finite set of generators of G, S' and write each element of S' in normal form with respect to the amalgam or the HNN-extension. Now take as new generating set S of G the set of all elements of A, B (and t in the HNN-extension case) that appear in these normal form expressions. Let Γ be the Cayley graph of G with respect to S and let T be the Bass-Serre tree of G for the splitting $G = A *_H B$ or $G = A *_H$. We consider the barycentric subdivisions Γ' of Γ and T' of T. We define now a simplicial map $p : \Gamma' \to T'$. Let e be the edge of T with stabilizer H. Let v be the midpoint of e. So v is a vertex of T'. We recall that the vertices of Γ are the elements of G. If g is a vertex of Γ define $p(g) = gv$. If (g, gs) is an edge of Γ (so $g \in G,\ s \in S$) then $d(sv, v)$ is either 2 or 0. So $d(gv, gsv)$ is 2 or 0 and we can extend p to (g, gs) either by mapping it to the 2 consecutive edges joining gv, gsv or by collapsing it to the vertex gv. This shows that the map p can be extended from the set of vertices of Γ to a simplicial map $p : \Gamma' \to T'$. Since the map p' is simplicial we have that $d(p(a), p(b)) \leq d(a, b)$ for all vertices of Γ. Further p is clearly onto. By our choice of v, $p^{-1}(v) = H$. Let $n \in \mathbb{N}$ and let v_1, v_2 be vertices of $T' - T$ lying in distinct connected components of $T' - v$ such that $d(v_1, v) \geq n$, $d(v_2, v) \geq n$. Let $g_1 \in p^{-1}(v_1)$, $g_2 \in p^{-1}(v_2)$. Then if α is a path in Γ' joining g_1 to g_2, v lies in $p(\alpha)$. It follows that α interects $p^{-1}(v) = H$. Further if h is the first vertex of α lying in H and $\alpha_1 = [g_1, h]$ the subpath of α with endpoints g_1, h, then $p(\alpha_1)$ joins v_1 to v. It follows that $length(p(\alpha_1)) \geq n$. Since p is distance non increasing we conclude that $d(g_1, H) \geq n$. Similarly $d(g_2, H) \geq n$. Since this is true for any n we conclude that H coarsely separates Γ. Here H is finite so it is a compact subset of the Cayley graph.

ii. Let $f : X \to Y$ be a quasi-isometry between two locally finite graphs. Let K be a compact subset of X such that $X - K$ has n ends. Let $r_1, ..., r_n$ be geodesic rays representing these ends, i.e. $d(r_i(t), K) \geq t$ for all t and there is no path joining $r_i(t)$ to $r_j(t)$ in $X - K$ for any $t > 0$.

If f is an (A, B)-quasi-isometry

$$d(f(r_i(t)), f(K)) \geq t/A - B$$

so for t big enough $f(r_i(t))$ is at least at distance $A + B$ from $f(K)$. It follows that we can join the images of successive vertices of $f(r_i)$ and obtain a path $p_i(t)$ such that

$$d(p_i(t), f(K)) \geq t/A - 2B - A$$

for all t.

Assume that for any $D > 0$ there is a path $q = (v_1, ..., v_r)$ (v_i vertices) joining some vertices $f(r_i(s)), f(r_j(t))$ ($i \neq j$) outside $N_D(f(K))$. We set $u_1 = r_i(s), u_r = r_j(t)$ and pick u_i such that $d(f(u_i), v_i) \leq B$. Then, for D big enough, the geodesic segments joining u_i, u_{i+1}, ($i = 1, ..., r - 1$) do no intersect K and we obtain thus is a path in $X \setminus K$ joining $r_i(t)$ to $r_j(t)$ a contradiction.

We conclude that there is some $D > 0$ such that Y minus the D-neighborhood of $f(K)$ has at least n unbounded connected components.

So $e(Y) \geq e(X)$. But we have the opposite inequality too using a quasi-isometry $g : Y \to X$. So $e(X) = e(Y)$.

Since any two Cayley graphs of the same f.g. group are quasi-isometric it doesn't matter which one we pick to define ends.

iii. $\mathbb{Z}_2$ has 0 ends, $\mathbb{Z}$ has 2 ends, $\mathbb{Z}^2$ has 1 end, and F_2 has ∞ ends. So all these are possible.

Let X be the Cayley graph of a f.g. group G. If X has more than 2 ends then there is a compact $K \subset X$ such that $X - K$ has at least 3 ends. We how now inductively that X has more than n ends for any $n \in \mathbb{N}$. Assume that M is compact and $X - M$ has n unbounded connected components. Let Y be an unbounded component of $X - M$. Let $v \in Y$ be a vertex such that $d(v, M) > diam(M)$. If w is a vertex of M (we may assume K contains vertices) take $g \in G$ such that $gw = v$. Then $L = M \cup gM$ is compact and $X - L$ has at least $2n - 1$ unbounded components. To see this note that $X - gM$ has n unbounded connected components. However gM is contained in Y so at least $n - 1$ of the connected components of $X - gM$ are contained in Y. Indeed if 2 components C_1, C_2 intersected $X - Y$ at the points a, b then we could join a to a point a' in M and b to a point b' in M by paths disjoint from Y. Finally we join a', b' by a geodesic which clearly is disjoint from gM producing a path joining C_1, C_2 in $X - gM$, a contradiction. This shows that $Y - gM$ has at least $n-1$ unbounded connected components and $X - L$ has at least $2n-1 \geq n+1$ unbounded connected components.

12. The objective of this exercise is to show that torsion free groups quasi-isometric to free groups are free.

Assume that a finitely generated group G is quasi-isometric to the free group F_n (with $n \geq 2$)

i. Show that G has infinitely many ends. (You may use the results of the previous exercise).

ii. Consider the Grusko decomposition of G as a free product: $G = G_1 * ... * G_k * F_s$. Show that none of the G_i's is 1-ended.

Hint: Note that if G_i is infinite then its Cayley graph contains a bi-infinite geodesic.

iii. Show that if H is a torsion free 2-ended group then H is isomorphic to $\mathbb{Z}$.

Hint: Use Stallings Theorem.

iv. Assume now that G is torsion free. Show that all G_i's are finite (and hence trivial) to conclude that $G \cong F_s$.

v. Deduce that if a f.g. torsion free group K has a finite index free subgroup then K is free.

Solution.

i. By the result of the previous exercise $e(G) = e(F_n)$ so $e(G) = \infty$.

ii. If G_i is finite then it has 0 ends. If G_i is infinite then its Cayley graph contains a bi-infinite geodesic. Let's pick a finite set of generators for G which is a union of the finite generating sets of G_i and F_s. Then G_i is isometrically embedded in G by the normal form theorem for free products. It follows that the Cayley graph Γ of G contains a bi-infinite geodesic L and all vertices of L are elements of G_i. Let $f : G \to F_n$ be a quasi-isometry. If $a_i (i \in \mathbb{Z})$ are the successive vertices of L then if we join for all i $f(a_i)$ to $f(a_{i+1})$ by a geodesic segment we obtain a quasi-geodesic (as f is a quasi-isometry). Clearly $f(L)$ is at finite distance from this quasigeodesic. However any quasi-geodesic in a tree (the Cayley graph of F_n) is at finite distance from a geodesic. So $f(L)$ is at finite distance from some geodesic L' and $f(G_i)$ contains $f(L)$. It follows that a point on L' separates $f(G_i)$ to at least 2 infinite components. Since f is a quasi-isometry G has at least 2 ends.

iii. Since $e(H) > 1$ H splits as $A*_C$ or $A *_C B$ with C finite. Since it is torsion free we have $C \cong \{e\}$. But H has 2 ends and $A * B$ has infinitely many ends so necessarily $H \cong A*_e \cong A * \mathbb{Z}$. However if A is infinite H has infinitely many ends so $A \cong \{e\}$ and $H \cong \mathbb{Z}$.

iv. Since G is torsion free if G_i has infinitely many ends then it splits as a free product by Stallings theorem. However G_i is indecomposable, a contradiction. So G_i has 2 ends. By the previous part $G_i \cong \mathbb{Z}$. But then G_i is contained in the free factor F_s (or put it differently there are no G_i's).

v. K is q.i. to a free group since it has a finite index free subgroup. So by the previous part it is free.