Supersymmetry & Supergravity
Lecture Notes

Michèle Levi

Mathematical Institute
University of Oxford

MMathPhys
HT 2026

levi@maths.ox.ac.uk
First published March 2024
Last updated January 18, 2026

Contents

Preface

These lecture notes were designed for the course Supersymmetry and Supergravity in the Mathematical and Theoretical Physics Master program in Oxford, initially given in Hilary term 2024. I have strived to make these lecture notes self-contained and well-structured, within the limited scope of the course. Nevertheless, it is critical to take the courses Groups and Representations, Quantum Field Theory (QFT), and General Relativity, prior to this course, and Advanced QFT simultaneously at the latest. While these lecture notes cover standard introductory material on supersymmetry, I chose to also include certain advanced topics, such as the superconformal algebra, and Kähler geometry. These topics are important in modern theoretical high energy physics, and beyond their pedagogic value in the present course, they serve as teasers to prospective studies in Conformal Field Theory, and Supergravity. The latter is not covered in this course, as opposed to what its title suggests, yet these lecture notes provide the foundation that is essential to proceed in this advanced direction of study.

The recommended textbooks to consult in parallel to these lectures notes, in order to broaden the view and deepen the understanding of the material presented here, are the References by Weinberg [1], and Wess & Bagger [2]. The last part of this course, specifically chapters 6, 7, 8, build on 3 topics, that ideally should have been encountered prior to this course within some advanced QFT courses: Renormalization group, non-Abelian gauge theories, and spontaneous symmetry breaking. Though these lecture notes provide proper preliminaries to these topics, it is also recommended to consult in parallel the QFT textbooks by Peskin & Schroeder[3], or Weinberg [4, 5]. Finally, this course does not cover the topic of Supersymmetry and the Standard Model. This phenomenological topic is properly covered in the dedicated courses Beyond the Standard Model I and II, which can be taken following this course. For those interested to pursue the route of Supergravity, it is recommended to refer to the textbook by Freedman and Van Proeyen [6].

Acknowledgements

My work has been funded by the UKRI Rutherford grant ST/V003895 *"Harnessing QFT for Gravity"*, and by the Mathematical Institute, University of Oxford. I extend my gratitude to James Sparks and Luis Fernando Alday for meaningfully and warmly welcoming me into the Mathematical Institute and the Mathematical Physics group.

MICHÈLE LEVI

Oxford
March 2024

1. Context and Motivation

Symmetries play an important role in physics. Continuous symmetries are associated to conservation laws by Noether's theorem. In a quantum field theory (QFT) in $D \equiv d+1$ spacetime dimensions with continuous Lie-group symmetries, the conserved Noether currents $j_I^\mu(x)$ are local operators, that satisfy

$$\partial_\mu j_I^\mu = 0\,. \tag{1.1}$$

Thus, the generator of the symmetry on the Hilbert space of the QFT is the charge operator:

$$Q_I = \int_{\Sigma_d} d^{\mathrm{d}}x\, j_I^0(x)\,, \qquad \frac{d}{dt}Q_I = 0\,, \tag{1.2}$$

where the integration is over a spatial slice Σ_d at constant time t.

What are the quantum field theories with the most extensive symmetry possible? Supersymmetric theories. This was shown through key theorems by Coleman and Mandula in 1967 [7], and by Haag et al. in 1975 [8], where the latter significantly generalized the former to include supersymmetry, that will be further discussed shortly as of section 3. Supersymmetric theories also have more degrees of freedom than the quantum field theories that form the Standard Model of Particle Physics. While the latter have been well-confirmed in experiments, there is no experimental evidence for supersymmetric theories. Yet, we expect to recover the Standard Model from supersymmetry in the appropriate limit of the relatively low energies that particle accelerators reach. Supersymmetry is then believed to spontaneously break in nature.

Historically, supersymmetry first appeared in few various publications from 1971, with the seminal Wess-Zumino model published in 1974 [9, 10]. This discovery launched a sustained investigation of supersymmetric QFTs (and supergravity theories), which is still ongoing 50 years later.

1.1 Motivations for Supersymmetry

For the above noted reasons supersymmetry requires some good motivations. We can broadly and historically classify them as follows:

1. **Grand Unified Theory (GUT).** This has been an attempt to unify the gauge symmetries of the Standard Model, i.e. the symmetry group:

$$G_{SM} = U(1) \times SU(2) \times SU(3)\,. \tag{1.3}$$

   In the attempt to unify the coupling constants of the Standard Model, better agreement is reached when supersymmetric versions of the Standard Model are considered. This motivation is concerned only with the 3 fundamental forces captured by the Standard Model.

2. **Hierarchy Problem.** The characteristic energy scale of electroweak interaction is “unnaturally” far from the Planck scale of quantum gravity. A new scale, such as a GUT scale with supersymmetry, can serve as the “natural” intermediate scale. This motivation is concerned with some missing link between the Standard Model and gravity, which is the only fundamental force that is not captured by the Standard Model.

3. **Qunatum Gravity.** Supergravity theories, which incorporate local supersymmetry, and are the supersymmetric generalization of the General Theory of Relativity, are useful in searching for a candidate theory of quantum gravity. For example, supergravity is an essential component in the gauge-gravity duality and in string theory. This motivation is concerned with uncovering a complete theory of gravity.

1.2 Coleman-Mandula Theorem

We then come back to the intriguing question: What is the most general symmetry of the S-matrix consistent with quantum field theory? We stated that it is supersymmetry. The proof of this statement is based on the Coleman-Mandula theorem (1967) [7], a powerful no-go theorem about the possible symmetries of the S-matrix. This theorem assumes Poincaré invariance in $D = 4$ spacetime, that is the symmetry group:

$$ISO(1,3) \cong SO(1,3) \ltimes \mathbb{R}^{1,3}\,, \tag{1.4}$$

with the following further assumptions:

1. Only a finite number of particles are associated with one-particle states of a given mass.
2. There are one-particle states of non-vanishing mass.
3. The S-matrix is analytic, i.e. scattering amplitudes are analytic functions of invariants/observables.

Let G be a symmetry generator of the theory, and consider its action on physical states to be as follows:

- $G|0\rangle = 0$, i.e. G keeps the vacuum state invariant (no spontaneous symmetry breaking).
- $G|i\rangle = \sum_{i'} G_{i'i}|i'\rangle$, that is one-particle states are taken into other one-particle states.
- $G|ij\rangle_{in} = G_{i'i}|i'j\rangle + G_{j'j}|ij'\rangle$, i.e. G acts on an in-state of more than one particle, e.g. two-particle state $|ij\rangle_{in} = |i\rangle|j\rangle$, as a direct sum of acting on the one-particle states in turn.

To further explain the last point, let us recall that the symmetry generators are the conserved charges, which are obtained according to Noether's theorem as a spatial integral of some current, $G = \int d^3x\, j_I(x)$. The state $|ij\rangle_{in}$ is when there is a wave packet of particle i, apart from another wave packet of particle j (everywhere else is vacuum), so that we can split space into 2 parts, one with i, the other with j, such that

$$\begin{aligned} G|ij\rangle_{in} &= \int d^3x\, j_I(x)|ij\rangle = \int_i d^3x\, j_I(x)|ij\rangle + \int_j d^3x\, j_I(x)|ij\rangle \\ &= G_{i'i}|i'\rangle|j\rangle + G_{j'j}|i\rangle|j'\rangle\,. \end{aligned} \tag{1.5}$$

The generalization to a multi-particle state is straightforward.

The action of a commutator of such operators on a multi-particle state is similar. Consider H, another operator that acts similarly on particle states. Then, we get:

$$\begin{aligned} GH|ij\rangle &= G\big(H_{i'i}|i'j\rangle + H_{j'j}|ij'\rangle\big) \\ &= H_{i'i}G_{i''i'}|i''j\rangle + H_{j'j}G_{j''j'}|ij''\rangle + H_{i'i}G_{j'j}|i'j'\rangle + H_{j'j}G_{i'i}|i'j'\rangle\,. \end{aligned} \tag{1.6}$$

The last 2 terms drop, if we act with the commutator $[G,H]$:

$$[G,H]|ij\rangle = [G,H]_{i''i}|i''j\rangle + [G,H]_{j''j}|ij''\rangle\,. \tag{1.7}$$

If G satisfies such form of action on physical states, and commutes with the S-matrix, i.e. $[G, S] = 0$ (that is the symmetry is preserved through the scattering), then the Coleman-Mandula theorem asserts that:

- Either G is some Poincaré generator, P;
- Or G commutes with the Poincaré algebra, i.e. $[G, P] = 0$, that is G is also a Lorentz scalar.

G can obviously also be a linear combination of these 2 options. Thus the most general symmetry group of the S-matrix is the product:

$$G_{\rm P} \times G_{\rm internal}\,, \tag{1.8}$$

where $G_{\rm internal}$ stands for some internal symmetries (as opposed to spacetime symmetries), that must commute with the Poincaré group, $G_{\rm P}$.

Note that if we drop our second assumption on the existence of massive particles, and consider a theory where all particles are massless, then the most general symmetry possible is more extensive. The Poincaré algebra, $\mathfrak{g}_{\rm P}$, is extended then to the conformal algebra, $\mathfrak{g}_{\rm C}$:

$$\{P_\mu, M_{\mu\nu}\} \subseteq \mathfrak{g}_{\rm P} \quad \longrightarrow \quad \{P_\mu, M_{\mu\nu}, D, K_\mu\} \subseteq \mathfrak{g}_{\rm C}\,, \tag{1.9}$$

where P_μ represents the 4 translation generators, and the antisymmetric $M_{\mu\nu}$, consists of the 6 Lorentz generators, whereas in the conformal algebra there are in addition the dilatation generator, D, and the 4 special conformal generators, K_μ. In this case, the most general symmetry group possible is extended to the product:

$$G_{\rm C} \times G_{\rm internal}\,. \tag{1.10}$$

We will return to the massless case later on, in section 3.2, where we will learn about the superconformal algebra.

Yet, there is a more critical generic loophole in the Coleman-Mandula theorem. If G is an operator of half-integral spin, rather than a bosonic operator, then the assumption about the action on multi-particle states should be modified according to the Dirac statistics. If G is a fermionic operator, then its action on a two-particle state, e.g., reads:

$$G|ij\rangle_{in} = G_{i'i}|i'j\rangle + (-1)^{f_i} G_{j'j}|ij'\rangle\,, \quad f_i \equiv \begin{cases} 0 & \text{i boson;} \\ 1 & \text{i fermion.} \end{cases} \tag{1.11}$$

It turns out that the Coleman-Mandula theorem did not treat the case of fermionic generators!

Thus, for G, H, some fermionic generators we get:

$$\begin{aligned} GH|ij\rangle &= G\big(H_{i'i}|i'j\rangle + (-1)^{f_i} H_{j'j}|ij'\rangle\big) \\ &= H_{i'i}G_{i''i'}|i''j\rangle + (-1)^{f_i} H_{j'j}(-1)^{f_i} G_{j''j'}|ij''\rangle \\ &\quad + H_{i'i}(-1)^{f_{i'}} G_{j'j}|i'j'\rangle + (-1)^{f_i} H_{j'j} G_{i'i}|i'j'\rangle \,, \end{aligned} \tag{1.12}$$

where

$$(-1)^{f_{i'}} = (-1)^{f_i+1} \,, \tag{1.13}$$

since if H is fermionic, then when it acts on i, it flips its statistics, so for i fermionic, i' is bosonic, and vice versa. Then the last 2 terms drop, if we take the anti-commutator $\{G, H\}$:

$$\{G, H\}|ij\rangle = \{G, H\}_{i''i}|i''j\rangle + \{G, H\}_{j''j}|ij''\rangle \,. \tag{1.14}$$

It is important to note that the anti-commutator itself is then a bosonic operator, that satisfies the Coleman-Mandula theorem.

To recap, the Coleman-Mandula theorem can be bypassed by fermionic symmetry generators, which are not Lorentz scalars, and are not forbidden by the theorem. As we shall see shortly, these would be the new supersymmetry generators!

2. Spinors Preliminary

Spinors play a starring role in supersymmetry, whose generators, as we shall see shortly in chapter 3, are fermionic. For this reason, it is essential to first make here a technical preliminary, in order to recall the spinorial representations of the Lorentz group, set up some notation and conventions, and enable algebraic manipulations of spinors.

2.1 Spinorial Lorentz Representations

We recall that the Lorentz generators encapsulated in the Lorentz tensor $M_{\mu\nu}$, can be traded for the Euclidean vector generators of rotations J_i, and of boosts K_i, where $i = 1, 2, 3$, whose commutation relations read:

$$[J_i, J_j] = i\epsilon_{ijk} J_k\,, \qquad [J_i, K_j] = i\epsilon_{ijk} K_k\,, \qquad [K_i, K_j] = -i\epsilon_{ijk} J_k\,, \tag{2.1}$$

where the latter relation is the famous Wigner rotation. To construct the spinorial representations of the Lorentz group, we note that we can consider instead the combinations:

$$L_i \equiv \frac{1}{2}(J_i + iK_i)\,, \qquad R_i \equiv \frac{1}{2}(J_i - iK_i)\,. \tag{2.2}$$

In this basis we get the commutation relations:

$$[L_i, L_j] = i\epsilon_{ijk} L_k\,, \qquad [R_i, R_j] = i\epsilon_{ijk} R_k\,, \qquad [L_i, R_j] = 0\,. \tag{2.3}$$

We see then that the representations of L_i, R_i, are each representations of angular momentum, of the form $|j, m\rangle$ characterized by j, and of dimension $2j + 1$ as $-j \leqslant m \leqslant j$, where $L_i + R_i = J_i$. So the spinorial representation of the Lorentz group is of the form (j_1, j_2) characterized by $j_{1,2} \in \frac{1}{2}\mathbb{N}_0$ with $j_1 + j_2 = j$, and is of dimension $(2j_1 + 1) \times (2j_2 + 1)$, where L_i, R_i, act only on m_1, m_2, respectively. Under conjugation $L_i \to R_i$, $R_i \to L_i$, or $(j_1, j_2)^* = (j_2, j_1)$. The smallest non-trivial representations are of dimension 2: $(\frac{1}{2}, 0)$ and its conjugate representation $(0, \frac{1}{2})$.

Let us construct these representations explicitly. Consider the group of 2×2 complex matrices, M, with $\det M = 1$, i.e. $SL(2,\mathbb{C})$. We show that $SL(2,\mathbb{C})$ is homomorphic to the restricted Lorentz group $SO^+(1,3)$ (the component of the Lorentz group connected to the identity element), through the use of the Pauli matrices, σ^μ. They are defined here as

$$\sigma^0 \equiv -\mathbb{I}_2\,, \quad \sigma^1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.4}$$

The σ matrices span 2×2 Hermitian matrices, such that any Hermitian matrix can be written as

$$H = v_\mu \sigma^\mu = \begin{pmatrix} -v_0 + v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_0 - v_3 \end{pmatrix}, \tag{2.5}$$

with v_μ real. The transformation $H \to H' = MHM^\dagger$, with $M \in SL(2,\mathbb{C})$, keeps H' Hermitian, so we can also write $H' = v'_\mu \sigma^\mu$. Moreover, there is an invariance of the determinants under the transformation:

$$\begin{aligned} \det H &= v_0^2 - v_1^2 - v_2^2 - v_3^2 \\ &= \det(MHM^\dagger) = (v'_0)^2 - (v'_1)^2 - (v'_2)^2 - (v'_3)^2\,. \end{aligned} \tag{2.6}$$

Thus $H \to MHM^\dagger$ transforms a 4-vector v_μ to a 4-vector v'_μ, and keeps the invariance of $\eta_{\mu\nu}v^\mu v^\nu = \eta_{\mu\nu}v'^\mu v'^\nu$. This is similar to a Lorentz transformation, $\Lambda_{4\times4}$:

$$v' = \Lambda v \quad \leftrightarrow \quad M v_\mu \sigma^\mu M^\dagger = v'_\mu \sigma^\mu\,, \quad M^* v_\mu \sigma^\mu M^T = v'_\mu \sigma^\mu\,, \tag{2.7}$$

where we also noted another transformation with M^*, which acts similarly, but is not equivalent. Thus this is the homomorphism between the $SO^+(1,3)$ and $SL(2,\mathbb{C})$ groups, where each element in the restricted Lorentz group corresponds to 2 elements in $SL(2,\mathbb{C})$, so that

$$SL(2,\mathbb{C})/Z_2 \cong SO^+(1,3)\,. \tag{2.8}$$

Furthermore, the Lie algebra of the group $SL(2,\mathbb{C})$ is spanned by traceless 2×2 matrices: $1 = \det e^A = e^{\operatorname{tr} A} \implies \operatorname{tr} A = 0$. Such matrices are spanned by 6 generators, which can be taken as: σ^i, $i\sigma^i$. Let us consider M close to the identity matrix:

$$M = \mathbb{I}_2 + i(\theta_i \sigma^i - i\eta_i \sigma^i)\,. \tag{2.9}$$

Then, the change in $H \to MHM^\dagger$ reads:

$$\delta H = MHM^\dagger - H = i(\sigma^i H - H\sigma^i)\theta_i + (\sigma^i H + H\sigma^i)\eta_i\,, \tag{2.10}$$

with

$$[\sigma^i, H] = [\sigma^i, v_\mu \sigma^\mu] = 2i\epsilon^{ijk} v_j \sigma^k \,, \tag{2.11}$$

$$\{\sigma^i, H\} = \{\sigma^i, v_\mu \sigma^\mu\} = -2(v_0 \sigma^i + v_i \sigma^0) \,, \tag{2.12}$$

where we used the identity:

$$\sigma^i \sigma^j = \delta^{ij} \mathbb{I}_2 + i\epsilon^{ijk}\sigma^k \,. \tag{2.13}$$

We see then that there is a rotation of v_μ in eq. (2.11), and a boost of v_μ in eq. (2.12). So if we map the Lorentz generators $J^i \to \frac{1}{2}\sigma^i$, $K^i \to -\frac{i}{2}\sigma^i$, then $L^i \to \frac{1}{2}\sigma^i$, $R^i \to 0$, and we land in the representation $(\frac{1}{2}, 0)$, whereas if we make the conjugate map, we land in the $(0, \frac{1}{2})$ representation, so we also have that

$$SL(2, \mathbb{C}) \cong SU(2) \times SU(2)^* \,. \tag{2.14}$$

2.2 Spinor Notation and Conventions

With the spinorial Lorentz representations at hand, it is time to introduce the Van der Waerden notation: The right-handed Weyl spinors, that sit in the conjugate representation $(0, \frac{1}{2})$, carry a dotted spinor index, e.g. $\dot{\alpha}$, whereas the left-handed spinor indices are undotted. The matrices $M \in SL(2, \mathbb{C})$ represent the action of the Lorentz group on left- and right-handed Weyl spinors, such that

$$\psi'_\alpha = M_\alpha^{\ \beta} \psi_\beta \,, \quad \bar{\psi}'_{\dot{\alpha}} = (M^*)_{\dot{\alpha}}^{\ \dot{\beta}} \bar{\psi}_{\dot{\beta}} \,. \tag{2.15}$$

Thus, the spinors with undotted indices transform under the $(\frac{1}{2}, 0)$ representation, whereas those with dotted indices transform under the conjugate representation $(0, \frac{1}{2})$. The left-handed Weyl spinor, ψ_α, that sits in $(\frac{1}{2}, 0)$, has then the following connection to a right-handed one:

$$(\psi_\alpha)^* = \bar{\psi}^{\dot{\alpha}} \,, \qquad (\psi_\alpha)^\dagger = \bar{\psi}_{\dot{\alpha}} \,. \tag{2.16}$$

Note that an undotted lower index is a row index, while an undotted upper index is a column index, whereas the dotted indices follow the opposite convention: an upper index is a row one, and a lower index is a column one. From eqs. (2.7) and (2.15) we then infer that σ^μ has the spinor index structure $\sigma^\mu_{\alpha\dot{\alpha}}$.

It is also easy to see that the totally antisymmetric tensors in $2D$, defined here as

$$\epsilon^{12} = -\epsilon^{21} \equiv 1 \,, \qquad \epsilon_{21} = -\epsilon_{12} \equiv 1 \,, \tag{2.17}$$

or

$$\epsilon^{\alpha\beta} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon_{\alpha\beta} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.18}$$

are invariant under the action of $SL(2,\mathbb{C})$, since

$$\epsilon^{\gamma\delta} M_\gamma^\alpha M_\delta^\beta = \epsilon^{\alpha\beta} \det M = \epsilon^{\alpha\beta}, \tag{2.19}$$

as $\det M = 1$, which is similar for lower and/or dotted indices. As the ϵ tensors are invariant tensors of $SL(2,\mathbb{C})$, they are used to raise and lower spinor indices:

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta, \qquad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta, \tag{2.20}$$

where $\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta_\alpha^\gamma$. The ϵ tensor can then also be used to raise the indices of the σ matrices:

$$\bar{\sigma}^{\mu\,\dot{\alpha}\alpha} \equiv \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot{\beta}}, \tag{2.21}$$

where it can be easily verified that

$$\bar{\sigma}^\mu = (\sigma^0, -\sigma^i). \tag{2.22}$$

From the definition of the σ matrices, we find:

$$\left(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu\right)_\alpha^{\ \beta} = -2\eta^{\mu\nu}\delta_\alpha^{\ \beta}, \qquad \left(\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu\right)^{\dot{\alpha}}_{\ \dot{\beta}} = -2\eta^{\mu\nu}\delta^{\dot{\alpha}}_{\ \dot{\beta}}, \tag{2.23}$$

where

$$\eta_{\mu\nu} \equiv \mathrm{diag}(-1,1,1,1), \tag{2.24}$$

as well as the completeness relations:

$$\mathrm{tr}\left(\sigma^\mu\bar{\sigma}^\nu\right) = -2\eta^{\mu\nu}, \tag{2.25}$$

$$\sigma^\mu_{\alpha\dot{\alpha}}\bar{\sigma}_\mu^{\dot{\beta}\beta} = -2\delta_\alpha^{\ \beta}\delta_{\dot{\alpha}}^{\ \dot{\beta}}. \tag{2.26}$$

Eqs. (2.23) make it easy to relate two-component to four-component spinors through the following realization of the Dirac γ matrices:

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \tag{2.27}$$

which satisfy the Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}\mathbb{I}_4. \tag{2.28}$$

This is the Weyl basis, in which Dirac spinors contain two Weyl spinors:

$$\Psi_D \equiv \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}. \tag{2.29}$$

The Lorentz generators are then given in terms of

$$\sigma^{\mu\nu}{}_{\alpha}{}^{\beta} \equiv \frac{1}{4}\big(\sigma^{\mu}_{\alpha\dot\alpha}\bar\sigma^{\nu\,\dot\alpha\beta} - \sigma^{\nu}_{\alpha\dot\alpha}\bar\sigma^{\mu\,\dot\alpha\beta}\big)\,, \quad \bar\sigma^{\mu\nu\,\dot\alpha}{}_{\dot\beta} \equiv \frac{1}{4}\big(\bar\sigma^{\mu\,\dot\alpha\alpha}\sigma^{\nu}_{\alpha\dot\beta} - \bar\sigma^{\nu\,\dot\alpha\alpha}\sigma^{\mu}_{\alpha\dot\beta}\big)\,, \tag{2.30}$$

as

$$M^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu,\gamma^\nu] = i\begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar\sigma^{\mu\nu} \end{pmatrix}, \tag{2.31}$$

thus $i\sigma^{\mu\nu}$ and $i\bar\sigma^{\mu\nu}$ are the Lorentz generators on left- and right-handed Weyl spinors, respectively.

Finally, we note that the completeness relation in eq. (2.25) can be used to convert a vector to a bispinor, and vice versa:

$$v_{\alpha\dot\alpha} = \sigma^{\mu}_{\alpha\dot\alpha}v_\mu\,, \qquad v^\mu = -\frac{1}{2}\bar\sigma^{\mu\,\dot\alpha\alpha}v_{\alpha\dot\alpha} = -\frac{1}{2}\,\mathrm{tr}(\bar\sigma^\mu v)\,. \tag{2.32}$$

2.2.1 Spinor Algebra

In line with the conventions on lower/upper indices being row/column ones for undotted indices, and vice versa for dotted indices, as noted e.g. following eq. (2.16), we shall use the following spinor contraction convention:

$$\psi\chi \equiv \psi^\alpha\chi_\alpha = -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi\,, \tag{2.33}$$

$$\bar\psi\bar\chi \equiv \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} = -\bar\psi^{\dot\alpha}\bar\chi_{\dot\alpha} = \bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha} = \bar\chi\bar\psi\,, \tag{2.34}$$

where we used the fact that spinors anticommute. This is also in line with

$$(\chi\psi)^\dagger = (\chi^\alpha\psi_\alpha)^\dagger = \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} = \bar\psi\bar\chi = \bar\chi\bar\psi\,. \tag{2.35}$$

Note that conjugation reverses the order of the spinors.

We conclude with some selected useful spinor identities, that can be easily verified (as in, e.g., the problem sheets):

$$\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta\,, \quad \theta_\alpha\theta_\beta = +\frac{1}{2}\epsilon_{\alpha\beta}\theta\theta\,, \tag{2.36}$$

$$\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = +\frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\bar\theta\bar\theta\,, \quad \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} = -\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\bar\theta\bar\theta\,, \tag{2.37}$$

$$\big(\theta\phi\big)\big(\theta\psi\big) = -\frac{1}{2}\big(\phi\psi\big)\theta\theta\,, \tag{2.38}$$

$$\theta\sigma^\mu\bar\theta\theta\sigma^\nu\bar\theta = -\frac{1}{2}\theta\theta\bar\theta\bar\theta\eta^{\mu\nu}\,, \tag{2.39}$$

the flip identity

$$\chi\sigma^\mu\bar\psi = -\bar\psi\bar\sigma^\mu\chi\,, \tag{2.40}$$

and finally the Fierz rearrangement formula

$$\big(\psi\phi\big)\bar\chi_{\dot\alpha} = -\frac{1}{2}\big(\phi\sigma^\mu\bar\chi\big)\big(\psi\sigma_\mu\big)_{\dot\alpha}\,. \tag{2.41}$$

3. Supersymmetry Algebra

Picking up the discussion from the end of section 1.2, in order to bypass the limitations of the Coleman-Mandula theorem, let us generalize the notion of a Lie algebra to include anti-commutators as well as commutators.

Let us define the super-commutator:

$$[\hat{O}_a, \hat{O}_b\} \equiv \hat{O}_a \hat{O}_b - (-1)^{f_a f_b} \hat{O}_b \hat{O}_a \,, \quad f_{a,b} \in \{0, 1\} = \mathbb{Z}_2 \,, \tag{3.1}$$

where $f_{a,b}$ is the $\mathbb{Z}_2$ grading of the operators, with 0 for bosonic (or even) elements of the algebra, and 1 for fermionic (or odd) ones. The brackets are then taken as square or curly, according to the grading of operators therein. Such an algebra is called a graded Lie algebra or a super-algebra, and its Jacobi identities read:

$$(-1)^{f_c f_a}[[\hat{O}_a, \hat{O}_b\}\hat{O}_c\} + (-1)^{f_a f_b}[[\hat{O}_b, \hat{O}_c\}\hat{O}_a\} + (-1)^{f_b f_c}[[\hat{O}_c, \hat{O}_a\}\hat{O}_b\} = 0 \,, \tag{3.2}$$

also called super-Jacobi identities.

Let us denote bosonic operators by B, and fermionic ones by F, that is $B \in \mathfrak{g}_0$, $F \in \mathfrak{g}_1$, where $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are the even and odd parts, respectively, of the graded Lie algebra, $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 . \tag{3.3}$$

Then schematically, the super-algebra takes the form:

$$[B, B'] = B'' \,, \qquad [B, F] = F'' \,, \qquad \{F, F'\} = B \,, \tag{3.4}$$

which can also be encapsulated as follows:

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j} \,, \qquad i, j \in \mathbb{Z}_2 \,. \tag{3.5}$$

It is evident that the bosonic algebra is a sub-algebra of the super-algebra.

With these new definitions we can now proceed to construct the super-algebra for a QFT in $D = 4$ spacetime dimensions.

3.1 Super-Poincaré Algebra

To go beyond the Coleman-Mandula theorem, let us then consider some fermionic generator Q. It can be decomposed into a sum of spinorial irreducible representations of the Lorentz group, $Q_{j_1,m_1;j_2,m_2}$, with $j_1+j_2 = k+\frac{1}{2}$, $k \in \mathbb{N}_0$. We will show now that k must be 0. Let us consider the anticommutator of Q, and its hermitian conjugate, with the highest projections m_1, m_2, in each:

$$\{Q_{j_1,j_1;j_2,j_2}, (Q_{j_1,j_1;j_2,j_2})^\dagger\} = \{Q_{j_1,j_1;j_2,j_2}, \bar{Q}_{j_2,j_2;j_1,j_1}\} = \hat{O}_{j_1+j_2,j_1+j_2}\,, \tag{3.6}$$

from the addition of angular momenta. As we already noted at the end of section 1.2, $\hat{O}$ is a bosonic operator, and thus satisfies the Coleman-Mandula theorem. Therefore it is either a Poincaré generator or it is vanishing. In order to identify $\hat{O}$, let us then recall the spinorial Lorentz representations of the Poincaré generators, $P = \{P_\mu, L_i, R_i\}$:

- The energy-momentum 4-vector P_μ transforms in the representation $(\frac{1}{2}, \frac{1}{2})$, i.e. it carries the spinorial indices $P_{\alpha\dot{\beta}}$.
- The Lorentz generators L_i and R_i are 3-vectors, which commute, thus they sit in the representations $(1, 0)$ and $(0, 1)$, respectively. In terms of spinorial indices they are represented as the bispinors $M_{\alpha\beta}$, $M_{\dot{\alpha}\dot{\beta}}$, respectively, where the bispinors are symmetric in their indices.

Then for $j_1 + j_2 > \frac{1}{2}$, $\hat{O}$ must vanish. But $\hat{O} = QQ^\dagger + Q^\dagger Q$ is a positive-definite operator with Q the "square root" of $\hat{O}$. So for $j_1 + j_2 > \frac{1}{2}$, we infer that $Q = 0$. Since the operator with the highest projections vanish, all other projections in the irreducible representation vanish as well, i.e. the whole irreducible representation vanishes.

We conclude then that the fermionic generators can only be Q_{j_1,j_2} with $j_1 + j_2 = \frac{1}{2}$, so that we can only have the pair Q_α, $\bar{Q}_{\dot{\alpha}} = (Q_\alpha)^\dagger$, sitting in the representations $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$, respectively, and we can have $\mathcal{N} \in \mathbb{N}$ such pairs. Thus the most extended symmetry possible for a general QFT in $D = 4$ is generated by the algebra:

$$\mathfrak{g}_{\text{SP}} = \mathfrak{g}_{\text{P}} \oplus \mathfrak{g}_{\text{internal}} \oplus Q^I_\alpha, \bar{Q}^I_{\dot{\alpha}}, \quad I \in \{1, \ldots, \mathcal{N}\}\,, \tag{3.7}$$

called the super-Poincaré algebra, which constitutes supersymmetry! This is the celebrated result of the Haag, Lopuszanski, and Sohnius theorem (1975) [8]. More precisely, for $\mathcal{N} = 1$ this is called simple supersymmetry, whereas for $\mathcal{N} > 1$ this is called extended supersymmetry. As we shall see later in section 3.4, $\mathcal{N}$ actually has maximal values, dependent on the spacetime dimensionality D.

Let us press on to uncover the (anti-)commutation relations of the super-Poincaré algebra. From representation considerations, we can infer:

$$\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} = H^{IJ}\, P_{\alpha\dot\beta}\,, \tag{3.8}$$

since the anti-commutator must sit in the representation $(\frac{1}{2},\frac{1}{2})$, and as noted the Coleman-Mandula theorem then asserts that this bosonic operator must be a Poincaré generator. Through conjugation of the anti-commutator, we can see that H^{IJ} is a hermitian matrix, so it can be diagonalized and normalized by a proper choice of basis. Thus the anti-commutator is taken as:

$$\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} = 2\delta^{IJ}\, P_{\alpha\dot\beta}\,. \tag{3.9}$$

Let us check the commutation relations of the new fermionic generators with the Poincaré generators. We start with the energy-momentum generator:

$$[P_{\alpha\dot\beta}, Q^I_\gamma] = \epsilon_{\alpha\gamma} X^{IJ} \bar{Q}^J_{\dot\beta} \quad\Longrightarrow\quad [P_{\beta\dot\alpha}, \bar{Q}^I_{\dot\gamma}] = \epsilon_{\dot\alpha\dot\gamma} (X^*)^{IJ'} Q^{J'}_\beta\,, \tag{3.10}$$

where from representation considerations the commutators must sit in $(0,\frac{1}{2})$, $(\frac{1}{2},0)$, respectively, since we saw that a spin 3/2 operator, as in e.g. $(1,\frac{1}{2})$, does not exist in the extended supersymmetric algebra, and the second commutator is obtained from the first by conjugation. We will now show that the matrix of coefficients X must be equal to 0. On the one hand, we can easily write:

$$[P_{\alpha\dot\beta}, [P_{\alpha\dot\beta}, \{Q^I_\gamma, \bar{Q}^J_{\dot\delta}\}]] = 0\,. \tag{3.11}$$

On the other hand, using the super-Jacobi identity:

$$[P, \{Q, Q'\}] = \{Q, [P, Q']\} + \{Q', [P, Q]\}\,, \tag{3.12}$$

we can also write:

$$\begin{aligned}
[P_{\alpha\dot\beta}, [P_{\alpha\dot\beta}, \{Q^I_\gamma, \bar{Q}^J_{\dot\delta}\}]] &= [P_{\alpha\dot\beta}, \big(\{Q^I_\gamma, [P_{\alpha\dot\beta}, \bar{Q}^J_{\dot\delta}]\} + \{\bar{Q}^J_{\dot\delta}, [P_{\alpha\dot\beta}, Q^I_\gamma]\}\big)] \\
&= [P_{\alpha\dot\beta}, \big(\epsilon_{\dot\beta\dot\delta}(X^*)^{JK}\{Q^I_\gamma, Q^K_\alpha\} + \epsilon_{\alpha\gamma} X^{IL}\{\bar{Q}^J_{\dot\delta}, \bar{Q}^L_{\dot\beta}\}\big)] \\
&= \epsilon_{\dot\beta\dot\delta}(X^*)^{JK}\big(\{Q^I_\gamma, [P_{\alpha\dot\beta}, Q^K_\alpha]\} + \{Q^K_\alpha, [P_{\alpha\dot\beta}, Q^I_\gamma]\}\big) \\
&\quad + \epsilon_{\alpha\gamma} X^{IL}\big(\{\bar{Q}^J_{\dot\delta}, [P_{\alpha\dot\beta}, \bar{Q}^L_{\dot\beta}]\} + \{\bar{Q}^L_{\dot\beta}, [P_{\alpha\dot\beta}, \bar{Q}^J_{\dot\delta}]\}\big) \\
&= \epsilon_{\dot\beta\dot\delta}\epsilon_{\alpha\gamma}(X^*)^{JK} X^{IL}\{Q^K_\alpha, \bar{Q}^L_{\dot\beta}\} \\
&\qquad + \epsilon_{\alpha\gamma}\epsilon_{\dot\beta\dot\delta} X^{IL}(X^*)^{JK}\{\bar{Q}^L_{\dot\beta}, Q^K_\alpha\} \\
&= 4\epsilon_{\alpha\gamma}\epsilon_{\dot\beta\dot\delta} P_{\alpha\dot\beta}\delta_{KL} X^{IL}(X^*)^{JK} = 4\epsilon_{\alpha\gamma}\epsilon_{\dot\beta\dot\delta} P_{\alpha\dot\beta}(XX^\dagger)^{IJ}.
\end{aligned} \tag{3.13}$$

Comparing equations (3.11) and (3.13), we infer that $XX^\dagger = 0$, but $XX^\dagger$ is positive-definite, thus $X = 0$.

We can then infer:

$$[P_{\alpha\dot{\beta}}, Q^I_\gamma] = [P_{\alpha\dot{\beta}}, \bar{Q}^I_{\dot{\gamma}}] = 0\,, \tag{3.14}$$

so the supersymmetric charges commute with the energy-momentum generator. Note that thus far these commutation relations of the new fermionic generators are similar to those that the Coleman-Mandula would have imposed on bosonic generators.

To see the differences then, let us check the commutation relations of the supersymmetric generators with the homogenous Lorentz generators, L_i and R_i. These relations are easier to find once we recall that the spinorial representations of these generators are the symmetric bispinors $M_{\alpha\beta}$, $M_{\dot{\alpha}\dot{\beta}}$, respectively. Then, we get for L_i:

$$[M_{\alpha\beta}, Q^I_\gamma] = i\big(\epsilon_{\alpha\gamma} Q^I_\beta + \epsilon_{\beta\gamma} Q^I_\alpha\big)\,, \tag{3.15}$$

since from representation considerations the commutator must sit in $(\frac{1}{2}, 0)$, and we saw that a spin 3/2 operator, as in $(\frac{3}{2}, 0)$, does not exist in the extended supersymmetric algebra, whereas:

$$[M_{\alpha\beta}, \bar{Q}^I_{\dot{\gamma}}] = 0\,, \tag{3.16}$$

since from representation considerations a non-vanishing commutator would sit in $(1, \frac{1}{2})$, and as we noted a spin 3/2 operator does not exist in the extended supersymmetric algebra. Similarly, for R_i we get:

$$[M_{\dot{\alpha}\dot{\beta}}, Q^I_\gamma] = 0\,, \qquad [M_{\dot{\alpha}\dot{\beta}}, \bar{Q}^I_{\dot{\gamma}}] = i\big(\epsilon_{\dot{\alpha}\dot{\gamma}} \bar{Q}^I_{\dot{\beta}} + \epsilon_{\dot{\beta}\dot{\gamma}} \bar{Q}^I_{\dot{\alpha}}\big)\,. \tag{3.17}$$

Thus, this is where we see the difference in the fermionic supersymmetric generators, which do not commute with the Lorentz generators of the Poincaré algebra. For completeness, we also include here the non-vanishing commutation relations of the supercharges with the Lorentz generators in their tensorial representation:

$$[M_{\mu\nu}, Q^I_\alpha] = i(\sigma_{\mu\nu} Q^I)_\alpha\,, \qquad [M_{\mu\nu}, \bar{Q}^{I\dot{\alpha}}] = i(\bar{\sigma}_{\mu\nu} \bar{Q}^I)^{\dot{\alpha}}\,. \tag{3.18}$$

Let us press on to uncover the new super-Poincaré algebra by also considering the commutation relations between the fermionic elements of the algebra. For the anti-commutator of the supercharges, we have:

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ} \cancel{+ M_{\alpha\beta} Y^{IJ}}\,, \tag{3.19}$$

where the first and second terms are the anti-symmetric and symmetric parts, respectively. From representation considerations and the Coleman-Mandula theorem, the second term would sit in $(1, 0)$, that is the Lorentz generator L_i. However, due to eq. (3.14), the LHS commutes with P_μ, whereas on the RHS the first term also does, but the second term does not commute with P_μ, and thus the latter would be inconsistent. It is also easy to get:

$$Z^{IJ} = -Z^{JI} \,, \tag{3.20}$$

since the LHS is symmetric under the simultaneous interchange of both the spinorial and supercharge indices. In particular for $\mathcal{N} = 1$, i.e. for simple supersymmetry, $Z^{IJ} = Z^{11} = 0$.

Let us consider the commutation relations of Z^{IJ} (for $\mathcal{N} > 1$). First, we can write:

$$[\bar{Q}^K_{\dot{\gamma}}, Z^{IJ}] \sim [\bar{Q}^K_{\dot{\gamma}}, \{Q^I_\alpha, Q^J_\beta\}] = 0 \,, \tag{3.21}$$

where the last equality is obtained from the Jacobi identity:

$$[\bar{Q}^K_{\dot{\gamma}}, \{Q^I_\alpha, Q^J_\beta\}] + \cancel{[Q^I_\alpha, \{Q^J_\beta, \bar{Q}^K_{\dot{\gamma}}\}]} + \cancel{[Q^J_\beta, \{\bar{Q}^K_{\dot{\gamma}}, Q^I_\alpha\}]} = 0 \,, \tag{3.22}$$

in which the last two terms drop due to eqs. (3.9) and (3.14). Next, we consider:

$$[Z^{IJ}, Q^K_\alpha] = X^{\underline{IJK}\,L} Q^L_\alpha \,, \tag{3.23}$$

from representation considerations as Z^{IJ} is a Lorentz scalar, and $X^{\underline{IJK}\,L}$ is some matrix of numerical coefficients. We will show now that $X = 0$. First, using Jacobi identity, we can write:

$$\{[Z^{IJ}, Q^K_\alpha], \bar{Q}^M_{\dot{\beta}}\} = -\cancel{[\{Q^K_\alpha, \bar{Q}^M_{\dot{\beta}}\}, Z^{IJ}]} + \cancel{\{[\bar{Q}^M_{\dot{\beta}}, Z^{IJ}], Q^K_\alpha\}} = 0 \,, \tag{3.24}$$

where the first term drops due to eq. (3.9), and since Z^{IJ} commutes with the Poincaré algebra, and the second term drops due to eq. (3.21). From eqs. (3.23), (3.24), we infer:

$$\{X^{\underline{IJK}\,L} Q^L_\alpha, \bar{Q}^M_{\dot{\beta}}\} = 0 \,, \tag{3.25}$$

which is true for any spinor index, and any supercharge indices, so we can write:

$$\{X^{\underline{IJK}\,L} Q^L_\alpha, (X^{\underline{IJK}\,M})^* \bar{Q}^M_{\dot{\alpha}}\} = 0 \,. \tag{3.26}$$

If we denote the first operator in the anti-commutator by Y, then we got $YY^\dagger + Y^\dagger Y = 0$, and since this is a sum of positive-definite operators, we infer that $Y = 0$, and thus $X = 0$. Therefore, we conclude:

$$[Z^{IJ}, Q^K_\alpha] = 0 \,, \tag{3.27}$$

and thus far we have seen then, that Z^{IJ} commutes with all supercharges, as well as with the Poincaré algebra.

Let us consider now the commutation relations of Z^{IJ} among themselves. First from eqs. (3.19), (3.27), and the use of a Jacobi identity, it is easy to see:

$$[Z^{IJ}, Z^{KL}] = [Z^{IJ}, \{Q_1^K, Q_2^L\}] = 0 \quad \Longrightarrow \quad [\bar{Z}^{IJ}, \bar{Z}^{KL}] = 0\,, \tag{3.28}$$

where the second equality is obtained from the first via conjugation (in the supercharge indices), using $\bar{Z} \equiv Z^\dagger$, and we also have:

$$\{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta}\bar{Z}^{IJ}\,. \tag{3.29}$$

Then similarly, from eqs. (3.21), (3.29), and the use of a Jacobi identity, we get:

$$[Z^{IJ}, \bar{Z}^{KL}] = [Z^{IJ}, \{\bar{Q}^K_{\dot 1}, \bar{Q}^L_{\dot 2}\}] = 0\,. \tag{3.30}$$

To conclude, Z^{IJ} and its conjugate, also commute among themselves.

According to the Coleman-Mandula theorem, the bosonic subalgebra is $\mathfrak{g}_{\mathrm{P}} \oplus \mathfrak{g}_{\mathrm{internal}}$, such that the generators of the internal symmetry group, G_{internal}, commute with the Poincaré algebra, i.e. they are also Lorentz scalars. Let us then proceed to consider the commutators of the generators of G_{internal} with Z^{IJ}. First, we denote the generators of G_{internal} in some representation r by $T_{(r)}$, and thus we can write:

$$[T_{(r)}, Q^I_\alpha] = X^{IJ}Q^J_\alpha\,, \tag{3.31}$$

from representation considerations. Using this together with eq. (3.19), and a Jacobi identity, we can then write:

$$\begin{aligned}
[T_{(r)}, Z^{IJ}] &= [T_{(r)}, \{Q_1^I, Q_2^J\}] = \{[T_{(r)}, Q_1^I], Q_2^J\} + \{[T_{(r)}, Q_2^J], Q_1^I\} \\
&= X^{IK}\{Q_1^K, Q_2^J\} + X^{JK}\{Q_2^K, Q_1^I\} \\
&= X^{IK}Z^{KJ} - X^{JK}Z^{KI} = M^{\underline{IJ}\,\underline{KL}}Z^{\underline{KL}}\,,
\end{aligned} \tag{3.32}$$

where M is some matrix of coefficients between pairs of supercharge indices. Thus we can already see that $[\mathfrak{g}_{\mathrm{internal}}, Z] = Z'$, and $Z \subseteq \mathfrak{g}_{\mathrm{internal}}$, so the Z operators form an Abelian subalgebra of $\mathfrak{g}_{\mathrm{internal}}$. Furthermore, we will show now that the Z operators commute with $\mathfrak{g}_{\mathrm{internal}}$ as well. We recall then, that $T_{(r)}$ and Z are both represented by finite matrices in the same representation, so we can consider the following trace:

$$\begin{aligned}
\mathrm{tr}\left([T_{(r)}, Z^{IJ}]\bar{Z}^{KL}\right) &= \mathrm{tr}\left(T_{(r)}Z^{IJ}\bar{Z}^{KL} - Z^{IJ}T_{(r)}\bar{Z}^{KL}\right) \\
&= \mathrm{tr}\left(Z^{IJ}\bar{Z}^{KL}T_{(r)} - \bar{Z}^{KL}Z^{IJ}T_{(r)}\right) \\
&= \mathrm{tr}\left([Z^{IJ}, \bar{Z}^{KL}]T_{(r)}\right) = 0\,,
\end{aligned} \tag{3.33}$$

where we used the cyclicity of the trace, and eq. (3.30). Using eqs. (3.32), (3.33), we can now write:

$$\mathrm{tr}\left(M^{\underline{IJ}\,\underline{MN}} Z^{\underline{MN}} \left(M^{\underline{IJ}\,\underline{KL}}\right)^* \bar{Z}^{\underline{KL}}\right) = 0\,. \tag{3.34}$$

But $(MZ)(MZ)^\dagger$ is positive-definite, and the trace is invariant, so we can infer that $MZ = 0$.

We can conclude then:

$$[T_{(r)}, Z^{IJ}] = 0 \quad \Longrightarrow \quad [\mathfrak{g}_{\text{internal}}, Z^{IJ}] = 0\,, \quad I, J \in \{1, \ldots, \mathcal{N}\}\,, \tag{3.35}$$

thus Z^{IJ} commute with all of the generators of the super-Poincaré algebra, so they belong to the center of the algebra, and are called central charges. The central charges then form an Abelian subalgebra of the super-Poincaré algebra, inside the algebra of the internal symmetry, $\mathfrak{g}_{\text{internal}}$.

3.1.1 R-Symmetry

Before we conclude our discussion on the super-Poincaré algebra for QFTs with massive particles, let us note here that there is another possible symmetry, which is implicit in the super-Poincaré algebra, called R-symmetry. This symmetry is an automorphism of the super-Poincaré algebra. In $D = 4$ spacetime dimensions it can make unitary rotations among the supercharges in the Weyl basis, while leaving the supersymmetric algebra invariant. Thus, the maximal R-symmetry possible for a supersymmetry of $\mathcal{N}$ supercharges is $U(\mathcal{N})$, with the supercharges transforming in the fundamental representation, and the conjugate supercharges in the anti-fundamental representation.

For example, in a simple supersymmetry with $\mathcal{N} = 1$ the maximal R-symmetry is $U(1)_R$. It acts on the supercharges as follows:

$$Q_\gamma \to Q'_\gamma = \exp(-i\alpha) Q_\gamma\,, \quad \bar{Q}_{\dot{\gamma}} \to \bar{Q}'_{\dot{\gamma}} = \exp(+i\alpha)\bar{Q}_{\dot{\gamma}}\,, \quad \alpha \in \mathbb{R}\,, \tag{3.36}$$

which clearly leaves the supersymmetric algebra invariant. We will get a better understanding of how this symmetry may be realized as of chapter 5, after we have introduced the concepts of superspace and superfields in chapter 4. Moreover, as we shall see shortly in section 3.2, when the super-Poincaré algebra is extended to a superconformal algebra, R-symmetry shows up explicitly as an additional generator in the algebra.

3.2 Superconformal Algebra

Let us now turn to the case of QFTs without massive particles, i.e. massless QFTs. We already noted in section 1.2 on the Coleman-Mandula theorem, that the most general symmetry of a theory without massive particles, excluding fermionic generators, is extended as follows:

$$\mathfrak{g}_{\mathrm{P}} \oplus \mathfrak{g}_{\mathrm{internal}} \quad \rightarrow \quad \mathfrak{g}_{\mathrm{C}} \oplus \mathfrak{g}_{\mathrm{internal}} \, . \tag{3.37}$$

With conformal symmetry there is no mass or length scale, hence there is an invariance with respect to changes of scale, namely scale invariance. In $D = 4$ spacetime dimensions the conformal algebra has 5 additional generators beyond the 10 Poincaré generators.

Before we proceed to uncover the extension of the conformal algebra to a superconformal algebra by the addition of fermionic generators, let us get a bit familiar with conformal symmetry by presenting its generators in differential form. For the Poincaré algebra we recall that we have:

$$P_\mu = -i\partial_\mu \, , \tag{3.38}$$

$$M_{\mu\nu} = i\big(x_\mu \partial_\nu - x_\nu \partial_\mu\big) \, . \tag{3.39}$$

In the conformal algebra the following generators are added:

$$D = -ix^\nu \partial_\nu \, , \tag{3.40}$$

$$K_\mu = i\big(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu\big) \, . \tag{3.41}$$

The first equation shows the dilatation generator, a Lorentz scalar, that generates rescaling transformations, e.g., $x^\mu \rightarrow \alpha x^\mu$, where $\alpha > 0$ is some rescaling factor. The second equation shows the generator of special conformal transformations, such as inversion, $x^\mu \rightarrow x^\mu / x^2$. Obviously, the special conformal generator is a 4-vector.

Using this differential form of the generators, the conformal algebra can be inferred, where we note only the non-vanishing commutation relations:

$$[M_{\mu\nu}, M_{\rho\sigma}] = i\big(\eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\nu\rho}\big) \, , \tag{3.42}$$

$$[M_{\mu\nu}, P_\rho] = -i\big(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu\big) \, , \tag{3.43}$$

from the Poincaré algebra, and the additional relations:

$$[M_{\mu\nu}, K_\rho] = -i\big(\eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu\big) \, , \tag{3.44}$$

$$[P_\mu, K_\nu] = -2i\big(\eta_{\mu\nu} D + M_{\mu\nu}\big) \, , \tag{3.45}$$

where the last relation may be further understood in terms of the spinorial representation:

$$[P_{\alpha\dot{\beta}}, K_{\gamma\dot{\delta}}] = 2i\big(2\epsilon_{\alpha\gamma}\epsilon_{\dot{\beta}\dot{\delta}} D - \epsilon_{\dot{\beta}\dot{\delta}} M_{\alpha\gamma} - \epsilon_{\alpha\gamma} M_{\dot{\beta}\dot{\delta}}\big) \, , \tag{3.46}$$

since from representation considerations, we should get on the RHS $(0,0)$, $(1,0)$ and $(0,1)$, but not $(1,1)$, since a spin 2 operator does not exist in the conformal algebra. Finally, the commutation relations with D can also be easily fixed from the combination of dimensional considerations, where $[P] = 1$, $[K] = -1$, $[D] = 0$, and the familiar considerations of spinorial representations. Thus, the non-vanishing relations with D read:

$$[D, P_{\alpha\dot{\beta}}] = +\, iP_{\alpha\dot{\beta}}\,, \tag{3.47}$$

$$[D, K_{\alpha\dot{\beta}}] = -\, iK_{\alpha\dot{\beta}}\,. \tag{3.48}$$

In a similar manner to the extension of the Poincaré algebra to a super-Poincaré algebra, the conformal algebra can be even further extended to the superconformal algebra through the addition of new fermionic generators. We begin then by considering the commutation relations of the additional bosonic generators in the conformal algebra with the Poincaré supercharges, defined via eq. (3.9). Recalling that we are also guided now by dimensional considerations, we note that from eq. (3.9), it is easy to infer that $[Q] = 1/2$. Then again by combining dimensional and spinor-representation considerations, it is easy to write the commutation relation with the dilatation generator:

$$[D, Q^I_\alpha] = \frac{i}{2} Q^I_\alpha\,, \qquad [D, \bar{Q}^I_{\dot{\alpha}}] = \frac{i}{2} \bar{Q}^I_{\dot{\alpha}}\,, \tag{3.49}$$

where the second relation is obtained from the first by conjugation.

For the special conformal generator we write the commutation relation:

$$[K_{\alpha\dot{\beta}}, Q^I_\gamma] = 2\epsilon_{\alpha\gamma} \bar{S}^I_{\dot{\beta}}\,, \tag{3.50}$$

where we uncover a new conformal supercharge! It has a new dimension of $[S] = -1/2$, and sits in $(0, \frac{1}{2})$, since the extension of a bosonic algebra to include fermionic operators does not allow for operators of spin $3/2$, as in $(1, \frac{1}{2})$. The new superconformal charge is then defined as follows:

$$\bar{S}^I_{\dot{\beta}} \equiv -\frac{1}{4} \epsilon^{\alpha\gamma} [K_{\alpha\dot{\beta}}, Q^I_\gamma]\,. \tag{3.51}$$

By conjugation of eq. (3.50) we obtain:

$$[K_{\alpha\dot{\beta}}, \bar{Q}^I_{\dot{\gamma}}] = 2\epsilon_{\dot{\beta}\dot{\gamma}} S^I_\alpha\,. \tag{3.52}$$

Thus, the superconformal algebra doubles the supercharges with respect to the super-Poincaré algebra.

Let us check then the commutation relations of the new supercharges with the Poincaré generators. First we consider:

$$\begin{aligned}[P_{\alpha\dot{\beta}}, \bar{S}^I_{\dot{\gamma}}] &= -\frac{1}{4}\epsilon^{\rho\sigma}[P_{\alpha\dot{\beta}}, [K_{\rho\dot{\gamma}}, Q^I_\sigma]] = -\frac{1}{4}\epsilon^{\rho\sigma}[[P_{\alpha\dot{\beta}}, K_{\rho\dot{\gamma}}], Q^I_\sigma] \\ &= -\frac{i}{2}\epsilon^{\rho\sigma}\big(2\epsilon_{\alpha\rho}\epsilon_{\dot{\beta}\dot{\gamma}}[D, Q^I_\sigma] - \epsilon_{\dot{\beta}\dot{\gamma}}[M_{\alpha\rho}, Q^I_\sigma]\big) \\ &= \frac{1}{2}\epsilon_{\dot{\beta}\dot{\gamma}}\big(Q^I_\alpha - \epsilon^{\rho\sigma}(\epsilon_{\alpha\sigma}Q^I_\rho + \epsilon_{\rho\sigma}Q^I_\alpha)\big) = 2\epsilon_{\dot{\beta}\dot{\gamma}}Q^I_\alpha\,, \end{aligned} \tag{3.53}$$

where in the first line for the second equality we used the Jacobi identity and eq. (3.14), in the second line we used eq. (3.46), and in the third line for the first equality we used eqs. (3.49), (3.15). In fact, the combination of dimensional, representation, and symmetry considerations, already fixes the result, up to the numerical coefficient of proportionality, for which the above computation was required. By conjugation of the last result we obtain:

$$[P_{\alpha\dot{\beta}}, S^I_\gamma] = 2\epsilon_{\alpha\gamma}\bar{Q}^I_{\dot{\beta}}\,. \tag{3.54}$$

It is easy to verify that the commutation relations of the conformal supercharges with the Lorentz generators, L_i, R_i, are analogous to those with the Poincaré supercharges, e.g. for L_i:

$$[M_{\alpha\beta}, S^I_\gamma] = i\big(\epsilon_{\alpha\gamma}S^I_\beta + \epsilon_{\beta\gamma}S^I_\alpha\big)\,, \qquad [M_{\alpha\beta}, \bar{S}^I_{\dot{\gamma}}] = 0\,, \tag{3.55}$$

where the relations with R_i are simply obtained by conjugation.

In fact, it is also easy to verify that similar to the commutation relation of Q with its conjugate $\bar{Q}$, the new supercharges S, $\bar{S}$, satisfy the following relation:

$$\{S^I_\alpha, \bar{S}^J_{\dot{\beta}}\} = 2\delta^{IJ}K_{\alpha\dot{\beta}}\,, \tag{3.56}$$

so that the superconformal charges are the "square root" of the special conformal generator. Similar to the Poincaré supercharges in the absence of central charges, which is necessarily implied in the superconformal algebra, as shown in the problem sheet, for the conformal supercharges we also have the relation:

$$\{S^I, S^J\} = \{\bar{S}^I, \bar{S}^J\} = 0\,. \tag{3.57}$$

From dimensional and representation considerations it is also easy to infer:

$$[K_{\rho\dot{\sigma}}, \bar{S}^I_{\dot{\beta}}] = 0\,, \tag{3.58}$$

since there is no generator of dimension $-3/2$. By conjugation we also get $[K, S^I] = [K, \bar{S}^I] = 0$, similar to $[P, Q^I] = [P, \bar{Q}^I] = 0$. Finally, it is also easy to find the commutation relations of the conformal supercharges with D:

$$[D, S^I_\alpha] = -\frac{i}{2} S^I_\alpha \,, \qquad [D, \bar{S}^I_{\dot\alpha}] = -\frac{i}{2} \bar{S}^I_{\dot\alpha} \,. \tag{3.59}$$

Our final task is to find the commutation relations between the Poincaré and the conformal supercharges. First, it is easy to infer:

$$\{Q^I_\alpha, \bar{S}^J_{\dot\beta}\} = \{\bar{Q}^I_{\dot\alpha}, S^J_\beta\} = 0 \,, \tag{3.60}$$

since there is no bosonic generator of dimension 0 that sits in $(\frac{1}{2}, \frac{1}{2})$. Proceeding to the last commutation relation between the supercharges, we write:

$$\{Q^I_\alpha, S^J_\beta\} = \epsilon_{\alpha\beta}\big(c_1 T^{IJ} + c_2 \delta^{IJ} D\big) + c_3 \delta^{IJ} M_{\alpha\beta} \,, \tag{3.61}$$

where the first bracketed term, and the second one, are the anti-symmetric, and symmetric parts, that sit in $(0, 0)$, and $(1, 0)$, respectively, where c_1, c_2, c_3, are some numerical coefficients.

What is T^{IJ}? It turns out that we need to introduce a new symmetry generator for the closure of the superconformal algebra. We shall see now that T^{IJ} is a Lorentz scalar, but unlike the dilatation generator, it actually commutes with the whole (bosonic) conformal algebra. Yet, it does not commute with the (fermionic) supercharges: neither with the super-Poincaré charges, nor with the superconformal ones.

Let us define then:

$$T^{IJ} \equiv \epsilon^{\alpha\beta}\{Q^I_\alpha, S^J_\beta\} + 4i\delta^{IJ} D \,. \tag{3.62}$$

The contraction of eq. (3.61) with $\epsilon^{\alpha\beta}$ removes the symmetric part of it, and by this definition: $c_1 = -1/2$, $c_2 = 2i$. Since we are left with the $(0, 0)$ part of eq. (3.61), T^{IJ} clearly commutes with the Lorentz generators. Let us check the commutator with $P_{\alpha\dot\beta}$:

$$\begin{aligned}
[T^{IJ}, P_{\gamma\dot\delta}] &= \epsilon^{\alpha\beta}[\{Q^I_\alpha, S^J_\beta\}, P_{\gamma\dot\delta}] + 4i\delta^{IJ}[D, P_{\gamma\dot\delta}] \\
&= -\epsilon^{\alpha\beta}\{[P_{\gamma\dot\delta}, S^J_\beta], Q^I_\alpha\} - 4\delta^{IJ} P_{\gamma\dot\delta} \\
&= 2\big(\{\bar{Q}^J_{\dot\delta}, Q^I_\gamma\} - 2\delta^{IJ} P_{\gamma\dot\delta}\big) = 0 \,,
\end{aligned} \tag{3.63}$$

where in the second equality we used the Jacobi identity and eqs. (3.14), (3.47), and in the third, and last equalities, we used eqs. (3.54), and (3.9), respectively. It is easy to show that in addition:

$$[T^{IJ}, K_{\alpha\dot\beta}] = [T^{IJ}, D] = 0 \,. \tag{3.64}$$

It remains to check the commutation relations of T^{IJ} with the supercharges, for example:

$$\begin{aligned}[T^{IJ},\bar{Q}^K_{\dot{\alpha}}] &= \epsilon^{\beta\gamma}[\{Q^I_\beta, S^J_\gamma\},\bar{Q}^K_{\dot{\alpha}}] + 4i\delta^{IJ}[D,\bar{Q}^K_{\dot{\alpha}}] \\ &= -\epsilon^{\beta\gamma}[\{Q^I_\beta,\bar{Q}^K_{\dot{\alpha}}\}, S^J_\gamma] - 2\delta^{IJ}\bar{Q}^K_{\dot{\alpha}} = -2\epsilon^{\beta\gamma}\delta^{IK}[P_{\beta\dot{\alpha}}, S^J_\gamma] - 2\delta^{IJ}\bar{Q}^K_{\dot{\alpha}} \\ &= -4\epsilon^{\beta\gamma}\epsilon_{\beta\gamma}\delta^{IK}\bar{Q}^J_{\dot{\alpha}} - 2\delta^{IJ}\bar{Q}^K_{\dot{\alpha}} = 8\delta^{IK}\bar{Q}^J_{\dot{\alpha}} - 2\delta^{IJ}\bar{Q}^K_{\dot{\alpha}}\,, \end{aligned} \tag{3.65}$$

where in the second equality we used the Jacobi identity. So we see that T^{IJ} rotates the supercharges among themselves: $\bar{Q}^K \to \bar{Q}^J$. It is also easy to show that T^{IJ} is Hermitian, and then by conjugation find the commutator with the supercharge Q^K, which similarly yields Q^I. Thus, T rotates Q, $\bar{Q}$, and it can be similarly shown to also rotate S, $\bar{S}$.

This symmetry, acting on the supercharges, is called R-symmetry, as we noted in section 3.1.1. The novelty here is that in the superconformal algebra a generator of R-symmetry is included, whereas in non-conformal supersymmetric theories R-symmetry is not a part of the algebra, nor does it even always hold. Finally, we can fully fix the numerical coefficients in eq. (3.61) by a proper computation.

To recap, our lessons in the transition from the super-Poincaré to the superconformal algebra can be summarized as follows:

- The supersymmetry is "doubled": For each of the $\mathcal{N}$ pairs Q^I, $\bar{Q}^I$, of fermionic super-Poincaré charges, there is an additional pair of fermionic superconformal charges, S^I, $\bar{S}^I$, of dimension $[S] = -1/2$.

- The commutator between the two types of supercharges, Q^I and S^J, yields a new bosonic generator of R-symmetry, T^{IJ}, which rotates each set of fermionic supercharges, and of their conjugates, onto itself. Thus, there is another generator, which is conformally invariant, in the internal symmetry algebra, $T^{IJ} \in \mathfrak{g}_{\text{internal}}$, of dimension $[T] = 0$.

3.3 Supersymmetric Representations

We return to the super-Poincaré algebra, and ask what are the representations of its one-particle states?

Let us first recall the irreducible representations of the Poincaré algebra. Using the Pauli-Lubanski pseudovector which is defined as follows:

$$W_\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}\,, \tag{3.66}$$

the only Casimir operators of the Poincaré algebra can be shown to be:

$$C_1 = P_\mu P^\mu \,, \qquad C_2 = W_\mu W^\mu \,, \tag{3.67}$$

which commute with every Poincaré generator. For the Poincaré algebra we know then that there are two kinds of irreducible representations:

1. Massive Particles. For $P_\mu P^\mu = -M^2 < 0$ the representation is labelled by its mass and spin, defined by the 2 Casimir operators in eq. (3.67). For a massive particle of energy-momentum p_μ we can then go to the rest frame:

$$p^\mu = (M, \vec{0}) \,. \tag{3.68}$$

These particles are then classified in terms of representations of the "little group" that leaves eq. (3.68) invariant, namely $SO(3)$. We then have:

$$W^\mu = (0, W^i) \,, \quad W^i = P_0 J^i \,, \tag{3.69}$$

where J_i is the $SO(3)$ spin operator, that satisfies:

$$J^i \equiv -\frac{1}{2}\epsilon^{ijk} M_{jk} \,, \quad [J^i, J^j] = i\epsilon^{ijk} J^k \,. \tag{3.70}$$

Therefore the massive particles are classified by their mass M and their spin $j \in \frac{1}{2}\mathbb{N}_0$:

$$C_1 = -M^2 \,, \qquad C_2 = M^2 J_i J^i = M^2 j(j+1) \,, \tag{3.71}$$

where j is the highest projection of angular momentum on the z axis. Thus at a fixed mass M^2 a representation of the Poincaré group is a representation of $SO(3)$, or more precisely of $SU(2)$, since the spin can be half-integer. The spin-j representation consists of $2j+1$ states, $|j, m\rangle$:

$$J^z |j, m\rangle = m |j, m\rangle \,, \quad -j \leqslant m \leqslant j \,. \tag{3.72}$$

2. Massless Particles. For $P_\mu P^\mu = 0$ the representation is labelled by its energy and helicity. For a massless particle we can go to the frame:

$$p^\mu = (E, 0, 0, E) \,, \qquad E > 0 \,. \tag{3.73}$$

These particles are then classified by their energy, and helicity, $\lambda \in \frac{1}{2}\mathbb{Z}$, which is a representation of the little group $SO(2)$, or more precisely of $U(1)$. The helicity operator corresponds to $J^z = -M_{xy}$ by definition:

$$J^z |E, \lambda\rangle = \lambda |E, \lambda\rangle \,, \tag{3.74}$$

which amounts to having only 1 state in the representation. Yet, since QFTs are typically CPT-invariant, and CPT reverses the sign of helicity, we actually get 2 states: $\lambda, -\lambda$.

Let us now build the supermultiplets, which are the collections of one-particles states, that form representations of the super-Poincaré algebra. First, we obviously have:

$$[C_1, Q_\alpha] = [P_\mu P^\mu, Q_\alpha] = 0\,, \quad [C_1, \bar{Q}_{\dot{\alpha}}] = [P_\mu P^\mu, \bar{Q}_{\dot{\alpha}}] = 0\,, \tag{3.75}$$

due to eq. (3.14), so all particles or irreducible representations in a supermultiplet have the same invariant mass, M^2. In other words, C_1 is also a Casimir of the full super-Poincaré algebra.

Yet, this is not the case for C_2. The action of J^z on the supercharges can be written in the compact form:

$$[J^z, Q_\alpha] = -\frac{1}{2}(\sigma^z)_\alpha^{\ \beta} Q_\beta = +\frac{(-1)^\alpha}{2} Q_\alpha\,, \tag{3.76}$$

$$[J^z, \bar{Q}_{\dot{\alpha}}] = +\frac{1}{2}\bar{Q}_{\dot{\beta}}(\sigma^z)^{\dot{\beta}}_{\ \dot{\alpha}} = -\frac{(-1)^\alpha}{2}\bar{Q}_{\dot{\alpha}}\,, \tag{3.77}$$

using eqs. (3.70), (3.18), (2.30), (2.21), (2.13), where $\alpha = 1, 2$.

Before we proceed then to treat each of the massive or massless cases, let us prove first that any representation of the supersymmetric algebra contains an equal number of bosonic and fermionic states, n_B and n_F, respectively, regardless of whether it is massive or massless. To this end we introduce the fermion number operator:

$$(-1)^F \equiv \begin{cases} (-1)^F|b\rangle = +|b\rangle \\ (-1)^F|f\rangle = -|f\rangle \end{cases}\,, \tag{3.78}$$

where $|b\rangle$ or $|f\rangle$ are bosonic or fermionic states, respectively. We are then interested to compute:

$$\mathrm{tr}((-1)^F) = n_B - n_F\,, \tag{3.79}$$

where the trace is over the whole finite-dimensional Hilbert space of the supersymmetric representation. Since P_μ is fixed for the whole supersymmetric representation due to eq. (3.14), we can write:

$$\begin{aligned} 2\delta^{11}P_{\alpha\dot{\beta}}\,\mathrm{tr}\left((-1)^F\right) &= \mathrm{tr}\left((-1)^F 2\delta^{11}P_{\alpha\dot{\beta}}\right) = \mathrm{tr}\left((-1)^F\{Q^1_\alpha, \bar{Q}^1_{\dot{\beta}}\}\right) \\ &= \mathrm{tr}\left((-1)^F\left(Q^1_\alpha\bar{Q}^1_{\dot{\beta}} + \bar{Q}^1_{\dot{\beta}}Q^1_\alpha\right)\right) \\ &= \mathrm{tr}\left((-1)^F Q^1_\alpha\bar{Q}^1_{\dot{\beta}} + Q^1_\alpha(-1)^F\bar{Q}^1_{\dot{\beta}}\right) \\ &= \mathrm{tr}\left((-1)^F\left(Q^1_\alpha\bar{Q}^1_{\dot{\beta}} - Q^1_\alpha\bar{Q}^1_{\dot{\beta}}\right)\right) = 0\,, \end{aligned} \tag{3.80}$$

where for the third line we used the cyclicity of the trace, and in the fourth line we used:

$$(-1)^F Q = -Q(-1)^F\,, \tag{3.81}$$

from the definition in eq. (3.78). Thus, for a non-vanishing p_μ of the supermultiplet we infer from eqs. (3.79), (3.80):

$$p_\mu \neq 0 \implies \text{tr}\left((-1)^F\right) = n_B - n_F = 0\,. \tag{3.82}$$

3.3.1 Massive Supermultiplets

Let us first construct representations of supersymmetry for massive one-particle states, $P^2 = -M^2$. Boosting to the rest frame, where $P^\mu = (M, \vec{0})$, the supersymmetric algebra takes the form:

$$\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} = 2M\delta_{\alpha\dot\beta}\delta^{IJ}\,, \tag{3.83}$$

$$\{Q^I_\alpha, Q^J_\beta\} = \{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\} = 0\,, \tag{3.84}$$

where we treat here the simple case without central charges. Let us define the rescaled generators:

$$a^I_\alpha \equiv \frac{1}{\sqrt{2M}} Q^I_\alpha\,, \qquad (a^I_\alpha)^\dagger \equiv \frac{1}{\sqrt{2M}} \bar{Q}^I_{\dot\alpha}\,, \tag{3.85}$$

which satisfy the following commutation relations:

$$\{a^I_\alpha, (a^J_\beta)^\dagger\} = \delta_{\alpha\beta}\delta^{IJ}\,, \tag{3.86}$$

$$\{(a^I_\alpha)^\dagger, (a^J_\beta)^\dagger\} = \{a^I_\alpha, a^J_\beta\} = 0\,. \tag{3.87}$$

This is similar to the Clifford algebra of $2\mathcal{N}$ fermionic creation and annihilation operators of Dirac fields, $(a^I_\alpha)^\dagger$ and a^I_α, respectively.

The representations of this algebra are well-known. They are constructed from the so-called Clifford vacuum Ω, defined by $a^I_\alpha \Omega = 0$ for any a^I_α, where in contrast to the usual vacuum, Ω satisfies $P^2\Omega = -M^2\Omega$. The states are built by applying the creation operators $(a^I_\alpha)^\dagger$ to Ω:

$$\Omega^{(n)}{}_{(I_1\alpha_1)\cdots(I_n\alpha_n)} = \frac{1}{\sqrt{n!}}(a^{I_1}_{\alpha_1})^\dagger \cdots (a^{I_n}_{\alpha_n})^\dagger \Omega\,. \tag{3.88}$$

Each pair of indices $(I_i\alpha_i)$ can take one of $2\mathcal{N}$ different values, since the $(a^{I_i}_{\alpha_i})^\dagger$ anti-commute, and $\Omega^{(n)}$ is anti-symmetric in the exchange of 2 such pairs of indices $(I_i\alpha_i)$, $(I_j\alpha_j)$. For any given n, there are then $\binom{2\mathcal{N}}{n}$ different states, and summing over all possible n gives the dimension of the representation:

$$\sum_{n=0}^{2\mathcal{N}} \binom{2\mathcal{N}}{n} = (1+1)^{2\mathcal{N}} = 2^{2\mathcal{N}}\,. \tag{3.89}$$

Since there is always an equal number of bosonic and fermionic states, then in this massive representation there are $2^{2\mathcal{N}-1}$ bosonic states, and $2^{2\mathcal{N}-1}$ fermionic states, adding up to the total $2^{2\mathcal{N}}$.

Note that from eqs. (3.77), (3.85), we have:

$$[J^z,(a_1^I)^\dagger] = +\frac{1}{2}(a_1^I)^\dagger\,, \qquad [J^z,(a_2^I)^\dagger] = -\frac{1}{2}(a_2^I)^\dagger\,, \tag{3.90}$$

which implies:

$$J^z(a_1^I)^\dagger|j,m\rangle = (m+\tfrac{1}{2})(a_1^I)^\dagger|j,m\rangle\,, \quad J^z(a_2^I)^\dagger|j,m\rangle = (m-\tfrac{1}{2})(a_2^I)^\dagger|j,m\rangle\,. \tag{3.91}$$

So the state with the highest spin in the representation is obtained by symmetrizing in as many supercharge indices as possible, $(a_1^1)^\dagger\cdots(a_1^{\mathcal{N}})^\dagger|j\rangle$:

$$J^z(a_1^1)^\dagger\cdots(a_1^{\mathcal{N}})^\dagger|j\rangle = \big(j+\tfrac{\mathcal{N}}{2}\big)(a_1^1)^\dagger\cdots(a_1^{\mathcal{N}})^\dagger|j\rangle\,. \tag{3.92}$$

For example, for $\mathcal{N}=1$, acting with $(a_\alpha)^\dagger$ on a spin-j set of states, where $j>0$, we obtain the states:

$$j\otimes\tfrac{1}{2} = (j-\tfrac{1}{2})\oplus(j+\tfrac{1}{2})\,. \tag{3.93}$$

Acting with the two creation operators, we obtain $\epsilon^{\alpha\beta}(a_\alpha)^\dagger(a_\beta)^\dagger|j,m\rangle$, which is of the same spin as $|j\rangle$. Thus the massive supermultiplet has the form:

$$\underbrace{|j\rangle}_{2j+1}\,, \quad a_\alpha^\dagger|j\rangle\sim\underbrace{|j-\tfrac{1}{2}\rangle}_{2j}\oplus\underbrace{|j+\tfrac{1}{2}\rangle}_{2j+2}\,, \quad \underbrace{a_1^\dagger a_2^\dagger|j\rangle}_{2j+1}\,, \tag{3.94}$$

where we noted the number of states in each set. As expected, the number of bosonic or fermionic states, each adds up to $4j+2$, which then adds up to the expected total of $(2j+1)\times 2^{2\times(\mathcal{N}=1)} = 8j+4$ states in the supermultiplet. Let us further specify this example to two key multiplets.

Massive chiral multiplet. Consider the spin-0 case for the Clifford vacuum. Then we get 2 scalar bosons, and a spin-$\frac{1}{2}$ fermion in the multiplet:

$$\begin{aligned} &\text{bosons:} && |0\rangle\,, \quad a_1^\dagger a_2^\dagger|0\rangle\,, \\ &\text{fermions:} && a_\alpha^\dagger|0\rangle\sim|\tfrac{1}{2}\rangle\,, \end{aligned} \tag{3.95}$$

with 2 bosonic and 2 fermionic states, thus 4 states in total, as expected.

Massive vector multiplet. This multiplet starts from a spin-$\frac{1}{2}$ fermion. We then get:

$$\begin{array}{lll} \text{fermions:} & |\tfrac{1}{2}\rangle\,, \quad a_1^\dagger a_2^\dagger|\tfrac{1}{2}\rangle\,, \\ \text{bosons:} & a_\alpha^\dagger|\tfrac{1}{2}\rangle \sim |0\rangle \oplus |1\rangle\,, \end{array} \tag{3.96}$$

thus this multiplet contains 2 spin-$\frac{1}{2}$ fermions, as well as a scalar and a $SO(3)$ vector, adding up to 4 fermionic and 4 bosonic states, as expected.

3.3.2 Massless Supermultiplets

Note that at high enough energies all massive particles seem massless. Let us proceed then to consider massless supersymmetric representations. First we go to the frame where $P^\mu = (E, 0, 0, E)$, $E > 0$. Then the supersymmetric algebra becomes:

$$\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} = 2\begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix}\delta^{IJ} \qquad \Longrightarrow \qquad \left\{ \begin{array}{l} \{Q^I_1, \bar{Q}^J_{\dot 1}\} = 4E\delta^{IJ}\,, \\ \{Q^I_2, \bar{Q}^J_{\dot 2}\} = 0\,, \end{array} \right. \tag{3.97}$$

and the rest of the algebra vanishes. The vanishing relation of Q^I_2 and $\bar{Q}^J_{\dot 2}$ implies that these operators as well as central charges must vanish on massless multiplets. So in this case we can only define $\mathcal{N}$ pairs of creation and annihilation operators:

$$a^I_1 \equiv \frac{1}{2\sqrt{E}} Q^I_1\,, \qquad (a^I_1)^\dagger \equiv \frac{1}{2\sqrt{E}} \bar{Q}^I_{\dot 1}\,, \tag{3.98}$$

which yield the following algebra:

$$\{a^I_1, (a^J_1)^\dagger\} = \delta^{IJ}\,, \tag{3.99}$$

$$\{(a^I_1)^\dagger, (a^J_1)^\dagger\} = \{a^I_1, a^J_1\} = 0\,. \tag{3.100}$$

From eqs. (3.76), (3.77), (3.98), we have:

$$[J^z, (a^I_1)^\dagger] = +\frac{1}{2}(a^I_1)^\dagger\,, \qquad [J^z, a^I_1] = -\frac{1}{2}a^I_1\,, \tag{3.101}$$

so the operators $(a^I_1)^\dagger$, a^I_1, raise and lower the helicity of a state by $\frac{1}{2}$, respectively. So $a^I_1\Omega_{\underline{\lambda}} = 0$, where $\underline{\lambda}$ is the state of lowest helicity, and $\Omega_{\underline{\lambda}}$ is the Clifford vacuum. As we already noted, for Q^I_2 we get from eq. (3.97):

$$Q^I_2|E, \lambda\rangle = \bar{Q}^I_{\dot 2}|E, \lambda\rangle = 0\,, \tag{3.102}$$

on any state. The states in the multiplet are then built from $\Omega_{\underline{\lambda}}$:

$$\Omega^{(n)}_{\underline{\lambda}\,(I_1)\cdots(I_n)} = \frac{1}{\sqrt{n!}}(a^{I_1}_1)^\dagger \cdots (a^{I_n}_1)^\dagger \Omega_{\underline{\lambda}}\,. \tag{3.103}$$

These states have helicity $\underline{\lambda} + \frac{n}{2}$, and they are $\binom{\mathcal{N}}{n}$ degenerate. The state of the highest helicity is $\underline{\lambda} + \frac{\mathcal{N}}{2}$, and the representation has dimension $2^{\mathcal{N}}$.

For example, for $\mathcal{N} = 1$, acting on a state of lowest helicity λ, the supermultiplets consist of pairs of states:

$$|E, \lambda\rangle \,, \quad a_1^\dagger |E, \lambda\rangle = |E, \lambda + \tfrac{1}{2}\rangle \,. \tag{3.104}$$

We shall further specify below the $\mathcal{N} = 1$ example to notable multiplets.

Let us note that for CPT invariance the number of states must in general be doubled (since CPT reverses the sign of helicity), unless the multiplets are automatically CPT-complete. So in general, if $-\lambda \neq \lambda + \frac{\mathcal{N}}{2}$, we also need to add the CPT-conjugate multiplet.

Massless chiral multiplet. The chiral multiplet, starting from $\lambda = 0$, is an important example. It consists of a massless scalar boson and a massless $\lambda = \frac{1}{2}$ fermion:

$$\text{boson:} \quad |E, 0\rangle \,, \qquad \text{fermion:} \quad |E, \tfrac{1}{2}\rangle = a_1^\dagger |E, 0\rangle \,. \tag{3.105}$$

A $\lambda = \pm\frac{1}{2}$ particle is a massless Weyl fermion, which is left-chiral, ψ_α, or right-chiral, $\bar{\psi}^{\dot{\alpha}}$, respectively. Thus the multiplet in eq. (3.105), which contains ψ_α, is referred to as the chiral multiplet, while the CPT-conjugate that contains $\bar{\psi}^{\dot{\alpha}}$, is called the anti-chiral multiplet:

$$\text{boson:} \quad |E, 0\rangle \,, \qquad \text{fermion:} \quad |E, -\tfrac{1}{2}\rangle = a_1 |E, 0\rangle \,. \tag{3.106}$$

It is also common to just refer to the CPT-complete pair of multiplets as the chiral multiplet.

Massless vector multiplet. Also known as the gauge multiplet, starting from $\lambda = \frac{1}{2}$, we obtain the pair:

$$\text{fermion:} \quad |E, \tfrac{1}{2}\rangle \,, \qquad \text{boson:} \quad |E, 1\rangle = a_1^\dagger |E, \tfrac{1}{2}\rangle \,. \tag{3.107}$$

The $\lambda = 1$ particle together with its CPT-conjugate, $\lambda = -1$, yield a massless vector. Thus a massless vector multiplet contains the on-shell DOFs of a 4-dimensional gauge field A^μ. Its fermionic superpartner in the multiplet, generally denoted by λ_α, $\bar{\lambda}^{\dot{\alpha}}$, is called the gaugino.

Supergravity multiplet. Starting from $\lambda = \frac{3}{2}$, and its CPT-conjugate, we get the states:

$$\text{fermion:} \quad |E, \pm\tfrac{3}{2}\rangle\,, \qquad \text{boson:} \quad |E, \pm 2\rangle\,. \tag{3.108}$$

A massless particle of helicity 2 is a graviton, which can only appear in a supersymmetric theory of gravity, known as a supergravity theory. The fermionic superpartner of the graviton in the multiplet is of helicity $\frac{3}{2}$, and is called the gravitino. This also promotes the supersymmetry from being global to local, as supersymmetric theories without massless particles of helicities $|\lambda| > 1$, are also called global supersymmetric theories.

3.4 Bounds on Extended Supersymmetry

To conclude this chapter, it is easy to infer important critical bounds for extended supersymmetry at 4-dimensional spacetime from the above analysis of massive and massless representations of supersymmetry.

We know that for renormalizable QFTs in the free limit the only allowed fields of elementary particles are massive of spin 0 and $\frac{1}{2}$, and massless of spin 1.

For a massive multiplet we found the maximal spin to be $j + \frac{\mathcal{N}}{2}$, thus a renormalizable QFT with supersymmetry should have:

$$\mathcal{N} = 1\,. \tag{3.109}$$

For a massless multiplet we found that the maximal helicity is $\underline{\lambda} + \frac{\mathcal{N}}{2}$, so starting from $\underline{\lambda} = -1$, we can only have:

$$\mathcal{N}^{\text{global}}_{\text{max}} = 4\,, \tag{3.110}$$

for a renormalizable theory with global supersymmetry.

If we are interested to also incorporate gravity, then we must allow for a non-renormalizable QFT, and switch on local supersymmetry, so starting from $\underline{\lambda} = -2$, we find:

$$\mathcal{N}^{\text{local}}_{\text{max}} = 8\,. \tag{3.111}$$

4. Superspace and Superfields

We uncovered the super-Poincaré algebra with $\mathcal{N}$ pairs of fermionic supercharges Q^I, $\bar{Q}^I$, $I \in \{1, \ldots, \mathcal{N}\}$. We found its one-particle representations in terms of on-shell supermultiplets, yet we would like to formulate supersymmetric QFTs in terms of fields. In order to require that such a QFT has supersymmetric invariance, we need to know the variation of fields under the action of the generators of supersymmetry Q, $\bar{Q}$, where from now on we specialize to $\mathcal{N} = 1$. To that end, the supersymmetric generators need to be represented as differential operators on some manifold, on which supersymmetric fields would be defined.

4.1 Coset Spaces

Let us then illustrate how to generically arrive at such a manifold, that is a domain of fields, specializing first to the familiar case of ordinary QFTs with Poincaré invariance, defined on Minkowski spacetime.

Consider a Lie group G with its Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. The Hermitian generators of $\mathfrak{g}$, denoted by T_A, are closed under commutation:

$$[T_A, T_B] = iC_{ABC}T_C \,, \tag{4.1}$$

with the indices $A, \ldots \in \{1, \ldots, \dim \mathfrak{g}\}$. The group elements $g \in G$ are unitary operators which are obtained via the exponential map:

$$g = \exp\left(it^A T_A\right) \,, \tag{4.2}$$

where t^A are some real parameters, called the group parameters in parameter space. Consider then that the group G has a Lie subgroup $H \subset G$, with its Lie algebra $\mathfrak{h} = \mathrm{Lie}(H)$. We can write then the Lie algebra as the direct sum:

$$\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{K} \,, \tag{4.3}$$

where $\mathfrak{K}$ is the complement of $\mathfrak{h}$. Let us denote the generators of $\mathfrak{h}$ and $\mathfrak{K}$, by M_a and K_I, respectively, with the indices $a \in \{1, \ldots, \dim \mathfrak{h}\}$, and

$I \in \{1, \ldots, \dim \mathfrak{K}\}$. Then we can also write general group elements $g \in G$ from eq. (4.2), and $h \in H$, in the form:

$$g = \exp\left(i\omega^a M_a + i\alpha^I K_I\right) \,, \qquad h = \exp\left(i\tilde{\omega}^a M_a\right) \,, \tag{4.4}$$

where $t^A = (\omega^a, \alpha^I)$, and $\tilde{\omega}^a$, are all real parameters.

The quotient space of left cosets, G/H, is defined as the set of equivalence classes under group multiplication of H from the left:

$$G/H = \{gH : g \in G \,,\, g' \sim g \iff \exists h \in H | g' = gh\} \,. \tag{4.5}$$

The group acts on this coset space, which is a differentiable manifold, also called the coset manifold, and is known as a homogenous space. We also have:

$$[\mathfrak{h}, \mathfrak{K}] \subset \mathfrak{K} \,, \tag{4.6}$$

or in terms of the generators:

$$[M_a, M_b] = iC_{abc}M_c \,, \qquad [M_a, K_I] = iC_{aIJ}K_J \,. \tag{4.7}$$

Let us denote the representatives of cosets that make up the coset space by g_c. Then we take such a representative:

$$g_c(z) = \exp\left(iz^I K_I\right) \in G \,, \tag{4.8}$$

with z some local coordinates on the coset manifold.

The induced action of the group on coset space is read from:

$$g\, g_c(z) = g_c(z')h(g, z) \,, \tag{4.9}$$

such that the group multiplication from the left on the LHS yields another element of an equivalence class in coset space. In order to evaluate this action from eq. (4.9), we need the Baker-Campbell-Hausdorff formula for the product of exponentiated generators:

$$\exp(A)\exp(B) = \exp\left(A + B + \tfrac{1}{2}[A, B] + \tfrac{1}{12}[A, [A, B]] + \cdots\right) \,, \tag{4.10}$$

where only the first commutator on the RHS is actually needed in what follows. The group multiplication in eq. (4.9) implies an induced shift of the coordinates on coset space, $g : z \to z'$. Thus for g near the identity, at first order in an infinitesimal group parameter, ϵ^A, we can write:

$$g = \exp\left(i\epsilon^A T_A\right) \simeq 1 + i\epsilon^A T_A \qquad \Longrightarrow \qquad z'^I - z^I \simeq \epsilon^A \frac{\partial z'^I}{\partial \epsilon^A} \,. \tag{4.11}$$

Then, the action of the group G is realized on scalar fields, e.g. $\phi(z)$, that is on functions of the coset space, via the generators represented as differential operators:

$$T_A = -i\frac{\partial z'^I}{\partial \epsilon^A}\frac{\partial}{\partial z^I}\,, \tag{4.12}$$

where this form is easily inferred from eqs. (4.9), (4.11). Thus the group element, which is a unitary operator, acts on such a field that depends on coset space, in terms of these differential operators:

$$U(g)\phi(z) = \exp\left(i\epsilon^A T_A\right)\phi(z) = (1 + i\epsilon^A T_A + \cdots)\phi(z) = \phi(z')\,. \tag{4.13}$$

In fact, the familiar Minkowski spacetime is a coset space. For the Poincaré group:

$$ISO(1,3) \cong SO(1,3) \ltimes \mathbb{R}^{1,3}\,, \tag{4.14}$$

any group element can be written in the form:

$$g = \exp\left(\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu} + ix^\mu P_\mu\right)\,, \tag{4.15}$$

for some real parameters $\omega^{\mu\nu}$, x^μ. Then Minkowski spacetime can be viewed as a coset space, with $G = ISO(1,3)$, and the Lorentz subgroup, $H = SO(1,3)$:

$$\mathbb{R}^{1,3} \cong ISO(1,3)/SO(1,3)\,. \tag{4.16}$$

Here, the generators K_I from eq. (4.8) are simply the translation generators, P_μ. Let us parametrize the coset space with the coordinates x^μ:

$$g_c(x^\mu) = \exp\left(ix^\mu P_\mu\right)\,. \tag{4.17}$$

Using eqs. (4.9), (4.12), it is then easy to find the induced action of translations, and of Lorentz transformations, on coset space:

$$g_T \equiv \exp\left(ia^\mu P_\mu\right)\,, \qquad g_L \equiv \exp\left(\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}\right)\,, \tag{4.18}$$

from which we can recover their form as differential operators to be that which is given in eqs. (3.38), (3.39), respectively, as expected. It can be readily verified that the differential operators for the Poincaré generators satisfy the Poincaré algebra on scalar fields, $\phi(x)$, defined on $\mathbb{R}^{1,3}$.

4.2 Superspace and Supergroups

With this notion of coset spaces in mind, we thus look for a manifold, on which supersymmetry transformations are represented "geometrically". We first consider then the so-called supergroup, obtained from exponentiating the super-Poincaré algebra, denoted by $ISO(1,3|\mathcal{N})$, for $\mathcal{N}$ pairs of supercharges Q^I, $\bar{Q}^I$, $I \in \{1,\ldots,\mathcal{N}\}$. A supergroup element, where we specialize here to $\mathcal{N}=1$, reads then:

$$g = \exp\left(i\left[\frac{1}{2}\omega^{\mu\nu}M_{\mu\nu} + x^\mu P_\mu + \theta^\alpha Q_\alpha + \bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}\right]\right) \in ISO(1,3|1)\,, \qquad (4.19)$$

where the contraction conventions of eqs. (2.33), (2.34), are used for the spinors. This can work, if the supergroup parameters θ^α, $\bar{\theta}_{\dot{\alpha}}$, are Grassmann numbers, which by definition anti-commute, e.g.:

$$\{\theta^\alpha,\theta^\beta\} = \{\theta^\alpha,\bar{\theta}^{\dot{\beta}}\} = \{\theta^\alpha, Q_\beta\} = 0\,. \qquad (4.20)$$

These anti-commuting supersymmetry parameters turn the anti-commutation relations of the supersymmetric algebra into commutation relations:

$$\begin{aligned}\left[\theta^\alpha Q_\alpha, \bar{\theta}_{\dot{\beta}}\bar{Q}^{\dot{\beta}}\right] &= -\theta^\alpha Q_\alpha \bar{\theta}^{\dot{\beta}}\bar{Q}_{\dot{\beta}} + \bar{\theta}^{\dot{\beta}}\bar{Q}_{\dot{\beta}}\theta^\alpha Q_\alpha = \theta^\alpha\bar{\theta}^{\dot{\beta}}\left(Q_\alpha\bar{Q}_{\dot{\beta}} + \bar{Q}_{\dot{\beta}}Q_\alpha\right)\\ &= 2\theta^\alpha\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}P_\mu\,, \qquad (4.21)\\ \left[\theta Q,\theta Q\right] &= \left[\bar{\theta}\bar{Q},\bar{\theta}\bar{Q}\right] = 0\,. \qquad (4.22)\end{aligned}$$

This guarantees the closure of the supergroup.

Thus now, similar to Minkowski spacetime as viewed in eq. (4.16), we define superspace as the following coset space:

$$\mathbb{R}^{1,3|4} \cong ISO(1,3|1)/SO(1,3)\,. \qquad (4.23)$$

The representative element of the quotient space is of the form:

$$g_c(x^\mu,\theta^\alpha,\bar{\theta}_{\dot{\alpha}}) = \exp\left(i\left[-x^\mu P_\mu + \theta^\alpha Q_\alpha + \bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}\right]\right)\,, \qquad (4.24)$$

over superspace coordinates:

$$z = (x^\mu,\theta^\alpha,\bar{\theta}_{\dot{\alpha}})\,. \qquad (4.25)$$

Superspace then contains the 4 ordinary bosonic spacetime coordinates, as well as 4 new fermionic coordinates θ^α, $\bar{\theta}_{\dot{\alpha}}$. Thus this coset manifold, which is sometimes called a supermanifold, has even and odd coordinates.

Let us then check the action of the supersymmetry "group element", which reads:

$$g_{SUSY} = \exp\left(i\left[\eta Q + \bar{\eta}\bar{Q}\right]\right) = \exp\left(i\left[\eta^\alpha Q_\alpha + \bar{\eta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}\right]\right) = g_c(0, \eta^\alpha, \bar{\eta}_{\dot{\alpha}})\,. \tag{4.26}$$

Let us start by applying on coset space the piece with Q:

$$\exp\left(i\eta^\alpha Q_\alpha\right)\exp\left(i\left[-x^\mu P_\mu + \theta Q + \bar{\theta}\bar{Q}\right]\right) = \exp\left(i\left[-x'^\mu P_\mu + \theta' Q + \bar{\theta}'\bar{Q}\right]\right)\,. \tag{4.27}$$

We find:

$$\begin{aligned}
&\exp\left(i\eta^\alpha Q_\alpha\right)\exp\left(i\left[-x^{\alpha\dot{\beta}}P_{\alpha\dot{\beta}} + \theta^\alpha Q_\alpha + \bar{\theta}_{\dot{\beta}}\bar{Q}^{\dot{\beta}}\right]\right)\\
&\quad= \exp\left(i\left[-x^{\alpha\dot{\beta}}P_{\alpha\dot{\beta}} + (\theta^\alpha + \eta^\alpha)Q_\alpha + \bar{\theta}_{\dot{\beta}}\bar{Q}^{\dot{\beta}}\right] - \tfrac{1}{2}\left[\eta^\alpha Q_\alpha, \bar{\theta}_{\dot{\beta}}\bar{Q}^{\dot{\beta}}\right]\right)\\
&\quad= \exp\left(i\left[-x^{\alpha\dot{\beta}}P_{\alpha\dot{\beta}} + (\theta^\alpha + \eta^\alpha)Q_\alpha + \bar{\theta}_{\dot{\beta}}\bar{Q}^{\dot{\beta}}\right] - \eta^\alpha\bar{\theta}^{\dot{\beta}}P_{\alpha\dot{\beta}}\right)\\
&\quad= \exp\left(i\left[-(x^{\alpha\dot{\beta}} - i\eta^\alpha\bar{\theta}^{\dot{\beta}})P_{\alpha\dot{\beta}} + (\theta^\alpha + \eta^\alpha)Q_\alpha + \bar{\theta}_{\dot{\beta}}\bar{Q}^{\dot{\beta}}\right]\right)\,,
\end{aligned} \tag{4.28}$$

where we used eq. (4.21) for the second equality. Thus we can see that $\exp\left(i\eta^\alpha Q_\alpha\right)$ induces the following action on superspace coordinates:

$$\exp\left(i\eta^\alpha Q_\alpha\right) : \left(x^{\alpha\dot{\beta}}, \theta^\alpha, \bar{\theta}_{\dot{\beta}}\right) \to \left(x^{\alpha\dot{\beta}} - i\eta^\alpha\bar{\theta}^{\dot{\beta}}, \theta^\alpha + \eta^\alpha, \bar{\theta}_{\dot{\beta}}\right)\,. \tag{4.29}$$

Similarly, we can find that the action of $\exp\left(-i\bar{\eta}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}}\right)$ induces the motion:

$$\exp\left(-i\bar{\eta}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}}\right) : \left(x^{\alpha\dot{\beta}}, \theta^\alpha, \bar{\theta}_{\dot{\beta}}\right) \to \left(x^{\alpha\dot{\beta}} + i\theta^\alpha\bar{\eta}^{\dot{\beta}}, \theta^\alpha, \bar{\theta}_{\dot{\beta}} + \bar{\eta}_{\dot{\beta}}\right)\,. \tag{4.30}$$

Taking $\eta^\alpha \to \epsilon^\alpha$, that is as an infinitesimal Grassmannian parameter, we can infer from eq. (4.12) and eq. (4.29), that the induced motion in superspace is generated by Q_α as the differential operator:

$$Q_\alpha = -i\left(\frac{\partial}{\partial\theta^\alpha} - i\bar{\theta}^{\dot{\beta}}\frac{\partial}{\partial x^{\alpha\dot{\beta}}}\right) = -i\left(\frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu\right)\,, \tag{4.31}$$

and similarly, using also eq. (4.30), we infer that the differential operator for $\bar{Q}_{\dot{\alpha}}$ reads:

$$\bar{Q}^{\dot{\alpha}} = -i\left(\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\beta}}\epsilon^{\dot{\beta}\dot{\alpha}}\partial_\mu\right) \quad\Longleftrightarrow\quad \bar{Q}_{\dot{\alpha}} = -i\left(-\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu\right)\,, \tag{4.32}$$

where some basic Grassmannian calculus, which is provided in section 4.2.1 below, was used. Using the differential form of the generators, we can check that the supersymmetric algebra is satisfied, with the non-vanishing commutation relation in eq. (3.9), where the energy-momentum operator, P_μ, is realized in superspace as in eq. (3.38). Note that while the differential form of P_μ in superspace remains the same as that in Minkowski spacetime, due to its trivial commutators with the supercharges, this is not the case for the Lorentz generators. The latter are represented differently on superspace compared to Minkowski spacetime (as shall be demonstrated in the problem sheet).

4.2.1 Grassmannian Calculus in Superspace

In this brief section we provide some basics of Grassmannian calculus essential for the treatment of superspace coordinates in what follows. First let us introduce the shorthand notation:

$$\partial_\alpha \equiv \frac{\partial}{\partial\theta^\alpha}\,, \quad \partial^\alpha \equiv \frac{\partial}{\partial\theta_\alpha}\,, \quad \bar{\partial}_{\dot\alpha} \equiv \frac{\partial}{\partial\bar{\theta}^{\dot\alpha}}\,, \quad \bar{\partial}^{\dot\alpha} \equiv \frac{\partial}{\partial\bar{\theta}_{\dot\alpha}}\,. \tag{4.33}$$

Then, we define Grassmannian differentiation:

$$\partial_\alpha\theta^\beta = \delta_\alpha^\beta\,, \qquad \bar{\partial}_{\dot\alpha}\bar{\theta}^{\dot\beta} = \delta_{\dot\alpha}^{\dot\beta}\,, \tag{4.34}$$

which implies:

$$\partial^\alpha\theta_\beta = -\delta_\beta^\alpha\,, \qquad \bar{\partial}^{\dot\alpha}\bar{\theta}_{\dot\beta} = -\delta_{\dot\beta}^{\dot\alpha}\,. \tag{4.35}$$

Grassmannian derivatives are taken from the left, so we must always move the relevant differentiated Grassmann number to the left, incurring possible minus signs along the way, before taking the derivative. In particular:

$$\partial_\alpha\theta\theta = 2\theta_\alpha\,, \qquad \bar{\partial}_{\dot\alpha}\bar{\theta}\bar{\theta} = -2\bar{\theta}_{\dot\alpha}\,. \tag{4.36}$$

We can expand functions of superspace coordinates in θ^α and $\bar{\theta}^{\dot\alpha}$ in a Taylor expansion, which truncates, since Grassmann numbers are anticommuting, so higher powers of θ, $\bar{\theta}$, vanish. For example, a function of x^μ and only a single Grassmann number, θ, reads:

$$F(x,\theta) = f_0(x) + \theta f_1(x)\,, \tag{4.37}$$

where f_0 and f_1 are arbitrary functions of x.

We also need to define integration over Grassmann numbers (in order to construct actions as of chapter 5), also known as Berezin integration:

$$\int d\theta\,\theta = 1\,, \qquad \int d\theta = 0\,, \tag{4.38}$$

so that in fact Grassmannian integration and differentiation act similarly. For $\mathcal{N} = 1$ we then define:

$$\int d^2\theta \equiv \frac{1}{2}\int d\theta_1 d\theta_2\,, \quad \int d^2\bar{\theta} \equiv \frac{1}{2}\int d\bar{\theta}_2 d\bar{\theta}_1\,, \quad d^4\theta = d^2\theta d^2\bar{\theta}\,, \tag{4.39}$$

and since $\theta\theta = 2\theta_2\theta_1$, $\bar{\theta}\bar{\theta} = 2\bar{\theta}_1\bar{\theta}_2$, then we have:

$$\int d^2\theta\, \theta\theta = 1\,, \quad \int d^2\bar{\theta}\, \bar{\theta}\bar{\theta} = 1\,. \tag{4.40}$$

In particular, an integral over the 4 Grassmannian coordinates is equivalent to collecting the $\theta\theta\bar{\theta}\bar{\theta}$ coefficient in the Taylor expansion of the integrand:

$$\int d^2\theta d^2\bar{\theta}\, F(x,\theta,\bar{\theta}) = F(x,\theta,\bar{\theta})\big|_{\theta\theta\bar{\theta}\bar{\theta}}\,. \tag{4.41}$$

4.3 Superfields and Component Fields

In supersymmetric QFTs there can then be expected some kind of fields, that are not only dependent on spacetime, but rather in superspace coordinates, $z = (x, \theta, \bar{\theta})$. Such functions, that depend on superspace, are called superfields, denoted by $\mathcal{S}(z)$. They should transform under infinitesimal translations and supersymmetry transformations, as follows:

$$\begin{aligned} \exp\left(i\left[a^\mu P_\mu + \epsilon Q + \bar{\epsilon}\bar{Q}\right]\right)\mathcal{S}(x,\theta,\bar{\theta}) &= \mathcal{S}(x',\theta',\bar{\theta}') \\ &= \mathcal{S}(x + a - i\epsilon\sigma\bar{\theta} + i\theta\sigma\bar{\epsilon}, \theta + \epsilon, \bar{\theta} + \bar{\epsilon})\,, \end{aligned} \tag{4.42}$$

i.e. the induced motion of their coordinates follows from eqs. (4.29), (4.30). It is thus clear that linear combinations of superfields are superfields, and that products of superfields are superfields, since the translations and supersymmetry generators are linear differential operators.

These functions should be understood in terms of their power series expansion in the Grassmannian coordinates, θ, $\bar{\theta}$, which as noted in section 4.2.1 truncates since θ, $\bar{\theta}$, are anti-commuting, so higher powers of θ, $\bar{\theta}$, vanish. Let us then write down a general superfield:

$$\begin{aligned} \mathcal{S}(x,\theta,\bar{\theta}) = B(x) + i\theta\chi(x) - i\bar{\theta}\bar{\omega}(x) + \tfrac{i}{2}\theta\theta F(x) - \tfrac{i}{2}\bar{\theta}\bar{\theta}G(x) - \theta^\alpha\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}A_\mu(x) \\ + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\rho(x) + \tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x)\,. \end{aligned} \tag{4.43}$$

The coefficients in these expansions of superfields are called component fields, and they are ordinary fields, as they depend only on spacetime. The component fields are assigned dimension and spin according to the θ and $\bar{\theta}$ powers that they accompany. Let us then fix the dimension of θ:

$$Q \sim \theta^{-1}\,, \quad [Q] = \tfrac{1}{2} \quad \Longrightarrow \quad [\theta] = -\tfrac{1}{2}\,. \tag{4.44}$$

The components of lowest and highest dimension in a superfield, denoted here as $B(x)$ and $D(x)$, are referred to as bottom and top components, respectively. Assuming, e.g., that B is bosonic, then F, G, A_μ, and D, are also bosonic, whereas χ, $\bar{\omega}$, $\bar{\lambda}$, and ρ, are fermionic.

From eq. (4.42) we then define the supersymmetry variation of a superfield as follows:

$$\delta\mathcal{S}[C] = i\left(\epsilon Q + \bar{\epsilon}\bar{Q}\right)\mathcal{S}[C] \equiv \mathcal{S}[\delta C(\epsilon,\bar{\epsilon})]\,, \tag{4.45}$$

where C stands for the component fields, and δC stands for the variations of the component fields. It is a straightforward though tedious task to derive the variation in components for general superfields, so we do not include it here. Yet, simply from dimensional considerations of ϵ, $\bar{\epsilon}$, as noted for θ, it is easily inferred that δC with $[C] = x$, always consists of the next-higher component fields, i.e. with dimension $x + \frac{1}{2}$, or of spacetime derivatives of the next-lower component fields, i.e. with dimension $x - \frac{1}{2}$. From similar dimensional and spin considerations, there is still an additional freedom to redefine the higher component fields of the superfield, in particular $\bar{\lambda}$, ρ, and D, in eq. (4.43), by adding terms with a spacetime derivative of χ, $\bar{\omega}$, and even 2 spacetime derivatives of B, respectively. In any case, this discussion implies that the supersymmetric variation of the top component of a superfield, $\delta C_{\rm top}(\epsilon,\bar{\epsilon})$, is necessarily a linear combination of spacetime derivatives of lower components. Since in global supersymmetry the supersymmetric parameters, ϵ, $\bar{\epsilon}$, are constant, then the variation $\delta C_{\rm top}$ is simply a total spacetime derivative altogether. This is a critical point for the formulation of supersymmetric actions, as we shall see soon in section 5.2.

Yet, general superfields, as off-shell representations of supersymmetry (SUSY), are highly reducible – they contain too many extra component fields, i.e. DOFs. For example, if we take the bottom component of the general superfield to be real, then it is easy to verify in eq. (4.43), that the superfield has 8 bosonic plus 8 fermionic DOFs. Such extra DOFs can be eliminated by imposing SUSY-covariant constraints, i.e. some appropriate constraints, which preserve supersymmetric invariance.

4.3.1 R-Symmetry in Superspace

We encountered R-symmetry at the level of supersymmetry algebra in section 3.1.1 as a $U(1)_R$ rotation of the supercharges in eq. (3.36). It is easy to see that R-symmetry is also realized in superspace as $Q \sim \theta^{-1}$, via the following transformation:

$$\theta \to \theta' = \exp(i\alpha)\theta\,, \qquad \bar{\theta} \to \bar{\theta}' = \exp(-i\alpha)\bar{\theta}\,. \tag{4.46}$$

From the definition of Grassmannian integration in eq. (4.38), we can then infer:

$$d^2\theta' = d^2\theta\,\exp(-2i\alpha)\,, \qquad d^2\bar{\theta}' = d^2\bar{\theta}\,\exp(+2i\alpha)\,. \tag{4.47}$$

Accordingly, the action of R-symmetry on superfields is defined as follows:

$$\mathcal{S}'(\theta') = \exp(iq_R\,\alpha)\mathcal{S}(\theta)\,, \tag{4.48}$$

where q_R is the R-symmetry charge that is assigned to $\mathcal{S}$.

4.4 Supersymmetric Covariant Derivatives

Going back to eq. (4.9), one can also study the induced motion on superspace, due to the right multiplication by the supergroup element in eq. (4.26), instead of the left multiplication in eq. (4.9).

It is easy to verify that such an analysis yields 2 additional differential operators of interest:

$$D_\alpha = -i\left(\partial_\alpha + i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu\right)\,, \qquad \bar{D}_{\dot{\alpha}} = -i\left(-\bar{\partial}_{\dot{\alpha}} - i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu\right)\,. \tag{4.49}$$

It is also easy to verify that these new operators satisfy the supersymmetry algebra:

$$\{D_\alpha, \bar{D}_{\dot{\beta}}\} = -2P_{\alpha\dot{\beta}}\,, \tag{4.50}$$

$$\{D_\alpha, D_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0\,, \tag{4.51}$$

with the opposite sign for the non-vanishing relation. Furthermore importantly, it is easy to verify that these differential operators anti-commute with the supercharges:

$$\{D_\alpha, Q_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0\,, \tag{4.52}$$

$$\{D_\alpha, \bar{Q}_{\dot{\beta}}\} = \{\bar{D}_{\dot{\alpha}}, Q_\beta\} = 0\,. \tag{4.53}$$

From eqs. (4.52), (4.53), we see that the operators D_α, $\bar{D}_{\dot{\alpha}}$, do not affect the action of SUSY transformations, i.e. they preserve supersymmetric invariance, which is why D_α, $\bar{D}_{\dot{\alpha}}$, are called SUSY-covariant derivatives. Naively, we could have just considered the partial derivatives $\partial_\alpha, \bar{\partial}_{\dot{\alpha}}$, as the supersymmetric differentiation operators, but e.g. $[\eta Q, \bar{\partial}_{\dot{\alpha}}] \neq 0$, unlike eq. (4.53), so the partial derivatives do not preserve SUSY invariance. Accordingly, $\partial_\alpha \mathcal{S}$ or $\bar{\partial}_{\dot{\alpha}} \mathcal{S}$ are not superfields, whereas $D_\alpha \mathcal{S}$ and $\bar{D}_{\dot{\alpha}} \mathcal{S}$ are both superfields.

5. Chiral Superfields and Supersymmetric Actions

We already noted that general superfields contain extra off-shell DOFs, which can be eliminated by imposing SUSY-covariant constraints. With the SUSY covariant derivatives at hand let us then consider the following constraint on a general superfield:

$$\bar{D}\Phi = 0\,. \tag{5.1}$$

Superfields which satisfy this condition are called chiral superfields. One can require instead that general superfields satisfy:

$$D\bar{\Phi} = 0\,, \tag{5.2}$$

and then they are called anti-chiral superfields. Naively, we could have simply required that $\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}}\Phi = 0$, i.e. that Φ depends only on θ, and not on $\bar{\theta}$, but similar to what was explained in section 4.4, such a derivative of the superfield would not preserve SUSY invariance, whereas due to eqs. (4.52), (4.53), the conditions in eqs. (5.1), (5.2), are SUSY covariant.

Note that Φ is trivial, if the 2 constraints in eqs. (5.1), (5.2), are imposed at the same time:

$$\bar{D}\Phi = D\Phi = 0 \qquad \Longrightarrow \qquad \Phi = \text{const}\,, \tag{5.3}$$

due to eq. (4.50). Moreover, since eq. (5.1) leads to eq. (5.2), then such a constrained superfield Φ cannot be real, or else it is trivial, as in eq. (5.3). Thus the chiral superfield must be complex, and corresponds to the on-shell chiral multiplets from sections 3.3.1, 3.3.2.

5.1 Chiral Superfields

Let us see the consequences of the chiral constraint, which is expected to reduce DOFs in the general superfield. From the definition of $\bar{D}$ in eq. (4.49), we get:

$$\begin{cases} \bar{D}_{\dot{\alpha}}\theta^{\beta} = 0 \\ \bar{D}_{\dot{\alpha}}x^{\mu} = -\theta^{\beta}\sigma^{\mu}_{\beta\dot{\alpha}} \\ \bar{D}_{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = i\delta^{\dot{\beta}}_{\dot{\alpha}} \end{cases} \implies \bar{D}_{\dot{\alpha}}(-\theta^{\beta}\sigma^{\mu}_{\beta\dot{\gamma}}\bar{\theta}^{\dot{\gamma}}) = i\theta^{\beta}\sigma^{\mu}_{\beta\dot{\alpha}} = -i\bar{D}_{\dot{\alpha}}x^{\mu}\,, \quad (5.4)$$

where for the first equality on the right, there was a sign flip in order to take the Grassmannian derivative. From this we see that it is useful to introduce the so-called chiral coordinates:

$$y^{\mu} \equiv x^{\mu} + i\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}} \implies \bar{D}_{\dot{\alpha}}y^{\mu} = 0\,. \quad (5.5)$$

Similarly, it is useful to define the anti-chiral coordinates:

$$\bar{y}^{\mu} \equiv x^{\mu} - i\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}} \implies D_{\alpha}\bar{y}^{\mu} = 0\,. \quad (5.6)$$

Thus a superfield Φ, that depends only on (y, θ), satisfies $\bar{D}\Phi = 0$. An expansion in θ of a chiral superfield in chiral coordinates is then simple:

$$\Phi(y^{\mu}, \theta^{\alpha}) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y)\,. \quad (5.7)$$

Similarly, for an anti-chiral superfield in anti-chiral coordinates we have:

$$\bar{\Phi}(\bar{y}^{\mu}, \bar{\theta}_{\dot{\alpha}}) = \bar{\phi}(\bar{y}) + \sqrt{2}\bar{\theta}\bar{\psi}(\bar{y}) + \bar{\theta}\bar{\theta}\bar{F}(\bar{y})\,. \quad (5.8)$$

It is easy to see then that the chiral superfield has significantly less DOFs than a general one in eq. (4.43). Yet, we are interested to express the chiral superfield in terms of ordinary spacetime coordinates x^{μ}, rather than the chiral ones y^{μ}. To that end, we use an expansion in θ, $\bar{\theta}$, and in y around x:

$$\begin{aligned} \Phi(x, \theta, \bar{\theta}) &= \phi(x) + i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}\phi(x) - \frac{1}{2}\theta\sigma^{\mu}\bar{\theta}\theta\sigma^{\nu}\bar{\theta}\partial_{\mu}\partial_{\nu}\phi(x) \\ &\quad + \sqrt{2}\theta\psi(x) + i\sqrt{2}\theta\sigma^{\mu}\bar{\theta}\theta\partial_{\mu}\psi(x) + \theta\theta F(x) \\ &= \phi(x) + i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}\phi(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial_{\mu}\partial^{\mu}\phi(x) \\ &\quad + \sqrt{2}\theta\psi(x) + \frac{i}{\sqrt{2}}\theta\theta\bar{\theta}\bar{\sigma}^{\mu}\partial_{\mu}\psi(x) + \theta\theta F(x), \end{aligned} \quad (5.9)$$

where for the second equality we used the spinor identities in eq. (2.39), and eqs. (2.38), (2.40), to simplify the third, and fifth terms, respectively. A similar form for the conjugate $\bar{\Phi}$ can be easily inferred. We can identify then the component fields in the chiral superfield by matching powers of θ, $\bar{\theta}$, in eq. (5.9) to the components of a general superfield in eq. (4.43):

$$\begin{aligned} &\phi \to \phi\,, \quad \psi \to -i\sqrt{2}\psi\,, \quad F \to -2iF\,, \quad \bar{\chi} = G = \rho = 0\,, \\ &A_\mu = -i\partial_\mu\phi\,, \quad \bar{\lambda} = \frac{1}{\sqrt{2}}\bar{\sigma}^\mu\partial_\mu\psi\,, \quad D = \frac{1}{2}\partial^2\phi\,. \end{aligned} \tag{5.10}$$

Thus out of the component fields of a general superfield, for the chiral superfield we are left with:

$$\Phi = (\phi, \psi_\alpha, F)\,, \tag{5.11}$$

namely only 3 fields that are spacetime-dependent:

- ϕ – a complex scalar field, which amounts to 2 bosonic scalar DOFs,
- ψ_α – a Weyl spinor, that is left-chiral, with 4 fermionic DOFs,
- F – a complex scalar field with 2 extra off-shell bosonic DOFs. This is an auxiliary field rather than a physical one (as we shall see shortly in section 5.3), which maintains the equality of bosonic and fermionic DOFs in the off-shell superfield, similar to that of the on-shell supermultiplet.

This is in agreement with our findings for the chiral multiplets in sections 3.3.1, 3.3.2.

If 2 superfields are chiral, then their sum and their product are chiral as well:

$$\bar{D}_{\dot{\alpha}}\Phi_1 = \bar{D}_{\dot{\alpha}}\Phi_2 = 0 \implies \bar{D}_{\dot{\alpha}}(\Phi_1 + \Phi_2) = \bar{D}_{\dot{\alpha}}(\Phi_1\Phi_2) = 0\,. \tag{5.12}$$

If a superfield is chiral, then its conjugate is anti-chiral:

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \implies D_\alpha\bar{\Phi} = 0\,. \tag{5.13}$$

Yet, the product of a chiral superfield and its conjugate, $\bar{\Phi}\Phi$, is not chiral, nor is it anti-chiral, as it depends both in θ and $\bar{\theta}$. It is however a real superfield:

$$(\bar{\Phi}\Phi)^\dagger = \bar{\Phi}\Phi\,. \tag{5.14}$$

5.2 Supersymmetric Actions

In order to construct a proper action, we need to first recall Grassmannian integration over superspace, as provided in section 4.2.1. First, obviously the action should be real. Second, in order to construct an action from chiral superfields as integrands, we might have considered to integrate as follows:

$$\int d^4x d^2\theta d^2\bar{\theta}\, \Phi\left(x,\theta,\bar{\theta}\right) = \int d^4y d^2\theta d^2\bar{\theta}\, \Phi\left(y,\theta\right) = 0\,, \tag{5.15}$$

where Φ is some chiral superfield, and in the first equality we changed integration variables from standard superspace to chiral coordinates. Thus, there are in fact only 2 possibilities to construct an action:

1. D-term. Take as an integrand a general superfield that is not chiral, yet is required to be real, and integrate over the whole superspace:

$$\int d^4x d^2\theta d^2\bar{\theta}\, \mathcal{S}\left(x,\theta,\bar{\theta}\right) = \int d^4x\, \mathcal{S}\left(x,\theta,\bar{\theta}\right)\big|_{\theta\theta\bar{\theta}\bar{\theta}} = \frac{1}{2}\int d^4x\, D(x)\,, \tag{5.16}$$

so the Lagrangian density reads:

$$\mathcal{L} = \tfrac{1}{2}D(x)\,. \tag{5.17}$$

Since this formulation picks up the top component, D, of the general real superfield, as the Lagrangian density in an ordinary spacetime integration, this formulation is called the "D-term" Lagrangian.

2. F-term. Consider the integration of some chiral superfield Φ over "half" superspace:

$$\int d^4y d^2\theta\, \Phi(y,\theta) = \int d^4y\, \Phi(y,\theta)|_{\theta\theta} = \int d^4y\, F(y) = \int d^4x\, F(x)\,. \tag{5.18}$$

Note that if this route is taken, then in order for the action to be real, we also need to add in the action the Hermitian conjugate:

$$\begin{aligned}\int d^4y d^2\theta\, \Phi(y,\theta)\; + \;\text{H.C.} &= \int d^4y d^2\theta\, \Phi(y,\theta) + \int d^4\bar{y} d^2\bar{\theta}\, \bar{\Phi}(\bar{y},\bar{\theta}) \\ &= \int d^4x \left[F(x) + \bar{F}(x)\right]\,,\end{aligned} \tag{5.19}$$

where H.C. stands for the Hermitian conjugate. This yields the Lagrangian density:

$$\mathcal{L} = \left[F(x) + \bar{F}(x)\right]\,. \tag{5.20}$$

Since this formulation picks up the top components, F, $\bar{F}$, of the chiral and anti-chiral superfields, as the Lagrangian density, this is called the "F-term" Lagrangian.

Recalling our discussion at the end of section 4.3 on the supersymmetric variation of superfields, defined in eq. (4.45), the variation of the top component, $\delta C_{\rm top}$, is always a total spacetime derivate in global supersymmetry. Therefore for the supersymmetric variation of the action, due to the Grassmannian integration, we obtain:

$$\delta S = \int d^4x\, \delta\mathcal{L} = \int d^4x\, \delta C_{\rm top}(\epsilon, \bar{\epsilon}) = \int d^4x\, \partial_\mu X(\epsilon, \bar{\epsilon}) = 0\,, \tag{5.21}$$

where X is some linear combination of lower component fields multiplied by ϵ or $\bar{\epsilon}$, and the spacetime integration of a total spacetime derivative simply vanishes. For this reason the supersymmetric variation of generic actions, as formulated above, automatically vanishes, and thus these actions properly preserve supersymmetric invariance.

5.3 Actions of Chiral Superfields

Let us consider then the following superspace integral of the real superfield from eq. (5.14):

$$\int d^4x\, \mathcal{L} = \int d^4x d^2\theta d^2\bar{\theta}\, \left(\bar{\Phi}\Phi\right) = \int d^4x\, \left(\bar{\Phi}\Phi\right)\big|_{\theta\theta\bar{\theta}\bar{\theta}}\,, \tag{5.22}$$

where we substitute in the general expression for Φ from eq. (5.9), and collect only the $\theta\theta\bar{\theta}\bar{\theta}$ term, which then yields an ordinary Lagrangian density. As we shall see, this is the simplest supersymmetric action constructed from chiral superfields.

It is straightforward to compute the Lagrangian density in eq. (5.22) as noted above, using the spinor identities in eqs. (2.39), (2.38), (2.40), and integration by parts. This computation yields the following Lagrangian density:

$$\mathcal{L} = -\partial_\mu\bar{\phi}\partial^\mu\phi - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + \bar{F}F\,. \tag{5.23}$$

We are left then only with derivative terms, except for F, which is thus an auxiliary field. This is therefore a kinetic action of the component fields, or an action of 2 free massless scalar particles in ϕ, $\bar{\phi}$, and 2 free massless Weyl fermions, ψ, $\bar{\psi}$. Recall that when we constructed massless SUSY representations in section 3.3.2, for $\mathcal{N}=1$ we found supermultiplets with only 2 states: $\lambda = \{0, \frac{1}{2}\}$ or $\{-\frac{1}{2}, 0\}$, each of which is SUSY complete. Yet, CPT invariance must also hold in a QFT, and we see here that both multiplets, which taken together are CPT complete, are automatically included in our theory.

Let us now turn to some dimensional analysis in order to consider the renormalizability of our QFT. Taking the action as dimensionless, then $\mathcal{L}$ is renormalizable, if and only if any of its operators is of a classical dimension that is less than or equal to 4, which ensures that the coupling constants have non-negative mass dimensions. Consider the dimensions in our action:

$$[\theta] = -\tfrac{1}{2}\,, \quad \int d^2\theta\, \theta\theta = 1 \quad \Longrightarrow \quad [d^2\theta] = 1\,, \tag{5.24}$$

$$[\phi] = 1\,, \quad \int d^4x\, d^4\theta\, \left(\bar{\Phi}\Phi\right) \quad \Longrightarrow \quad [\Phi] = 1\,, \tag{5.25}$$

$$\Longrightarrow \quad [\psi] = \tfrac{3}{2}\,, \quad [F] = 2\,. \tag{5.26}$$

We can then in principle put in the action higher powers of the real product $\Phi\bar{\Phi}$, but from dimensional analysis we see that such higher powers would be non-renormalizable.

It is thus straightforward to write the most general basic action for n chiral superfields, which is still renormalizable:

$$\mathcal{L}_{\text{kin}} = \int d^2\theta d^2\bar{\theta}\left(G_{ab}\bar{\Phi}^a\Phi^b\right)\,, \quad a, b \in \{1, \ldots, n\}\,, \tag{5.27}$$

where G_{ab} is a constant Hermitian matrix, so that the integrand can also be recast in the form $\bar{\Phi}^a\Phi^a$, referred to as the canonical kinetic term. More generally, if we allow for non-renormalizability, then the kinetic part of the action of n chiral superfields can be generalized to the so-called Kähler potential, $K(\Phi^a, \bar{\Phi}^a)$, on which we shall elaborate in the following section 5.4.

Let us then proceed to consider further contributions to a supersymmetric action of chiral superfields. As we noted in section 5.2, there is another way to construct a supersymmetric action, rather than the "D-term" formulation, which we implemented in eq. (5.22) thus far. We can take the "F-term" route, which is to integrate a chiral integrand over "half" superspace, together with the Hermitian conjugate of this integral. We take then as an integrand $W(\Phi^a)$, which is a holomorphic function of the chiral superfields, namely it depends only on Φ^a, analytically. Then we have:

$$\int d^4x \left[\int d^2\theta\, W(\Phi^a) + \int d^2\bar{\theta}\, \overline{W}(\bar{\Phi}^a)\right] = \int d^4x \left[W(\Phi^a)|_{\theta\theta} + \overline{W}(\bar{\Phi}^a)|_{\bar{\theta}\bar{\theta}}\right]\,, \tag{5.28}$$

where the function $\overline{W}$ is anti-holomorphic in $\bar{\Phi}^a$. The superfield $W(\Phi^a)$ is called the superpotential, and as we shall see shortly, it encodes the interactions of the theory.

We can then expand $W(\Phi^a)$ around the bottom component ϕ^a, using eq. (5.7), which amounts to an expansion in θ, and then collect only the top component of the integrand, as follows:

$$\begin{aligned}\int d^2\theta\, W(\Phi^a) &= \Big[W(\phi^a) + \partial_a W(\phi^a)\,(\Phi^a - \phi^a) \\ &\qquad + \frac{1}{2}\partial_a\partial_b W(\phi^a)(\Phi^a-\phi^a)(\Phi^b-\phi^b)\Big]\Big|_{\theta\theta} \\ &= \partial_a W(\phi^a) F^a - \frac{1}{2}\partial_a\partial_b W(\phi^a)\psi^a\psi^b\,, \end{aligned} \tag{5.29}$$

where we introduced the notation:

$$\partial_a \equiv \frac{\partial}{\partial\phi^a}\,, \tag{5.30}$$

and in the second equality in eq. (5.29) we used the spinor identity from eq. (2.38). Therefore, eq. (5.28) for the superpotential yields:

$$\begin{aligned}\mathcal{L}_{\mathrm{SPot}} = \int d^2\theta\, W(\Phi^a) + \mathrm{H.C.} &= \left(\partial_a W F^a - \frac{1}{2}\partial_a\partial_b W \psi^a\psi^b\right) \\ &\quad + \left(\partial_{\bar a}\overline{W}\bar F^a - \frac{1}{2}\partial_{\bar a}\partial_{\bar b}\overline{W}\bar\psi^a\bar\psi^b\right)\,, \end{aligned} \tag{5.31}$$

where similar to eq. (5.30), we use:

$$\partial_{\bar a} \equiv \frac{\partial}{\partial\bar\phi^a}\,. \tag{5.32}$$

The total action for chiral superfields is then the sum of the canonical kinetic terms as in eq. (5.22), and the superpotential in eq. (5.31). From this total action we can solve for the auxiliary fields F^a, $\bar F^a$, using their Euler-Lagrange equations:

$$F_a = -\partial_{\bar a}\overline{W}\,, \qquad \bar F_a = -\partial_a W\,. \tag{5.33}$$

Using this to eliminate F^a, $\bar F^a$, from the total Lagrangian, we finally obtain:

$$\begin{aligned}\mathcal{L} &= \int d^2\theta d^2\bar\theta\,\left(\bar\Phi^a\Phi^a\right) + \int d^2\theta\, W(\Phi^a) + \int d^2\bar\theta\,\overline{W}(\bar\Phi^a) \\ &= \Big[-\partial_\mu\bar\phi^a\partial^\mu\phi^a - i\bar\psi^a\bar\sigma^\mu\partial_\mu\psi^a \\ &\qquad - \partial_a W\partial_{\bar a}\overline{W} - \frac{1}{2}\partial_a\partial_b W\psi^a\psi^b - \frac{1}{2}\partial_{\bar a}\partial_{\bar b}\overline{W}\bar\psi^a\bar\psi^b\Big]\,, \end{aligned} \tag{5.34}$$

from which we can identify the scalar potential of ϕ^a:

$$V_0(\phi^a, \bar{\phi}^a) \equiv \partial_a W \partial_{\bar{a}} \overline{W} = |\partial_a W|^2 = |F^a|^2 \geqslant 0 \,, \tag{5.35}$$

which is non-negative. The total potential in eq. (5.34), which involves both bosons and fermions, then reads:

$$V = V_0(\phi^a, \bar{\phi}^a) + \frac{1}{2}\partial_a \partial_b W \psi^a \psi^b + \frac{1}{2}\partial_{\bar{a}} \partial_{\bar{b}} \overline{W} \bar{\psi}^a \bar{\psi}^b \,. \tag{5.36}$$

Considering eqs. (5.24), (5.28), we get from dimensional analysis that the dimension of any operator in the superpotential is bound:

$$[W] \leqslant 3 \,, \tag{5.37}$$

for a renormalizable theory. Due to the locality property of QFTs, the holomorphic potential must be a polynomial, and as we already noted in eq. (5.25), $[\Phi^a] = 1$, so the polynomial is up to cubic order in the chiral superfields, namely it is of the general form:

$$W(\Phi_a) = f_a \Phi^a + m_{ab} \Phi^a \Phi^b + \lambda_{abc} \Phi^a \Phi^b \Phi^c \,, \tag{5.38}$$

where f_a, m_{ab}, λ_{abc}, are some complex coupling constants, whose mass dimensions are non-negative. Substituting this into eq. (5.34), we can obtain after a straightforward computation the most general renormalizable Lagrangian for a supersymmetric theory of chiral superfields.

5.3.1 R-Symmetry in Chiral Models

Using the definition of R-symmetry charge in eq. (4.48), let us assign to the chiral superfield the R-charge $q_R[\Phi^a] = 1$, so that the action of R-symmetry reads:

$$\Phi^a \to \exp(+i\alpha)\Phi^a \,, \qquad \bar{\Phi}^a \to \exp(-i\alpha)\bar{\Phi}^a \,, \tag{5.39}$$

and it is easy to see that the canonical kinetic term in eq. (5.27) always remains invariant under R-symmetry. From this R-charge assignment to the superfield, and due to eq. (4.46), we can fix the R-charges of the component fields of any chiral superfield from eq. (5.7):

$$q_R[\phi^a] = 1 \,, \qquad q_R[\psi^a] = 0 \,, \qquad q_R[F^a] = -1 \,. \tag{5.40}$$

From eq. (4.47) for the R-symmetry transformation of the Grassmannian integration measure, and eq. (5.28), it is easy to see that R-symmetry constrains the R-charge of the superpotential:

$$q_R[W] = 2 \,, \tag{5.41}$$

in order for the supersymmetric action to be R-symmetric.

5.4 Chiral Models and Kähler Geometry

More generally, for a possibly non-renormalizable theory, which can represent an effective field theory (EFT) at low energies, we can write the following general D-term Lagrangian of chiral superfields:

$$\mathcal{L}_{\mathrm{K}} = \int d^2\theta d^2\bar{\theta}\, K\left(\Phi^a, \bar{\Phi}^a\right) , \quad a \in \{1, \ldots, n\} , \tag{5.42}$$

where K is a real superfield called the Kähler potential, which is some function of n chiral superfields, Φ^a, $\bar{\Phi}^a$. This function encodes the kinetic part of the theory, and it has a geometrical interpretation, as we shall see shortly.

To that end, let us then expand this function around the bottom component ϕ^a, which amounts to an expansion in θ, $\bar{\theta}$, as follows:

$$\begin{aligned}\int d^2\theta d^2\bar{\theta}\, K\left(\Phi^a, \bar{\Phi}^a\right) = \Big[& K(\phi^a, \bar{\phi^a}) \\ & + \partial_a K(\phi^a, \bar{\phi^a})\,(\Phi^a - \phi^a) + \partial_{\bar{a}} K(\phi^a, \bar{\phi^a})\,(\bar{\Phi}^a - \bar{\phi}^a) \\ & + \frac{1}{2}\partial_a \partial_{\bar{b}} K(\phi^a, \bar{\phi^a})(\Phi^a - \phi^a)(\bar{\Phi}^b - \bar{\phi}^b) + \cdots \\ & + \frac{1}{4}\partial_a \partial_b \partial_{\bar{c}} \partial_{\bar{d}} K(\phi^a, \bar{\phi^a})(\Phi^a - \phi^a)(\Phi^b - \phi^b)(\bar{\Phi}^c - \bar{\phi}^c)(\bar{\Phi}^d - \bar{\phi}^d)\Big]\Big|_{\theta\theta\bar{\theta}\bar{\theta}} .\end{aligned} \tag{5.43}$$

This is a tedious but straightforward computation, and it is easy to see that the lowest derivative of K that survives in the result reads:

$$\partial_a \partial_{\bar{b}} K \equiv \frac{\partial^2 K}{\partial \phi^a \partial \bar{\phi}^b} . \tag{5.44}$$

Let us then define this derivative of K as a metric:

$$g_{a\bar{b}} \equiv \partial_a \partial_{\bar{b}} K . \tag{5.45}$$

This represents a metric on an n-dimensional complex manifold, called a Kähler manifold, which is parametrized in terms of complex coordinates, that are in this case the scalar fields, ϕ^a:

$$(z^a, \bar{z}^a) = (\phi^a, \bar{\phi}^a) . \tag{5.46}$$

The Kähler manifold is then endowed with a Hermitian metric $g_{a\bar{b}}$:

$$ds^2 = \partial_a \partial_{\bar{b}} K d\phi^a d\bar{\phi}^b . \tag{5.47}$$

The higher derivatives of K determine the associated connection and curvature, where the non-vanishing components of the connection are defined as follows:

$$g_{a\bar{b},c} \equiv g_{d\bar{b}}\Gamma^{d}_{ac}\,, \qquad g_{a\bar{b},\bar{c}} \equiv g_{a\bar{d}}\Gamma^{\bar{d}}_{\bar{b}\bar{c}}\,, \tag{5.48}$$

and the only non-vanishing curvature component reads:

$$R_{a\bar{b}c\bar{d}} = g_{e\bar{d}}\,(\Gamma^{e}_{ac})_{,\bar{b}}\,. \tag{5.49}$$

We can then eliminate the auxiliary fields, F^a, $\bar{F}^a$, from the result of eq. (5.43), using their Euler-Lagrange equations. The resulting Lagrangian reads:

$$\mathcal{L}_{\rm K} = -g_{a\bar{b}}\partial_\mu\phi^a\partial^\mu\bar{\phi}^b + ig_{a\bar{b}}D_\mu\psi^a\sigma^\mu\bar{\psi}^b + \tfrac{1}{4}R_{a\bar{b}c\bar{d}}\psi^a\psi^c\bar{\psi}^b\bar{\psi}^d\,, \tag{5.50}$$

where the covariant derivative of the spinor fields is defined as follows:

$$D_\mu\psi^a \equiv \partial_\mu\psi^a + \Gamma^a_{bc}\,\partial_\mu\phi^b\,\psi^c\,, \tag{5.51}$$

namely it is a covariant spacetime derivative, with the spinor fields, ψ^a, transforming as contravariant tensors in the tangent space of the Kähler manifold.

To get a better geometric interpretation of the latter, let us highlight the analogy with General Relativity (GR). In GR we are used to think of a worldline, which maps a real curve parameter, λ, to local coordinates on a Lorentzian manifold, that is spacetime (Minkowski spacetime in special relativity):

$$x^\mu : \mathbb{R} \to \mathbb{R}^{1,3}\,; \quad \lambda \mapsto x^\mu\,. \tag{5.52}$$

A curve on our Kähler manifold is then the following mapping:

$$\phi^a : \mathbb{R}^{1,3} \to \mathbb{C}^{\,n}\,; \quad x^\mu \mapsto \phi^a\,. \tag{5.53}$$

Thus the spinor field is parallel transported in eq. (5.51), using a covariant derivative along a curve, similar to the covariant derivative of a tensor, say of a particle's spin, $S^\mu(\lambda)$, that is defined only along a curve, $x^\mu(\lambda)$:

$$\frac{DS^\mu}{d\lambda} \equiv \frac{dS^\mu}{d\lambda} + \Gamma^\mu_{\nu\rho}\,\frac{dx^\nu}{d\lambda}S^\rho\,, \tag{5.54}$$

which is familiar from GR. So in our case the points of spacetime along a curve, $x \to x+\Delta x$, play the role of the curve parameter, $\lambda \to \lambda+\Delta\lambda$, and the scalar field, ϕ^a, plays the role of the a-th coordinate of the Kähler manifold. To recap, in eq. (5.50) we obtained the most general D-term Lagrangian of chiral superfields, and we uncovered its geometric interpretation in terms of a Kähler manifold.

Yet, the F-term Lagrangian of chiral superfields can also be further generalized from the superpotential in eq. (5.31), using the notion of Kähler manifolds. This is achieved by promoting summations over indices of conjugate superfields to Kähler metric contractions, and partial derivatives, as in eqs. (5.30), (5.32), to Kähler covariant derivatives, namely:

$$X^a Y^{\bar{a}} \to g_{a\bar{b}} X^a Y^{\bar{b}} \,, \tag{5.55}$$

$$\partial_a \to D_a \,, \qquad \partial_{\bar{a}} \to D_{\bar{a}} \,, \tag{5.56}$$

with $g_{a\bar{b}}$ in eq. (5.45), and Kähler covariant derivatives, which are defined as expected, in particular, for a Kähler scalar s, and a Kähler vector v_a, the covariant derivatives read, respectively:

$$D_a s = \partial_a s \,, \tag{5.57}$$

$$D_a v_b = \partial_a v_b - \Gamma^c_{ab} v_c \,. \tag{5.58}$$

If we apply these upgrades in eqs. (5.31), (5.34), then the interaction potential of chiral superfields in eq. (5.36) is generalized to the following form:

$$V_K = g_{a\bar{b}} D_a W D_{\bar{b}} \overline{W} + \tfrac{1}{2} D_a D_b W \psi^a \psi^b + \tfrac{1}{2} D_{\bar{a}} D_{\bar{b}} \overline{W} \bar{\psi}^a \bar{\psi}^b \,. \tag{5.59}$$

To conclude, adding up eq. (5.50) and eq. (5.59) yields the most general supersymmetric coupling of chiral superfields, and constitutes a supersymmetric version of the non-linear sigma model, expressed in geometrical terms.

By definition the metric $g_{a\bar{b}}$ is Hermitian, thus it can be locally diagonalized. But if it represents a flat Kähler manifold, then it can be diagonalized globally, and we get:

$$\partial_a \partial_{\bar{b}} K = g_{a\bar{b}} = \delta_{a\bar{b}} \implies K\left(\Phi^a, \bar{\Phi}^a\right) = \bar{\Phi}^a \Phi^a \,, \tag{5.60}$$

which is just a summation over n superfield indices. The latter is called the canonical Kähler potential, as noted after eq. (5.22), and it constitutes the simplest (kinetic) action of chiral superfields, as already discussed in section 5.3.

In fact, the chiral theory is renormalizable, only if the Kähler manifold is flat. This can be seen from the curvature term in the Lagrangian in eq. (5.50), wherein the curvature plays the role of a coupling constant, and it is easy to see that its dimension is negative, since as we saw in eq. (5.26), $[\psi] = \frac{3}{2}$. If the Kähler manifold is flat, then the general interaction potential of chiral superfields in eq. (5.59) also reduces in form to eq. (5.36), yet as noted in eq. (5.37), this is not sufficient for the theory to be renormalizable. To conclude, the flatness of the Kähler manifold is a necessary, yet insufficient condition for the renormalizability of the chiral theory.

6. Renormalization in Chiral Models

We saw that a renormalizable theory of chiral superfields must have a canonical Kähler potential as its kinetic action. Such $\mathcal{N}=1$ supersymmetric theories are referred to as Wess-Zumino models [10, 9], since in 1974 (which marks the birth of supersymmetry) Wess and Zumino first presented such a model, which is the simplest supersymmetric theory in 4 spacetime dimensions. In order to get a good sense of the powerful quantum renormalization properties of supersymmetric theories compared to non-supersymmetric QFTs, we shall consider the original Wess-Zumino model.

6.1 Renormalization in Wess-Zumino Model

Let us briefly review first the 2 types of quantum corrections to the effective action, which can be referred to more simply as the renormalized action for the purpose of the present analysis, at some renormalization scale, μ.

1. Field strength. The field strength is the quantum correction that is fixed from the field's kinetic term in the renormalized Lagrangian, for example:

$$\mathcal{L}_{\text{kin}}^{\text{eff}} = -Z_\phi \partial_\mu \bar{\phi} \partial^\mu \phi - i Z_\psi \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi \,, \tag{6.1}$$

where the field strength is a function of the renormalization scale μ:

$$Z_\phi = Z_\phi(\mu) \,, \qquad Z_\psi = Z_\psi(\mu) \,. \tag{6.2}$$

We can then define the renormalized fields:

$$\phi_R \equiv \sqrt{Z_\phi}\, \phi \,, \qquad \psi_R \equiv \sqrt{Z_\psi}\, \psi \,, \tag{6.3}$$

to rescale the kinetic terms back to their canonical form.

With the field strength we also define the so-called anomalous dimension of the field:

$$\gamma \equiv \mu \frac{\partial}{\partial \mu} \log \sqrt{Z} \,. \tag{6.4}$$

The quantum or scaling dimension of the field is then defined as follows:

$$\Delta \equiv \Delta_{\text{cl}} - \gamma \,, \tag{6.5}$$

where Δ_{cl} is the classical or engineering dimension of the field, which is fixed from dimensional analysis.

2. Coupling constants. There are independent quantum corrections to any operator in the renormalized Lagrangian, that is preceded by some coupling constant, including mass terms, of the form:

$$\mathcal{L}_g = g\mathcal{O}(x) \rightarrow \mathcal{L}_g^{\text{eff}} = Z_g g\mathcal{O}(x) = g(1 + \cdots)\mathcal{O}(x) \,. \tag{6.6}$$

In the Wess-Zumino (WZ) model the superpotential reads:

$$W_{\text{WZ}}(\Phi) = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3 \,, \tag{6.7}$$

with a single chiral superfield, and we take $m, \lambda \in \mathbb{R}$ for simplicity.

After we eliminate the auxiliary fields, F, $\bar{F}$, using the Euler-Lagrange equations, the full Lagrangian of this model reads:

$$\begin{aligned} \mathcal{L}_{\text{WZ}} = &- \partial_\mu\bar{\phi}\partial^\mu\phi - m^2\bar{\phi}\phi - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi - \frac{m}{2}\left(\psi\psi + \bar{\psi}\bar{\psi}\right) \\ &- m\lambda\left(\bar{\phi}\phi^2 + \bar{\phi}^2\phi\right) - \lambda^2\bar{\phi}^2\phi^2 - \lambda\left(\phi\psi\psi + \bar{\phi}\bar{\psi}\bar{\psi}\right) \,. \end{aligned} \tag{6.8}$$

The first line can be rewritten in the form:

$$\mathcal{L}_{\text{WZ}}^{\text{free}} = \bar{\phi}\left(\partial_\mu\partial^\mu - m^2\right)\phi - \frac{1}{2}\left(\psi^\alpha, \bar{\psi}_{\dot{\alpha}}\right)\begin{pmatrix} m\delta_\alpha^{\ \beta} & i\sigma^\mu_{\ \alpha\dot{\beta}}\partial_\mu \\ i\bar{\sigma}^{\mu\dot{\alpha}\beta}\partial_\mu & m\delta^{\dot{\alpha}}_{\ \dot{\beta}} \end{pmatrix}\begin{pmatrix} \psi_\beta \\ \bar{\psi}^{\dot{\beta}} \end{pmatrix} \,. \tag{6.9}$$

From this form it is easy to read the scalar propagator:

$$\bar{\phi} \;\dashrightarrow\; \phi = \frac{-i}{p^2 + m^2} \,, \tag{6.10}$$

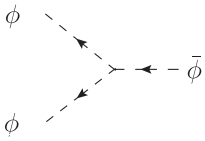
and the fermion propagators:

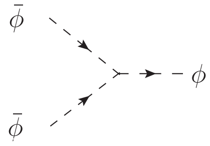
$$\bar{\psi} \longrightarrow \psi = \frac{-i\bar{\sigma}^\mu p_\mu}{p^2 + m^2} \,, \quad \psi \longleftarrow \bar{\psi} = \frac{-i\sigma^\mu p_\mu}{p^2 + m^2} \,, \tag{6.11}$$

$$\psi \longleftrightarrow \psi = \frac{-im}{p^2 + m^2} \,, \quad \bar{\psi} \rightarrow\!\leftarrow \bar{\psi} = \frac{-im}{p^2 + m^2} \,, \tag{6.12}$$

where the fermion propagators are given here in the 2-component Weyl notation, unlike the 4-component Dirac notation usually seen in common QFT textbooks. Note that the $\langle\psi\psi\rangle$ and $\langle\bar{\psi}\bar{\psi}\rangle$ propagators reverse the fermion chirality.

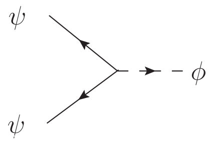
The Feynman rules of the interaction vertices are easily read from the second line of the Lagrangian in eq. (6.8). There are 2 cubic scalar vertices:

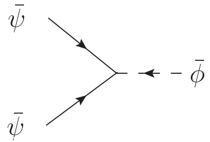


$=$


$$= -im\lambda\,, \tag{6.13}$$

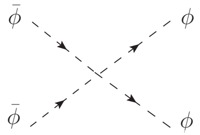
2 Yukawa couplings:



$=$


$$= -i\lambda\,, \tag{6.14}$$

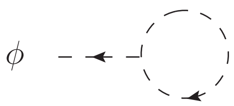
and a single quartic vertex:



$$= -i\lambda^2\,. \tag{6.15}$$

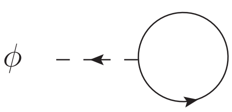
Let us evaluate then the n-point functions of our simple Wess-Zumino model at one-loop order, so as to identify the above quantum corrections.

Scalar tadpole. To evaluate the 1-point function $\langle\phi\rangle$ at one-loop order, we note that there are 2 contributing Feynman diagrams. The first diagram, that contains a scalar loop, reads:



$$= -im\lambda \int \frac{d^4q}{(2\pi)^4}\,\frac{-i}{q^2+m^2}\,, \tag{6.16}$$

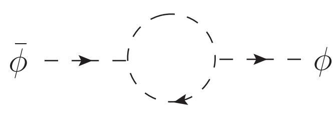
and the second diagram, that contains a fermion loop, reads:



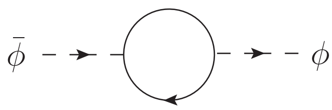
$$= (-1)(-i\lambda) \int \frac{d^4q}{(2\pi)^4}\,\frac{-im}{q^2+m^2}\,, \tag{6.17}$$

where we recall that a fermion loop accounts for an additional minus sign. It is easy to see that eqs. (6.16) and (6.17) cancel each other, adding up to a vanishing scalar tadpole. The overall cancellation of the tadpole is due to supersymmetry, where the bosonic and fermionic loops cancel each other. Note that in non-supersymmetric QFT, we usually have to put in a counterterm, in order to cancel the tadpole.

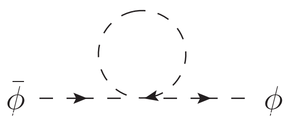
Scalar self-energy. To evaluate the 2-point function $\langle \bar{\phi}\phi \rangle$ at one-loop order, we need to consider 3 contributing Feynman diagrams:



$$= (-im\lambda)^2 \int \frac{d^4q}{(2\pi)^4} \left(\frac{-i}{q^2+m^2} \right)^2 , \tag{6.18}$$



$$= -(-i\lambda)^2 \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \frac{\mathrm{tr}\left(-\sigma^\mu \bar{\sigma}^\nu\right) q_\mu q_\nu}{\left(q^2+m^2\right)^2} , \tag{6.19}$$



$$= -i\lambda^2 \int \frac{d^4q}{(2\pi)^4} \frac{-i}{q^2+m^2} , \tag{6.20}$$

where the so-called BPHZ renormalization scheme is used here, namely taking the limit, where the external momentum goes to 0. It is easy to verify that when the 3 diagrams are added up, the sum vanishes:

$$\lim_{p^2 \to 0} \Pi_{\bar{\phi}\phi}(p^2) = 0 \,. \tag{6.21}$$

Thus, there is no renormalization of the scalar field-strength or mass m^2, thanks to supersymmetry.

Interaction vertices. One can similarly study the higher-point functions $\langle \bar{\phi}\phi\phi \rangle$ and $\langle \bar{\phi}\bar{\phi}\phi\phi \rangle$ with some more work, and verify that the one-loop corrections to the scalar cubic and quartic vertices also vanish in the renormalization scheme of vanishing external momentum.

To get a better understanding of the renormalization in the WZ model, let us consider the model, when the auxiliary fields are kept at the Lagrangian:

$$\begin{aligned} \mathcal{L}_{\mathrm{WZ}} = &- \partial_\mu \bar{\phi} \partial^\mu \phi - i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi + \bar{F}F + m\left(F\phi + \bar{F}\bar{\phi}\right) - \frac{m}{2}\left(\psi\psi + \bar{\psi}\bar{\psi}\right) \\ &+ \lambda\left(F\phi^2 + \bar{F}\bar{\phi}^2\right) - \lambda\left(\phi\psi\psi + \bar{\phi}\bar{\psi}\bar{\psi}\right) \,. \end{aligned} \tag{6.22}$$

In this formulation of the model there are additional scalar propagators including the auxiliary fields, where the former propagators are unchanged. The second line of eq. (6.22) yields only cubic vertices of 2 types, so in that sense this perturbation theory for the WZ model seems simpler.

In this formulation one finds at one-loop order that the 2-point functions are renormalized, but all with the same field-strength factor:

$$Z_\Phi \equiv Z_\phi = Z_\psi = Z_F\,. \tag{6.23}$$

Moreover, the mass m and the cubic coupling constant λ are also renormalized in terms of this single factor, Z_Φ:

$$m_R \equiv Z_\Phi^{-1} m\,, \qquad \lambda_R \equiv Z_\Phi^{-\frac{3}{2}} \lambda\,. \tag{6.24}$$

Accordingly, there is a single anomalous dimension, γ_Φ, for the components of the chiral superfield, such that the quantum dimensions of the component fields read:

$$\Delta_\phi = 1 - \gamma_\Phi\,, \quad \Delta_\psi = \frac{3}{2} - \gamma_\Phi\,, \quad \Delta_F = 2 - \gamma_\Phi\,. \tag{6.25}$$

The renormalization results that we have seen at one-loop order, generalize to all orders in perturbation theory. In fact, the effective action can always be written in the form:

$$\begin{aligned}\mathcal{L}^{\text{eff}}_{WZ} =& Z_\Phi \left(-\partial_\mu \bar\phi \partial^\mu \phi - i\bar\psi \bar\sigma^\mu \partial_\mu \psi + \bar F F\right) + m\left(F\phi + \bar F \bar\phi\right) - \frac{m}{2}\left(\psi\psi + \bar\psi\bar\psi\right) \\ &+ \lambda\left(F\phi^2 + \bar F \bar\phi^2\right) - \lambda\left(\phi\psi\psi + \bar\phi\bar\psi\bar\psi\right)\,,\end{aligned} \tag{6.26}$$

so that the part with coupling constants remains unchanged. Further, with the definition:

$$\Phi_R \equiv \sqrt{Z_\Phi}\,\Phi\,, \tag{6.27}$$

the effective action takes the form:

$$\begin{aligned}\mathcal{L}^{\text{eff}}_{\text{WZ}} =& -\partial_\mu \bar\phi_R \partial^\mu \phi_R - i\bar\psi_R \bar\sigma^\mu \partial_\mu \psi_R + \bar F_R F_R \\ &+ m_R\left(F_R\phi_R + \bar F_R \bar\phi_R\right) - \frac{m_R}{2}\left(\psi_R\psi_R + \bar\psi_R\bar\psi_R\right) \\ &+ \lambda_R\left(F_R\phi_R^2 + \bar F_R \bar\phi_R^2\right) - \lambda_R\left(\phi_R\psi_R\psi_R + \bar\phi_R\bar\psi_R\bar\psi_R\right)\,,\end{aligned} \tag{6.28}$$

which is identical to the initial form of the bare Lagrangian in eq. (6.22)!

6.2 Non-Renormalization in Chiral Models

Our analysis of the renormalization of the WZ model can be generalized to any theory of chiral superfields with a canonical kinetic term. In fact, there is a more general famous theorem for $\mathcal{N} = 1$ supersymmetric theories, which says that the effective (or renormalized) action of a chiral theory is always of the form:

$$\mathcal{L}^{\text{eff}}_{\text{chiral}} = \int d^2\theta d^2\bar{\theta} \sum_a Z_{\Phi^a} \bar{\Phi}^a \Phi^a + \int d^2\theta\, W(\Phi^a) + \int d^2\bar{\theta}\, \overline{W}(\bar{\Phi}^a) \,. \tag{6.29}$$

This form only has a field-strength renormalization factor for each chiral superfield, whereas the superpotential, $W(\Phi^a)$, is not renormalized at all! This result for supersymmetric theories is the non-renormalization theorem, elegantly proven by Seiberg in 1993 [11, 12], based entirely on symmetry arguments.

Before we outline the proof of this theorem, let us first shed some light on the concept of Wilsonian effective actions, see also, e.g. [3], for a useful introduction to the Wilsonian approach to renormalization.

Wilsonian effective action. The idea is to start with a theory which has an action defined at a UV energy scale $\mu_0 = \Lambda_{UV}$, which might be sent to infinity if the theory is renormalizable. Then the objective is to compute the effective action, S^{eff}_μ, at a scale $\mu < \mu_0$ by "integrating out" all degrees of freedom from μ_0 down to μ. In momentum space the fields are split into high and low momentum modes:

$$\varphi(q) \equiv \begin{cases} \varphi_H(q) & \mu < |q| \leqslant \mu_0 \\ \varphi_L(q) & |q| \leqslant \mu \end{cases} , \tag{6.30}$$

so that the functional integral can be written in terms of the 2 distinct modes:

$$\int D\varphi \, \exp\left(iS[\varphi]\right) = \int D\varphi_L D\varphi_H \, \exp\left(iS[\varphi_L, \varphi_H]\right) \,. \tag{6.31}$$

The Wilsonian effective action is then explicitly defined by functional integration only over the high momentum modes:

$$\exp\left(iS^{\text{eff}}_\mu[\varphi_L]\right) \equiv \int D\varphi_H \, \exp\left(iS[\varphi_L, \varphi_H]\right) \,. \tag{6.32}$$

On the other hand, the effective action at low energy μ can also be written as a generic sum over infinitely many operators:

$$S^{\text{eff}}_\mu = \int d^4x \sum_{i\in\mathbb{N}} g_i(\mu;\mu_0)\, \mathcal{O}_i(x) \,, \tag{6.33}$$

where the $\mathcal{O}_i$ operators are constrained by the symmetries that survive at low energies, and the coupling constants g_i encapsulate the dependence in the UV scale, which was suppressed.

Returning to the proof of Seiberg's non-renormalization theorem, supersymmetric invariance constrains the effective theory to also have the form:

$$\mathcal{L}_\mu^{\text{eff}} = \int d^2\theta d^2\bar{\theta}\, \bar{\Phi}_R^a \Phi_R^a + \left[\left[\int d^2\theta\, W_\mu^{\text{eff}}(\Phi^a) + \text{H.C.}\right]\right] . \qquad (6.34)$$

Recall that the superpotential at the UV scale reads:

$$W_{\mu_0}(\Phi^a) = \lambda_a \Phi^a + \lambda_{ab}\Phi^a\Phi^b + \lambda_{abc}\Phi^a\Phi^b\Phi^c\,, \qquad (6.35)$$

where λ_a, λ_{ab}, λ_{abc}, are some complex coupling constants.

A critical point is that the coupling constants can also be regarded as chiral superfields. In fact coupling constants can generally be considered as dynamical fields of some very massive particles in a more complete theory at higher energies. The couplings then appear as background or dormant fields, that are "frozen" in their vacuum expectation values (VEVs, see also in chapter 8).

Thus, the superpotential is holomorphic in the chiral superfields, and in the coupling constants, which also holds for the effective superpotential, that has the general form:

$$W_\mu^{\text{eff}}(\Phi^a) = \sum_{m\in\mathbb{N}} \sum_{\{a_1\cdots a_m\}} g_{a_1\cdots a_m} \Phi^{a_1}\cdots\Phi^{a_m}\,, \qquad (6.36)$$

where $g_{a_1\cdots a_m}$ are some coupling constants preceding monomials of m-th power, denoted by a set $\{a_1\cdots a_m\}$ of m indices, where m can be any positive integer. The effective coupling constants are holomorphic functions of the coupling constants at the scale μ_0:

$$g_{a_1\cdots a_m} = g_{a_1\cdots a_m}(\lambda_a, \lambda_{ab}, \lambda_{abc})\,. \qquad (6.37)$$

We can thus consider further symmetries beyond supersymmetry, say $U(1)\times U(1)_R$, with some global $U(1)$ symmetry, and $U(1)_R$ R-symmetry. This $U(1)\times U(1)_R$ symmetry, which clearly holds in the free theory, when the superpotential vanishes, also persists in the superpotential, if the couplings are regarded as fields, and further then in the effective theory. Let us then consider each of these symmetries at a time.

Global $U(1)$ symmetry. Consider some global $U(1)$ symmetry group, for simplicity, which transforms any superfield as follows:

$$\Phi^a \to \exp(iq_a\alpha)\Phi^a\,, \qquad (6.38)$$

where q_a is some arbitrary symmetry charge that we assign to Φ^a. Clearly, $\bar{\Phi}^a\Phi^a$ is invariant under this symmetry. Since we regard the coupling constants as superfields, we can assign to them symmetry charges as well, so that they transform under the $U(1)$ group as follows:

$$\lambda_a \to \lambda_a \exp(-iq_a\alpha)\,, \tag{6.39}$$

$$\lambda_{ab} \to \lambda_{ab} \exp(-i(q_a + q_b)\alpha)\,, \tag{6.40}$$

$$\lambda_{abc} \to \lambda_{abc} \exp(-i(q_a + q_b + q_c)\alpha)\,. \tag{6.41}$$

With this assignment of charges to the chiral superfields and coupling constants, it is evident that the UV superpotential in eq. (6.35) is invariant under the global $U(1)$ symmetry.

The effective superpotential is also invariant under this global $U(1)$ symmetry, which we write in the following short form:

$$W^{\text{eff}}_\mu(g', \Phi') = W^{\text{eff}}_\mu(g, \Phi)\,, \tag{6.42}$$

where g, Φ, are just shorthand notations for the sets of coupling constants and superfields, respectively, in the superpotential. Similarly to eqs. (6.39)-(6.41), we assign $U(1)$ charges to the coupling constants in the effective superpotential, so that they transform as follows:

$$g_{a_1\cdots a_m} \to \exp\left(-i(q_{a_1} + \cdots + q_{a_m})\alpha\right) g_{a_1\cdots a_m}\,, \tag{6.43}$$

so that the monomial terms in eq. (6.36) are also each invariant under this symmetry, and indeed the effective superpotential is altogether invariant.

Combining eqs. (6.37), (6.39)-(6.43), we get the following equation for the coupling constants in the effective superpotential:

$$\begin{aligned} &g_{a_1\cdots a_m}\left(\lambda_a \exp(-iq_a\alpha), \lambda_{ab}\exp(-i(q_a+q_b)\alpha), \lambda_{abc}\exp(-i(q_a+q_b+q_c)\alpha)\right) \\ &\qquad = g_{a_1\cdots a_m}\left(\lambda_a, \lambda_{ab}, \lambda_{abc}\right) \exp\left(-i(q_{a_1} + \cdots + q_{a_m})\alpha\right)\,, \end{aligned} \tag{6.44}$$

where the LHS is obtained by first transforming under $U(1)$, and then integrating out down to the scale μ, and the RHS is obtained by first integrating out and then transforming. From eq. (6.44) we infer that g must be a linear combination of products of λ couplings, where the set of indices of g is replicated in each such product by the disjoint union of its λ indices as subsets, each denoted here by $\{a_i\}$:

$$g_{a_1\cdots a_m} = \sum_j n_j \left[\prod_{\{a_i\}} \lambda_{\{a_i\}}\right]_j\,, \qquad \{a_1, \ldots, a_m\} = \sqcup\{a_i\}\,, \tag{6.45}$$

where n_j stand for some numerical coefficients of such products. So for example, a certain g coupling can be of the form:

$$g_{1124} = n_1 \lambda_1^2 \lambda_{24} + n_2 \lambda_{12} \lambda_{14} + n_3 \lambda_2 \lambda_{114} \,. \tag{6.46}$$

There can only be positive powers of λ in the products, since in the weak-coupling limit, when we take each $\lambda \to 0$, the theory should be free.

$U(1)$ **R-symmetry.** We already saw in section 5.3.1 that the canonical kinetic term in chiral models is R-symmetric, regardless of the R-charges that are assigned to the superfields. Following the definitions in section 4.3.1, in particular in eq. (4.48), let us assign to the superfields Φ^a arbitrary R-charges q_R^a. Recall that due to eq. (4.47), $U(1)$ R-symmetry also affects the Grassmannian integration measure in the integral of the superpotential, which fixes the R-charge of the superpotential in eq. (5.41).

Here too, we can assign R-charges to the coupling constants, so that they transform under the $U(1)_R$ R-symmetry as follows:

$$\lambda_a \to \lambda_a \exp(i\alpha[2 - q_R^a]) \,, \tag{6.47}$$

$$\lambda_{ab} \to \lambda_{ab} \exp(i\alpha[2 - (q_R^a + q_R^b)]) \,, \tag{6.48}$$

$$\lambda_{abc} \to \lambda_{abc} \exp(i\alpha[2 - (q_R^a + q_R^b + q_R^c)]) \,. \tag{6.49}$$

With this assignment of R-charges to the chiral superfields and coupling constants, it is evident that the action of the superpotential in eq. (6.35) is R-symmetric, and in agreement with eq. (5.41) the superpotential transforms as follows:

$$W_{\mu_0}(\lambda', \Phi') = \exp(2i\alpha) W_{\mu_0}(\lambda, \Phi) \,, \tag{6.50}$$

where λ, Φ, are again just for shorthand notation.

The effective action of the superpotential is also R-symmetric, so we have:

$$W_\mu^{\text{eff}}(g', \Phi') = \exp(2i\alpha) W_\mu^{\text{eff}}(g, \Phi) \,. \tag{6.51}$$

Similarly to eqs. (6.47)-(6.49), we assign R-charges to the coupling constants in the effective superpotential, so that they transform as follows:

$$g_{a_1 \cdots a_m} \to \exp\left(i\alpha[2 - (q_R^{a_1} + \cdots + q_R^{a_m})]\right) g_{a_1 \cdots a_m} \,, \tag{6.52}$$

so that the monomial terms in eq. (6.36) each yield the proper R-charge of the superpotential. Combining eqs. (6.37), (6.47)-(6.49), (6.52), we get that the coupling constant g must be linear in the λ couplings:

$$g_{a_1 \cdots a_m} = n_{a_1 \cdots a_m} \lambda_{a_1 \cdots a_m} \,, \tag{6.53}$$

	$U(1)$	$U(1)_R$
Φ	$+1$	$+1$
$\tilde{m}$	-2	0
λ	-3	-1

Table 6.1: The charges assigned to the superfields and couplings under the respective symmetries of the action.

where $n_{a_1 \cdots a_m}$ is some numerical coefficient.

Combining the implications of both symmetries, $U(1) \times U(1)_R$, as formulated in eqs. (6.45), (6.53), we obtain for each coupling constant in the effective superpotential:

$$g_{a_1 \cdots a_m} \equiv \begin{cases} n_{a_1 \cdots a_m} \lambda_{a_1 \cdots a_m} & m \leqslant 3 \\ 0 & m > 3 \end{cases} . \tag{6.54}$$

Finally, to find the numerical proportion coefficients $n_{a_1 \cdots a_m}$, we just take the weak-coupling limit, namely the limit of vanishing coupling constants, and match by comparison in perturbation theory. The numerical coefficients are found to equal 1, from matching with the free theory, that is when the superpotential vanishes, and with the tree-level theory. All in all then, the effective superpotential is found to be identical in form to the UV superpotential:

$$W_\mu^{\text{eff}} = W_{\mu_0} . \tag{6.55}$$

Although we outlined here the general proof of Seiberg's theorem, let us specialize to our simple Wess-Zumino model as an example, in order to get a better sense of how this powerful result is obtained. Let us first write the model in eq. (6.7) in terms of dimensionless coupling constants:

$$\tilde{m}(\mu) \equiv m/\mu \qquad \Longrightarrow \qquad [\tilde{m}] = [\lambda] = 0 \,, \tag{6.56}$$

so that the UV model in eq. (6.7) takes the form:

$$W_{\text{WZ}}(\mu_0) = \mu_0 \frac{\tilde{m}}{2} \Phi^2 + \frac{\lambda}{3} \Phi^3 . \tag{6.57}$$

The free theory, where $W = 0$, has our $U(1) \times U(1)_R$ symmetry. The symmetry charges that are assigned here to the superfields and couplings are listed in table 6.1. With these symmetry charges the superpotential is also $U(1) \times U(1)_R$ invariant.

Then the most general form for the effective superpotential at scale $\mu < \mu_0$ reads:

$$W_{\text{WZ}}^{\text{eff}}(\mu) = \mu \tilde{m} \Phi^2 f \left(\frac{\lambda \Phi}{\mu \tilde{m}}, \frac{\mu}{\mu_0} \right) , \tag{6.58}$$

where f is a holomorphic function of dimensionless, uncharged parameters. This form keeps the action $U(1) \times U(1)_R$ invariant, and of the correct dimension. The function f should be analytic in its first argument, and regular in the weak-coupling limit, $\tilde{m}, \lambda \to 0$. Expanding f accordingly, we find:

$$W_{\rm WZ}^{\rm eff}(\mu) = \sum_{i=0}^{\infty} c_i(\mu/\mu_0) \frac{\lambda^i}{(\mu\tilde{m})^{i-1}} \Phi^{i+2} \,, \tag{6.59}$$

where the c_i coefficients are functions of μ/μ_0 to be matched with the UV theory.

From the regularity of $\tilde{m} \to 0$, $i > 1$ is eliminated, and we are left with:

$$W_{\rm WZ}^{\rm eff}(\mu) = c_0 \mu \tilde{m} \Phi^2 + c_1 \lambda \Phi^3 \,. \tag{6.60}$$

From $\lambda \to 0$, the theory is free and classical, and the mass terms at the scales μ_0 and μ can be matched:

$$\tilde{m}(\mu) \equiv m/\mu = \tilde{m}(\mu_0)\mu_0/\mu \,. \tag{6.61}$$

This fixes $c_0 = 1/2$. At $\lambda \neq 0$ we can match c_1 by comparing perturbation theory at tree level. This fixes $c_1 = 1/3$. In conclusion we found:

$$W_{\rm WZ}^{\rm eff}(\mu) = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3 \,. \tag{6.62}$$

7. Supersymmetric Gauge Theories

We proceed to consider the supersymmetric versions of gauge theories. As the Standard Model of Particle Physics, which has been well-confirmed, is comprised of gauge theories, with the symmetry group in eq. (1.3), this is a first essential step towards the real world.

7.1 Gauge Theory Primer

Before we delve into the incorporation of gauge symmetry in supersymmetric theories, let us review first the basics of gauge theory without supersymmetry.

Consider a Lie group G with its Lie algebra $\mathfrak{g} = \text{Lie}(G)$, and T_A its Hermitian generators, which satisfy the commutation relations:

$$[T_A, T_B] = i f_{ABC} T_C \,, \tag{7.1}$$

where f_{ABC} are the structure constants. The orthogonality condition of the normalized generators reads:

$$\text{tr}\,(T_A T_B) = c_{(r)} \delta_{AB} \,, \qquad c_{(r)} > 0 \,, \tag{7.2}$$

where $c_{(r)}$ is a positive constant, that depends on the representation r of the Lie group.

Let A_μ denote a gauge field related with the gauge group G. It is a covector field, which is valued in the adjoint representation of G:

$$A_\mu(x) = A_\mu^A(x) T_A \,. \tag{7.3}$$

Let g(x) be a group-valued function:

$$g(x) : \mathbb{R}^{1,3} \to G \,. \tag{7.4}$$

Then a gauge transformation on the gauge field reads:

$$A'_\mu = g(x) \left(A_\mu + i\partial_\mu \right) g^{-1}(x) \,, \tag{7.5}$$

which keeps A'_μ physically equivalent to A_μ. Consider then a general group element:

$$g(x) = \exp\left(i\alpha(x)\right) \,, \qquad \alpha(x) \equiv \alpha_A(x) T_A \in i\mathfrak{g} \,, \tag{7.6}$$

where α_A are the group parameters. Then if we consider a gauge transformation of the gauge field to first order in the group parameters, we get:

$$\delta A_\mu = A'_\mu - A_\mu = \partial_\mu \alpha + i[\alpha, A_\mu] \,. \tag{7.7}$$

The field-strength of the gauge field is then defined as follows:

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] = F^A_{\mu\nu} T_A \,, \tag{7.8}$$

so that by construction its gauge transformation reads:

$$F'_{\mu\nu} = g F_{\mu\nu} g^{-1} \,, \tag{7.9}$$

and to first order in the gauge parameters we obtain:

$$\delta F_{\mu\nu} = F'_{\mu\nu} - F_{\mu\nu} = i[\alpha, F_{\mu\nu}] \,. \tag{7.10}$$

A gauge theory based on a compact Lie group, such as a special unitary group, SU(N), is called a Yang-Mills (YM) theory, and its Lagrangian reads:

$$\mathcal{L}_{\text{YM}} = \text{tr}\left(-\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu}\right) = -\frac{c_{(r)}}{4g^2} F^A_{\mu\nu} F^{\mu\nu}_A \,, \tag{7.11}$$

where g^2 is the YM gauge coupling, and in the second equality we used the orthogonality condition in eq. (7.2). Considering eq. (7.9) and the cyclicity of trace, it is easy to see that the YM action is gauge-invariant. The YM action is the canonical kinetic term for a gauge field, and it is easy to see that for the gauge group U(1), YM theory turns into Maxwell's theory, if we just make the replacement:

$$c_{(r)}/g^2 \to 1 \,. \tag{7.12}$$

Given a gauge group G, we can introduce charged matter fields φ_i, which might be scalars or spinors, in a representation r of G, so that they sit in m-plets of dimension:

$$i \in \{1, \ldots, m \equiv \dim r\} \,. \tag{7.13}$$

Under the gauge transformation in eq. (7.5), the matter field φ_i transforms as follows:

$$\varphi'_i = [g_{(r)}]_{ij} \varphi_j \,, \tag{7.14}$$

where $[g_{(r)}]_{ij}$ is the group element as a matrix in the representation r. To first order in the group parameters, we have then:

$$\delta\varphi_i = \varphi_i' - \varphi_i = i\alpha_{ij}\varphi_j = i\alpha_A[T_{A(r)}]_{ij}\varphi_j \,, \tag{7.15}$$

where $T_{A(r)}$ are the generators in the representation r of φ_i.

The gauge-covariant derivative is defined as follows:

$$D_\mu\varphi_i \equiv (\partial_\mu - iA_\mu)\,\varphi_i \,, \tag{7.16}$$

with A_μ a matrix in the representation r of φ_i. This is the so-called "minimal coupling" of the gauge field to the matter fields. By construction $D_\mu\varphi_i$ transforms covariantly in the same representation of φ_i:

$$\delta D_\mu\varphi_i = i\alpha D_\mu\varphi_i \,. \tag{7.17}$$

Finally, the gauge-invariant kinetic terms of matter fields can easily be written by replacing ordinary derivatives with gauge-covariant derivatives, e.g. for a scalar ϕ we have:

$$\mathcal{L}_{\text{matter}} = -D_\mu\bar{\phi}D^\mu\phi \,. \tag{7.18}$$

7.2 Abelian Vector Superfields

We would like to consider first the $\mathcal{N}=1$ supersymmetric version of QED, a supersymmetric QFT with the Abelian gauge group:

$$G = U(1) \,. \tag{7.19}$$

Such a theory contains a gauge field A_μ, subject to gauge transformations of the form:

$$A'_\mu(x) \equiv A_\mu(x) + \partial_\mu\alpha(x) \,, \tag{7.20}$$

where the commutator term in eq. (7.7) drops out.

Recall from eq. (4.43), that a general superfield includes a notable term, $\theta\sigma^\mu\bar{\theta}A_\mu$, which seems natural to yield a gauge field. Since a gauge field is real, let us then first require such a vector superfield, which we denote by V, to be real:

$$V = V^\dagger \,. \tag{7.21}$$

Applying this reality condition to the general superfield in eq. (4.43), we get:

$$\begin{aligned} V(x,\theta,\bar{\theta}) = {} & B(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) + \tfrac{i}{2}\theta\theta G(x) - \tfrac{i}{2}\bar{\theta}\bar{\theta}\bar{G}(x) - \theta\sigma^\mu\bar{\theta}A_\mu(x) \\ & + i\theta\theta\bar{\theta}\left(\bar{\lambda}(x) + \tfrac{i}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)\right) - i\bar{\theta}\bar{\theta}\theta\left(\lambda(x) + \tfrac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\right) \\ & + \tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left(D(x) + \tfrac{1}{2}\partial^2 B(x)\right) \,, \end{aligned} \tag{7.22}$$

where as we noted at the end of section 4.3, we incorporated here the freedom to redefine the higher component fields, λ, $\bar{\lambda}$, and D, by adding terms with a spacetime derivative of χ, $\bar{\chi}$, and 2 spacetime derivatives of B, respectively. Thus for a real superfield, the bottom component, B, is taken to be real, as well as the vector field, A_μ, and the top component, D, whereas the component fields χ, G, λ, and $\bar{\chi}$, $\bar{G}$, $\bar{\lambda}$, are conjugates of each other, respectively, as implied by their notation. We then call the above a vector superfield after its real component field A_μ.

Yet, to actually combine gauge invariance with supersymmetry, we should find a superfield generalization of the gauge transformation of the gauge field. Thus, let us consider the difference between a chiral superfield, Ω, and its conjugate, using eq. (5.9):

$$\begin{aligned}\tfrac{i}{2}\left(\Omega-\Omega^\dagger\right) = \tfrac{i}{2}\Big(&\left(\phi-\bar{\phi}\right) + \sqrt{2}\left(\theta\psi-\bar{\theta}\bar{\psi}\right) + \left(\theta\theta F-\bar{\theta}\bar{\theta}\bar{F}\right) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\left(\phi+\bar{\phi}\right) \\ &+ \tfrac{i}{\sqrt{2}}\left(\theta\theta\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi - \bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi}\right) + \tfrac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\,\partial^2\left(\phi-\bar{\phi}\right)\Big)\,. \end{aligned} \tag{7.23}$$

This difference is clearly real, and we identify the gradient in the gauge transformation in eq. (7.20), as the 4-vector component field. This motivates us to define the vector superfield transformation as follows:

$$V' \equiv V + \frac{i}{2}\left(\Omega-\bar{\Omega}\right)\,, \tag{7.24}$$

which is the supersymmetric generalization of eq. (7.20).

Under this transformation with the chiral superfield:

$$\Omega = \left(\phi, \psi_\alpha, F\right), \tag{7.25}$$

the component fields of the vector superfield in eq. (7.22) become:

$$B \to B + \frac{i}{2}\left(\phi-\bar{\phi}\right)\,, \tag{7.26}$$

$$\chi \to \chi + \frac{1}{\sqrt{2}}\psi\,, \qquad \bar{\chi} \to \bar{\chi} + \frac{1}{\sqrt{2}}\bar{\psi}\,, \tag{7.27}$$

$$G \to G + F\,, \qquad \bar{G} \to \bar{G} + \bar{F}\,, \tag{7.28}$$

$$A_\mu \to A_\mu + \partial_\mu\left(\frac{\phi+\bar{\phi}}{2}\right)\,, \tag{7.29}$$

$$\lambda \to \lambda\,, \qquad \bar{\lambda} \to \bar{\lambda}\,, \tag{7.30}$$

$$D \to D\,. \tag{7.31}$$

It is easy to see then, that B, χ, $\bar{\chi}$, G, $\bar{G}$, are pure gauge, i.e. they can be made to vanish by a proper choice of Ω, whereas λ, $\bar{\lambda}$, and D, are gauge-invariant. As to the gauge of the vector field from eq. (7.20), we can identify the real gauge function as follows:

$$\alpha(x) \equiv \frac{\phi(x) + \bar{\phi}(x)}{2} \,. \tag{7.32}$$

It is useful then to specify the so-called Wess-Zumino (WZ) gauge [13]:

$$B = \chi = \bar{\chi} = G = \bar{G} = 0 \,, \tag{7.33}$$

in which there is still residual gauge invariance left of the vector field, as in eq. (7.29) via eq. (7.32), with the chiral superfield:

$$\Omega_{\mathrm{WZ}} = \bar{\Omega}_{\mathrm{WZ}} = (\alpha, 0, 0) \,. \tag{7.34}$$

The vector superfield in the WZ gauge is then of the form:

$$V_{\mathrm{WZ}} = (A_\mu, \lambda_\alpha, \bar{\lambda}_{\dot{\alpha}}, D) \,, \tag{7.35}$$

and its expansion reads:

$$V_{\mathrm{WZ}} = -\theta\sigma^\mu\bar{\theta}A_\mu + i\theta\theta\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\theta\lambda + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D \,. \tag{7.36}$$

In this superfield we identify the bosonic component A_μ as the photon, and the components λ, $\bar{\lambda}$, as its fermionic superpartner, referred to as the photino. More generally, for a general (e.g. non-Abelian) gauge group, these components are referred to as the gauge boson, and gaugino, respectively. As we shall see shortly in section 7.3, the bosonic component D is an auxiliary field.

Finally, let us briefly comment on the compatibility of the WZ gauge with supersymmetric invariance. It can be easily verified that a supersymmetric transformation does not preserve the WZ gauge. Yet, one can then define a proper gauge transformation that restores the WZ gauge, so that the combined supersymmetric and gauge transformations preserve the WZ gauge.

7.3 Supersymmetric Abelian Gauge Theories

Let us now turn to consider a fully gauge-invariant superfield, rather than the vector superfield in the WZ gauge in eq. (7.36). Since λ, $\bar{\lambda}$, and D, are already gauge-invariant, let us just switch from A_μ to the field-strength from eq. (7.8):

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{7.37}$$

where the commutator term drops in an Abelian gauge group, and the field-strength is then gauge-invariant.

Thus, in order to construct a superfield from λ, $\bar\lambda$, D, and $F_{\mu\nu}$, we first note their dimensions:

$$[D] = [F_{\mu\nu}] = [\lambda] + \frac{1}{2} \,. \tag{7.38}$$

The field λ could then be a bottom component of some chiral spinor superfield, W_α, of the form:

$$W_\alpha = (\phi_\alpha, \psi_{\alpha\beta}, F_\alpha) \,, \tag{7.39}$$

with the components:

$$\phi_\alpha \equiv \lambda_\alpha \,, \tag{7.40}$$

$$\psi_{\alpha\beta} \equiv i\epsilon_{\alpha\beta} D + \epsilon_{\beta\gamma} (\sigma^{\mu\nu})_\alpha{}^\gamma F_{\mu\nu} \,, \tag{7.41}$$

$$F_\alpha \equiv i(\sigma^\mu \partial_\mu \bar\lambda)_\alpha \,, \tag{7.42}$$

where the bispinor component, $\psi_{\alpha\beta}$, and the top component, F_α, can be constructed from dimensional and spinorial considerations, and we recall that this is in the WZ gauge, and in chiral coordinates.

It can be verified that the above representation of W_α also matches the following definition, using the generic vector superfield V:

$$W_\alpha \equiv -\frac{i}{4} \bar D \bar D D_\alpha V \,, \qquad \overline{W}_{\dot\alpha} \equiv +\frac{i}{4} D D \bar D_{\dot\alpha} V \,. \tag{7.43}$$

This spinor superfield is chiral and gauge-invariant by construction. Chirality can also be seen to follow immediately from the definition in eq. (7.43):

$$\bar D_{\dot\beta} W_\alpha = D_\beta \overline{W}_{\dot\alpha} = 0 \,. \tag{7.44}$$

Gauge invariance can also be easily shown from the definition in eq. (7.43):

$$W'_\alpha = -\frac{i}{4} \bar D \bar D D_\alpha \left(V + \frac{i}{2} \left(\Omega - \bar\Omega \right) \right) = W_\alpha - \frac{1}{8} \bar D^{\dot\beta} \{ \bar D_{\dot\beta}, D_\alpha \} \Omega = W_\alpha \,, \tag{7.45}$$

where we used the chirality, and anti-chirality, of Ω, $\bar\Omega$, respectively, and in the last equality we also used the identity:

$$[\bar D, \{\bar D, D\}] = 0 \,. \tag{7.46}$$

In addition, this spinor superfield also satisfies the identity:

$$D^\alpha W_\alpha = \bar{D}_{\dot{\alpha}} \overline{W}^{\dot{\alpha}} \,, \tag{7.47}$$

which can be verified from the definition in eq. (7.43).

Since W_α is chiral, its Lorentz scalar is also chiral, and we can take the gauge-invariant F-term Lagrangian as follows:

$$\mathcal{L} = \frac{1}{4} \left(W^\alpha W_\alpha |_{\theta\theta} + \overline{W}_{\dot{\alpha}} \overline{W}^{\dot{\alpha}} \Big|_{\bar{\theta}\bar{\theta}} \right) \,, \tag{7.48}$$

which is the supersymmetric generalization of the Lagrangian for a free vector field. After direct computation of this Lagrangian with some integration by parts at the level of the action, we get the supersymmetric version of Maxwell's theory:

$$\mathcal{L}_{\text{SMaxwell}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - i\bar{\lambda}\bar{\sigma}^\mu \partial_\mu \lambda + \frac{1}{2} D^2 \,, \tag{7.49}$$

where as before, we identify the first and second terms as the kinetic terms of the photon and photino, respectively, and D as an auxiliary field.

Finally, one can also add the mass term $m^2 V^2$ to the free Lagrangian in eq. (7.48). This mass term is not gauge-invariant, so it should be computed from the general form of V (rather than in the WZ gauge). We then get for a mass term in the Lagrangian:

$$V^2 \big|_{\theta\theta\bar{\theta}\bar{\theta}} = -\frac{1}{2} A_\mu A^\mu - \chi\lambda - \bar{\chi}\bar{\lambda} - i\bar{\chi}\bar{\sigma}^\mu \partial_\mu \chi + \frac{1}{2} B \Box B + BD + \frac{1}{2} \bar{G} G \,. \tag{7.50}$$

Note that this term not only gives mass to the vector field, but also introduces the additional degrees of freedom, B and χ, that are required for a massive multiplet, $\{B, \chi, \bar{\chi}, \lambda, \bar{\lambda}, A_\mu\}$, with all component fields of an equal mass. It is easy to see that G is just an additional auxiliary field. We shall see in section 8.3.1, and in the problem sheet, when symmetry breaking in the ground states of gauge theories is considered, that the spontaneous breaking of U(1) gauge symmetry is in fact characterized by the vector field acquiring a mass.

Let us now turn our attention to matter fields in $\mathcal{N} = 1$ supersymmetric gauge theories. They sit in chiral multiplets that are charged with the generators of the gauge group, which constitute the conserved charges of the gauge symmetry. For U(1) symmetry, a chiral superfield Φ is transformed under U(1) rotations as follows:

$$\Phi' = \exp\left(i\Omega T\right) \Phi \,, \qquad \bar{D}_{\dot{\alpha}} \Omega = 0 \,, \tag{7.51}$$

where T is the real U(1) charge of Φ, and Ω constitutes the rotation parameter. Ω is also taken to be a chiral superfield, in order to assure that Φ remains chiral.

Let us check then what happens with the canonical kinetic term of chiral superfields in supersymmetric gauge theories, considering U(1) gauge transformations:

$$\bar{\Phi}'\Phi' = \bar{\Phi}\Phi \exp\left(iT\left(\Omega - \bar{\Omega}\right)\right) , \tag{7.52}$$

which is not gauge-invariant. Yet, an Abelian vector superfield transforms according to eq. (7.24), so we can take the kinetic term as follows:

$$\bar{\Phi}' \exp\left(-2TV'\right) \Phi' = \bar{\Phi} \exp\left(-2TV\right) \Phi , \tag{7.53}$$

which is clearly gauge-invariant. This is the so-called "minimal coupling" of the vector superfield to the chiral superfields, which is analogous to the replacement of ordinary derivatives by gauge-covariant derivatives on matter fields, for example:

$$\partial_\mu \phi \to D_\mu \phi = \partial_\mu \phi - iA_\mu \phi , \tag{7.54}$$

which is similar to eq. (7.16) in non-supersymmetric gauge theories.

At first, the term in eq. (7.53) may seem non-renormalizable, due to the high and infinite powers of V, coming from the power series of the exponential. Yet, it is easy to verify that in the WZ gauge:

$$(V_{\mathrm{WZ}})^n = 0 , \quad \forall n \geqslant 3 , \tag{7.55}$$

i.e. the powers of V vanish as of the third power. Thus, when evaluated in the WZ gauge, the minimal coupling of chiral superfields in the Lagrangian takes the form:

$$\begin{aligned}\mathcal{L}_{\mathrm{SMC}} &= \bar{\Phi} \exp\left(-2TV\right) \Phi\Big|_{\theta\theta\bar{\theta}\bar{\theta}} \\ &= -D^\mu \bar{\phi} D_\mu \phi - i\bar{\psi}\bar{\sigma}^\mu D_\mu \psi + \bar{F}F - T\bar{\phi}D\phi - i\sqrt{2}T\left(\bar{\phi}\lambda\psi - \phi\bar{\lambda}\bar{\psi}\right) ,\end{aligned} \tag{7.56}$$

wherein in addition to A_μ, the vector superfield components, λ and D, also couple to matter. It is thus easy to verify that this Lagrangian is in fact renormalizable.

Thus, the supersymmetric extension of QED is constructed with 2 chiral superfields:

$$\Phi'_+ = \exp\left(-ie\Omega\right) \Phi_+ , \qquad \Phi'_- = \exp\left(+ie\Omega\right) \Phi_- , \tag{7.57}$$

so that the supersymmetric QED Lagrangian reads:

$$\begin{aligned}\mathcal{L}_{\text{SQED}} = \frac{1}{4}\left(W^\alpha W_\alpha|_{\theta\theta} + \overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}\Big|_{\bar\theta\bar\theta}\right) \\ + \left[\bar\Phi_+ \exp\left(+2eV\right)\Phi_+ + \bar\Phi_- \exp\left(-2eV\right)\Phi_-\right]\Big|_{\theta\theta\bar\theta\bar\theta} \\ + m\left(\Phi_+\Phi_-|_{\theta\theta} + \bar\Phi_+\bar\Phi_-|_{\bar\theta\bar\theta}\right)\,,\end{aligned} \tag{7.58}$$

where the Weyl spinors, ψ_+, ψ_-, combine to form one massive Dirac spinor, the electron, and the scalars, ϕ_+, ϕ_-, constitute the electron's bosonic superpartner – the so-called selectron.

7.4 Non-Abelian Vector Superfields

It is straightforward to generalize the gauge transformations of vector and chiral superfields to a compact non-Abelian gauge group, with its Lie algebra $\mathfrak{g}$. As the generalization of vector superfields is motivated by their coupling to chiral superfields, we start with the latter by generalizing eq. (7.51). The chiral superfield multiplet, Φ_i, is transformed as follows:

$$\Phi_i' = [\exp\left(i\Omega\right)]_{ij}\Phi_j\,, \tag{7.59}$$

with an algebra-valued Ω_{ij}:

$$\Omega : \mathbb{R}^{1,3|4} \to i\mathfrak{g}\,, \qquad [\Omega]_{ij} \equiv \Omega_A\,[T_A]_{ij}\,, \tag{7.60}$$

where Ω_A are chiral superfields, and the generators are in the representation r of the chiral field Φ_i.

For the minimal coupling of the chiral superfields in eq. (7.53) to remain gauge-invariant for a non-Abelian gauge group, the gauge transformation of the non-Abelian vector superfield is extended from eq. (7.24) as follows:

$$\exp\left(-2V'\right) = \exp\left(i\bar\Omega\right)\exp\left(-2V\right)\exp\left(-i\Omega\right)\,, \tag{7.61}$$

with Ω and V valued in the adjoint representation of the algebra:

$$\Omega, V : \mathbb{R}^{1,3|4} \to i\mathfrak{g}\,, \qquad \Omega \equiv [\Omega]_{ij} \equiv \Omega_A\,[T_A]_{ij}\,, \quad V \equiv [V]_{ij} \equiv V_A\,[T_A]_{ij}\,, \tag{7.62}$$

where Ω_A, V_A, are chiral and real superfields, respectively, and the generators that are in the adjoint representation of dimension m:

$$i \in \{1, \ldots, m = \dim G\}\,. \tag{7.63}$$

In computing the product of exponentials in eq. (7.61), using the Baker-Campbell-Hausdorff formula, we encounter only commutators of group generators, evaluated via eq. (7.1), which allows to express the transformed V' in the same form as in eq. (7.62):

$$V' \equiv [V']_{ij} \equiv V'_A [T_A]_{ij} \,. \tag{7.64}$$

In the problem sheet we show that to linear order in the "group parameter" Ω, the gauge transformation of V in eq. (7.61) yields the shift:

$$\delta V = V' - V = \frac{i}{2}\left(\Omega - \bar{\Omega}\right) + \frac{i}{2}[\Omega + \bar{\Omega}, V] + \mathcal{O}(\Omega^2)\,, \tag{7.65}$$

which reduces to eq. (7.24) in the Abelian case, as required.

This gauge transformation still allows a WZ gauge, similar to the Abelian case, where eq. (7.55), which becomes a matrix equation in the non-Abelian case, still holds, and $V_{\rm WZ}$ is identical in form to eq. (7.36), only that now $V_{\rm WZ}$ is valued in the adjoint representation of the non-Abelian Lie algebra.

From eqs. (7.59), (7.61), it is also clear now that the generalization of the minimal coupling of the chiral superfield multiplet to general vector superfields reads:

$$\begin{aligned}\mathcal{L}_{\rm SMC} &= \bar{\Phi}\exp\left(-2V\right)\Phi\big|_{\theta\theta\bar{\theta}\bar{\theta}} \\ &= -D^\mu\bar{\phi}D_\mu\phi - i\bar{\psi}\bar{\sigma}^\mu D_\mu\psi + \bar{F}F - \bar{\phi}D\phi - i\sqrt{2}\left(\bar{\phi}\lambda\psi - \phi\bar{\lambda}\bar{\psi}\right)\,,\end{aligned} \tag{7.66}$$

wherein all the representation indices, both of the chiral superfield multiplets and of the vector superfield matrices, are suppressed, and similar to the Abelian case in eq. (7.56), D and λ of the vector superfield also couple to matter. For n chiral multiplets, n such minimal-coupling terms should be included for each of the chiral multiplets.

7.5 Supersymmetric Gauge Theories

We now look to generalize the Abelian field-strength superfield in eq. (7.43) to non-Abelian gauge groups. We then define the non-Abelian field-strength superfield as follows:

$$W_\alpha \equiv \frac{i}{8}\bar{D}\bar{D}\exp\left(2V\right)D_\alpha\exp\left(-2V\right)\,, \tag{7.67}$$

which properly reduces to the Abelian case, and satisfies chirality as in eq. (7.44), i.e. $\bar{D}_{\dot{\beta}}W_\alpha = 0$.

Under non-Abelian gauge transformations this field-strength superfield is transformed as follows:

$$\begin{aligned}
W'_\alpha &= \\
&= \frac{i}{8}\bar{D}\bar{D}\left[\exp(i\Omega)\exp(2V)\exp\left(-i\bar{\Omega}\right)D_\alpha\left[\exp\left(i\bar{\Omega}\right)\exp(-2V)\exp(-i\Omega)\right]\right] \\
&= \exp(i\Omega)\,W_\alpha\exp(-i\Omega) + \frac{i}{8}\exp(i\Omega)\,\bar{D}\{\bar{D},D_\alpha\}\exp(-i\Omega) \\
&= \exp(i\Omega)\,W_\alpha\exp(-i\Omega)\ ,
\end{aligned} \tag{7.68}$$

where we used the chirality and anti-chirality of Ω, $\bar{\Omega}$, respectively, and in the last equality also the identity in eq. (7.46). Yet, note that the chirality of W_α is maintained, thanks to the chirality of both exponents to the left and right.

The explicit form of W_α in chiral coordinates is identical to the Abelian case in eqs. (7.39)-(7.42), except that in the latter equation, which is of the top component, the ordinary derivative should be explicitly replaced by the gauge-covariant derivative:

$$F_\alpha = i\left(\sigma^\mu D_\mu\bar{\lambda}\right)_\alpha\ , \qquad D_\mu\bar{\lambda} = \partial_\mu\bar{\lambda} - i[A_\mu,\bar{\lambda}]\,, \tag{7.69}$$

since $\bar{\lambda}$ is in the adjoint representation, in which the gauge-covariant derivative from eq. (7.16) is equal to the ordinary derivative in the Abelian case (unlike matter fields), but has to be modified in the non-Abelian case.

Let us consider then the gauge transformation of the trace of the Lorentz scalar of the field-strength:

$$\mathrm{tr}\left(W'^\alpha W'_\alpha\right) = \mathrm{tr}\left(\exp(i\Omega)\,W^\alpha W_\alpha\exp(-i\Omega)\right) = \mathrm{tr}\left(W^\alpha W_\alpha\right)\ , \tag{7.70}$$

where the last equality is due to the trace cyclicity. Thus, this trace is gauge-invariant.

We are now ready to write the super Yang-Mills (SYM) Lagrangian, that holds in particular for a simple gauge group SU(N), by taking the above trace as the gauge-invariant F-term Lagrangian:

$$\mathcal{L}_{\text{SYM}} = -\frac{\tau}{16\pi i}\,\mathrm{tr}\left(W^\alpha W_\alpha\right)\Big|_{\theta\theta} + \frac{\bar{\tau}}{16\pi i}\,\mathrm{tr}\left(\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}\right)\Big|_{\bar\theta\bar\theta}\ , \tag{7.71}$$

where we introduced a complex coupling constant, τ, defined as follows:

$$\tau \equiv \frac{4\pi i}{g^2} + \frac{\Theta}{2\pi}\,, \tag{7.72}$$

which enters the Lagrangian holomorphically, namely it is a holomorphic gauge coupling, with g^2 the real YM coupling, and Θ the so-called theta angle. After direct computation and some integration by parts at the level of the action, we get for the SYM Lagrangian:

$$\mathcal{L}_{\text{SYM}} = \frac{1}{g^2}\,\text{tr}\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - i\bar{\lambda}\bar{\sigma}^{\mu}D_{\mu}\lambda + \frac{1}{2}D^2\right) + \frac{\Theta}{64\pi^2}\,\text{tr}\left(\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right)\,, \tag{7.73}$$

where the last term is the topological Chern-Pontryagin density, which is a total spacetime derivative that can have a non-vanishing integral, due to the existence of instanton solutions in non-Abelian gauge theories. For each gauge group, there is such a theory with an independent gauge coupling, as in eq. (7.72).

It is easy to verify that the SYM Lagrangian preserves $U(1)_R$ R-symmetry, where the superfield W_α, and similarly its bottom component, λ, are assigned the R-charge 1.

Finally, we are ready to write down the most general Lagrangian for a renormalizable supersymmetric gauge theory, with a simple compact Lie group, consisting of scalar, spinor and vector fields:

$$\begin{aligned}\mathcal{L} = &-\frac{\tau}{16\pi i}\,\text{tr}\left(W^\alpha W_\alpha\right)\big|_{\theta\theta} + \frac{\bar{\tau}}{16\pi i}\,\text{tr}\left(\overline{W}_{\dot{\alpha}}\overline{W}^{\dot{\alpha}}\right)\Big|_{\bar{\theta}\bar{\theta}} + \bar{\Phi}\exp\left(-2V\right)\Phi\big|_{\theta\theta\bar{\theta}\bar{\theta}} \\ &+\left[\left(\frac{m_{ab}}{2}\,\Phi_a\Phi_b + \frac{g_{abc}}{3}\,\Phi_a\Phi_b\Phi_c\right)\Big|_{\theta\theta} + \text{H.C.}\right]\,,\end{aligned} \tag{7.74}$$

where H.C. stands for the anti-holomorphic superpotential of the anti-chiral superfields. Note that here the superpotential, with the interactions among chiral superfields and their coupling constants, are further subject to the constraints of gauge symmetry.

8. Spontaneous Symmetry Breaking

When a symmetry is preserved at the level of the action, but the ground state is not invariant under the action of the corresponding charges, we say that spontaneous symmetry breaking (SSB) takes place. As noted already in the opening chapter 1, there is no experimental evidence for supersymmetry as yet. None of the supersymmetric partners to the particles, which comprise the Standard Model, has ever been observed. Thus, it is believed that supersymmetry holds at very high energies, whereas in our presently-accessible real-world low energies, supersymmetric theories are in their ground state, and supersymmetry is spontaneously broken. This final chapter is thus clearly a critical step in connecting supersymmetry to the real world.

We normally first learn about SSB in non-supersymmetric QFTs with gauge symmetry. Here we add supersymmetry to various theories, first without gauge symmetry, and then also with gauge symmetry. Thus, in the following we first study about ground states of supersymmetric theories in section 8.1, and analyse these vacua in various supersymmetric models. Then, we consider the simple case of SSB in supersymmetric theories without gauge symmetry, namely in chiral models, in section 8.2, in order to understand spontaneous breaking of supersymmetry by itself. Finally, we proceed to consider SSB in supersymmetric gauge theories in section 8.3, where there can be SSB of the gauge symmetry, or of supersymmetry, or of both symmetries at the same time.

8.1 Supersymmetric Vacuum

From the $\mathcal{N} = 1$ supersymmetry algebra, we have for any state $|\Psi\rangle$:

$$\langle\Psi|2\sigma^{\mu}_{\alpha\dot{\beta}}P_{\mu}|\Psi\rangle = \langle\Psi|Q_{\alpha}\bar{Q}_{\dot{\beta}} + \bar{Q}_{\dot{\beta}}Q_{\alpha}|\Psi\rangle\,. \tag{8.1}$$

Taking the trace on spinorial indices, we get:

$$\mathrm{tr}\left(2\sigma^{\mu}_{\alpha\dot{\beta}}P_{\mu}\right) = \sum_{\alpha=1}^{2}\left(|Q^{\dagger}_{\alpha}|\Psi\rangle|^2 + |Q_{\alpha}|\Psi\rangle|^2\right) = -4\langle\Psi|P_0|\Psi\rangle = 4E \geqslant 0\,, \quad (8.2)$$

so the energy of any state is non-negative. We can infer then that *zero-energy states, $|0\rangle$, are supersymmetric ground states of the theory.*

First, zero-energy states are ground states since their vanishing expectation value of H, is the smallest one possible:

$$\langle 0|H|0\rangle = \langle 0|P^0|0\rangle = 0\,. \quad (8.3)$$

Second, zero-energy states are supersymmetric since eq. (8.2) yields:

$$\langle\Psi|H|\Psi\rangle = 0 \iff Q_{\alpha}|\Psi\rangle = \bar{Q}_{\dot{\alpha}}|\Psi\rangle = 0\,, \quad (8.4)$$

thus a state is left invariant under the supersymmetry group, if and only if it is a zero-energy state. Therefore, whereas ground states of zero energy preserve supersymmetry, those of positive energy, spontaneously break supersymmetry. This actually holds in any global supersymmetric theory in any spacetime dimension.

8.1.1 Supersymmetric Vacuum and the Superpotential

For $\mathcal{N} = 1$ supersymmetric theory of n chiral superfields, we obtained an interaction potential, which contains a scalar potential, $V_0(\phi^a, \bar{\phi}^a)$, in eq. (5.35). For zero-energy ground states the scalar potential must vanish, and as the latter is a sum of squares, a supersymmetric vacuum exists, if and only if:

$$\partial_a W = F^a = 0\,, \quad \forall a\,, \quad (8.5)$$

where we recall that these derivatives are evaluated on the bottom components ϕ^a. This system of n equations is therefore called the supersymmetric vacuum equations. The solutions to these equations determine the possible vacuum expectation values (VEVs) for the scalar fields ϕ^a. A supersymmetric vacuum is then a configuration of constant VEVs:

$$\phi^a = \langle\phi^a\rangle = \langle 0|\phi^a|0\rangle\,, \quad (8.6)$$

which solve eqs. (8.5), and thus yield $V_0 = 0$.

The supersymmetric vacuum equations thus constitute n equations for n unknowns. Depending on the superpotential W, there are 3 possibilities:

1. There are no solutions. In this case supersymmetry is spontaneously broken.

2. There is a finite number of solutions, k, corresponding to k discrete supersymmetric vacua, local minima of the scalar potential with $V_0 = 0$.

3. There is a continuum of solutions, which is called a supersymmetric vacuum moduli space.

In the problem sheet we consider several chiral models that demonstrate each of these 3 possibilities.

In the case of a vacuum moduli space, if the solutions of the vacuum equations are not related by a symmetry of the action, then the physics around each local minimum is different. In an ordinary QFT any such "flat directions" in the classical potential would generally be lifted by the quantum corrections in renormalization. In supersymmetric theories however, due to the non-renormalization theorem that we proved in section 6.2 for chiral models, supersymmetry actually preserves the moduli space to all orders in perturbation theory.

Let us consider an illustrative example for such a moduli space, via a model of 3 chiral superfields with the superpotential:

$$W = \frac{m}{2}\Phi_3^2 + \lambda\Phi_1\Phi_2\Phi_3 \,. \tag{8.7}$$

The supersymmetric vacuum equations in eq. (8.5) then yield:

$$\begin{cases} \partial_1 W &= \lambda\phi_2\phi_3 &= 0 \\ \partial_2 W &= \lambda\phi_1\phi_3 &= 0 \\ \partial_3 W &= m\phi_3 + \lambda\phi_1\phi_2 &= 0 \end{cases} \quad \Longrightarrow \quad \begin{cases} \phi_1 = \phi_3 &= 0 \\ \quad \text{or} & \\ \phi_2 = \phi_3 &= 0 \end{cases} , \tag{8.8}$$

so the solution is a union of 2 complex planes, where the scalar potential vanishes:

$$\{\phi_1 = \phi_3 = 0, \quad \forall\phi_2\} \cup \{\phi_2 = \phi_3 = 0, \quad \forall\phi_1\} \,, \tag{8.9}$$

which is the vacuum moduli space. The corresponding scalar potential reads:

$$V_0 = |\partial_a W|^2 = |\lambda|^2|\phi_2\phi_3|^2 + |\lambda|^2|\phi_1\phi_3|^2 + |\lambda\phi_1\phi_2 + m\phi_3|^2 \,. \tag{8.10}$$

Let us take the plane $\{\phi_2 = \phi_3 = 0, \quad \forall\phi_1\}$, and expand the scalar potential around it. Since on the vacuum plane the potential is at its minimum:

$$\partial_a V_0 = \partial_{\bar{a}} V_0 = 0 \,, \quad \forall a \,, \tag{8.11}$$

then we need to go beyond first order, and expand the potential by going to second order in all fields, which yields:

$$\begin{aligned}\Delta V_0 &= |\lambda|^2|\Delta\phi_2\Delta\phi_3|^2 + |\lambda|^2|(\phi_1+\Delta\phi_1)\Delta\phi_3|^2 + |\lambda(\phi_1+\Delta\phi_1)\Delta\phi_2 + m\Delta\phi_3|^2\\ &= |\lambda|^2|\phi_1\Delta\phi_3|^2 + |\lambda|^2|\phi_1\Delta\phi_2|^2 + |m|^2|\Delta\phi_3|^2\\ &\quad + \lambda\bar{m}\phi_1\Delta\phi_2\overline{\Delta\phi}_3 + \bar{\lambda}m\bar{\phi}_1\overline{\Delta\phi}_2\Delta\phi_3\,, \qquad (8.12)\end{aligned}$$

so we recover the massive fields $\Delta\phi_2$, $\Delta\phi_3$, whereas $\Delta\phi_1$ is massless, since a change in ϕ_1 remains in the vacuum plane. The masses of $\Delta\phi_2$, $\Delta\phi_3$, depend on $|\phi_1|^2$, which shows that this vacuum moduli are not physically equivalent – in each point ϕ_1 there is different physics (different spectrum).

Why is there a flat direction? The superpotential W has a symmetry, but it is not a symmetry of the Kähler potential, and thus it is not a symmetry of the whole action. W depends on ϕ_1 and ϕ_2 together, rather than separately. Consider then $\phi_1 \to c\phi_1$, $\phi_2 \to \phi_2/c$, $\phi_3 \to \phi_3$. If $c \neq 1$, then $|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2$ is not invariant. So, on the vacuum moduli all points have the same physics, since there is a symmetry there on the whole action, yet around the moduli the physics is different near different points.

8.2 SSB in Chiral Models

Let us consider the case, where there is spontaneous symmetry breaking (SSB) in the simplest supersymmetric theories, namely in chiral models. This is when there is no solution to the supersymmetric vacuum equations in eq. (8.5). If the number of unknowns is actually less than the number of equations, it can be that there is no solution. When R-symmetry holds, then this is the case. Thus R-symmetry entails SSB.

Possibly the simplest example of such a theory, where supersymmetry is spontaneously broken, is the O'Raifeartaigh model (1975) [14] with 3 chiral superfields and the superpotential:

$$W_{\mathrm{O'R}} = \alpha\Phi_1 + \beta\Phi_2\Phi_3 + \gamma\Phi_1\Phi_2^2\,. \qquad (8.13)$$

The supersymmetric vacuum equations yield:

$$\begin{cases}\partial_1 W &= \alpha + \gamma\phi_2^2 &= 0\\ \partial_2 W &= \beta\phi_3 + 2\gamma\phi_1\phi_2 &= 0\\ \partial_3 W &= \beta\phi_2 &= 0\end{cases}\,, \qquad (8.14)$$

and it is easy to see that the first and third equations clash, so that there is no solution. The corresponding scalar potential reads:

$$V_0 = |\alpha + \gamma\phi_2^2|^2 + |\beta\phi_3 + 2\gamma\phi_1\phi_2|^2 + |\beta\phi_2|^2\,. \qquad (8.15)$$

The second term can be made to vanish, but those with only ϕ_2 cannot. Since the minimum of $V_0 \neq 0$, then supersymmetry is spontaneously broken. Classically, there is the pseudo vacuum moduli:

$$\{\phi_2 = \phi_3 = 0, \quad \forall \phi_1\}, \tag{8.16}$$

so any VEV of ϕ_1 is allowed, but since supersymmetry is broken, this "degeneracy" is lifted at one-loop quantum corrections.

Note that such models of supersymmetry breaking always seem very fine-tuned. Let us recall R-symmetry, its transformation of Grassmannian integration measure in eq. (4.47), and the definition of R-charges in eq. (4.48): If the chiral superfields Φ_1, Φ_2, Φ_3, are assigned the R-charges $q_1 = 2$, $q_2 = 0$, $q_3 = 2$, respectively, then it is easy to see that the O'Raifeartaigh model is R-symmetric. Yet, if we only add, e.g. the mass term $m\Phi_3^2$, to the O'Raifeartaigh model in eq. (8.13), then it is easy to verify that supersymmetric vacuum can be restored, whereas R-symmetry no longer holds.

8.2.1 Goldstino and the Mass Sum Rule

Let us now consider the Lagrangian of a general chiral model in eq. (5.34), and extract only the contribution with fermions:

$$\mathcal{L}_F = -i\bar{\psi}^a \bar{\sigma}^\mu \partial_\mu \psi^a - \frac{1}{2}\partial_a \partial_b W \psi^a \psi^b - \frac{1}{2}\partial_{\bar{a}} \partial_{\bar{b}} \overline{W} \bar{\psi}^a \bar{\psi}^b, \tag{8.17}$$

where the interaction terms can be recast using a matrix as follows:

$$\mathcal{L}_F \supset -\frac{1}{2}\left(\bar{\psi}^a, \psi^a\right) \begin{pmatrix} 0 & \partial_{\bar{a}}\partial_{\bar{b}}\overline{W} \\ \partial_a \partial_b W & 0 \end{pmatrix} \begin{pmatrix} \psi^b \\ \bar{\psi}^b \end{pmatrix}, \tag{8.18}$$

which defines the mass matrix of the fermions:

$$[m_F]_{ab} \equiv \begin{pmatrix} 0 & \partial_{\bar{a}}\partial_{\bar{b}}\overline{W} \\ \partial_a \partial_b W & 0 \end{pmatrix}. \tag{8.19}$$

To find the mass spectrum of the fermions, we thus need to diagonalize the mass matrix.

Let us assume that supersymmetry is spontaneously broken:

$$\exists a, \quad \partial_a W = -\bar{F}^a \neq 0 \quad \Longleftrightarrow \quad \exists a, \quad \partial_{\bar{a}} \overline{W} = -F^a \neq 0. \tag{8.20}$$

Since the potential is at its minimum in the ground state, as in eq. (8.11) we can write:

$$\partial_b V_0 = \partial_b \left(\partial_a W \partial_{\bar{a}} \overline{W}\right) = \partial_b \partial_a W \partial_{\bar{a}} \overline{W} = 0, \tag{8.21}$$

where in the second equality we used the fact that W, $\overline{W}$, are holomorphic and anti-holomorphic, respectively. Similarly, we also have:

$$\partial_{\bar{b}} V_0 = \partial_{\bar{b}} \left(\partial_a W \partial_{\bar{a}} \overline{W} \right) = \partial_a W \partial_{\bar{b}} \partial_{\bar{a}} \overline{W} = 0 \,. \tag{8.22}$$

Eqs. (8.21) and (8.22) can be put together into the matrix equation:

$$\begin{pmatrix} 0 & \partial_{\bar{a}} \partial_{\bar{b}} \overline{W} \\ \partial_a \partial_b W & 0 \end{pmatrix} \begin{pmatrix} \partial_{\bar{a}} \overline{W} \\ \partial_a W \end{pmatrix} = 0 \,, \tag{8.23}$$

or using eqs. (8.19), (8.20):

$$[m_F]_{ab} \begin{pmatrix} F^a \\ \bar{F}^a \end{pmatrix} = 0 \,. \tag{8.24}$$

Recall that SSB is generally characterized by the appearance of a so-called Goldstone particle, which is massless. In non-supersymmetric QFTs we encounter Goldstone bosons (specifically scalars), corresponding to the SSB of some gauge symmetry, namely of some bosonic (scalar) symmetry. Here we find a Goldstone fermion:

$$\chi \equiv \begin{pmatrix} F^a \psi^a \\ \bar{F}^a \bar{\psi}^a \end{pmatrix} \,, \tag{8.25}$$

as χ is an eigenvector of the fermion mass-matrix with an eigenvalue 0. The Goldstone fermion, called the goldstino, which characterizes the SSB of supersymmetry, is a spinor spanned by the chiral spinor fields with scalar coefficients F^a, as the VEV of F^a does not vanish, $\langle F^a \rangle \neq 0$, see eq. (8.20). Thus, here we see a Goldstone fermion of spin $1/2$, corresponding to the SSB of supersymmetry, namely of fermionic spin $1/2$ symmetries.

Note that the square of the fermion mass matrix in eq. (8.19) reads:

$$[m_F^2]_{ab} \equiv \begin{pmatrix} \partial_{\bar{a}} \partial_{\bar{c}} \overline{W} \, \partial_c \partial_b W & 0 \\ 0 & \partial_a \partial_c W \, \partial_{\bar{c}} \partial_{\bar{b}} \overline{W} \end{pmatrix} \,. \tag{8.26}$$

Let us then consider in comparison the square-mass matrix of the bosons in a general chiral model. The latter can be identified from the Lagrangian in eq. (5.34), by expanding the scalar potential to second order, similar to our arguments around eq. (8.12), so that its contribution to the Lagrangian is written as follows:

$$\mathcal{L}_B \supset -\frac{1}{2} \left(\bar{\phi}^a, \phi^a \right) \begin{pmatrix} \partial_{\bar{a}} \partial_b V_0 & \partial_{\bar{a}} \partial_{\bar{b}} V_0 \\ \partial_a \partial_b V_0 & \partial_a \partial_{\bar{b}} V_0 \end{pmatrix} \begin{pmatrix} \phi^b \\ \bar{\phi}^b \end{pmatrix} \,, \tag{8.27}$$

with the square-mass matrix of the bosons defined as follows:

$$[m_B^2]_{ab} \equiv \begin{pmatrix} \partial_{\bar{a}}\partial_b V_0 & \partial_{\bar{a}}\partial_{\bar{b}} V_0 \\ \partial_a \partial_b V_0 & \partial_a \partial_{\bar{b}} V_0 \end{pmatrix} . \tag{8.28}$$

Using eq. (5.35), the boson square-mass matrix is then rewritten as follows:

$$\begin{aligned} [m_B^2]_{ab} &= \begin{pmatrix} \partial_{\bar{a}}\partial_b \left(\partial_c W \partial_{\bar{c}} \overline{W}\right) & \partial_{\bar{a}}\partial_{\bar{b}} \left(\partial_c W \partial_{\bar{c}} \overline{W}\right) \\ \partial_a \partial_b \left(\partial_c W \partial_{\bar{c}} \overline{W}\right) & \partial_a \partial_{\bar{b}} \left(\partial_c W \partial_{\bar{c}} \overline{W}\right) \end{pmatrix} \\ &= \begin{pmatrix} \boxed{\partial_b \partial_c W \, \partial_{\bar{a}} \partial_{\bar{c}} \overline{W}} & \partial_c W \, \partial_{\bar{a}} \partial_{\bar{b}} \partial_{\bar{c}} \overline{W} \\ \partial_a \partial_b \partial_c W \, \partial_{\bar{c}} \overline{W} & \boxed{\partial_a \partial_c W \, \partial_{\bar{b}} \partial_{\bar{c}} \overline{W}} \end{pmatrix} , \end{aligned} \tag{8.29}$$

where in the second equality we applied the derivatives on W or $\overline{W}$, according to their holomorphic or anti-holomorphic nature, respectively. Notice that the diagonal blocks of the square of the fermion mass-matrix in eq. (8.26), and of the boson square-mass matrix in eq. (8.29), are identical!

Therefore, if supersymmetry is not broken, then it is easy to see from the supersymmetric vacuum equations in eq. (8.5), substituted into eqs. (8.26), (8.29), that the mass spectrum of bosons and fermions in the ground state is identical. This is similar to what we have already seen for non-zero energy states, when we discussed supermultiplets in section 3.3, in particular from eqs. (3.75) and (3.82). On the other hand, if supersymmetry is broken, then in the ground state the mass spectrum of bosons is different than that of fermions, due to the non-vanishing off-diagonal blocks in eq. (8.29).

Yet, whether supersymmetry is broken or not, the invariant trace of the square-mass matrices of bosons and of fermions is equal:

$$\mathrm{tr}\left(m_B^2\right) = \mathrm{tr}\left(m_F^2\right) , \tag{8.30}$$

which amounts to the following:

$$\sum_{\text{bosons}} m_B^2 = \sum_{\text{fermions}} m_F^2 \, , \tag{8.31}$$

where the sum is over all bosonic or fermionic vacuum states, respectively. This is the supersymmetric mass sum rule, which says that even if supersymmetry gets spontaneously broken, the average square masses of bosons is still equal to that of fermions. This is a remainder of the supersymmetry that got broken. Such mass sum rules are common in general supersymmetric theories.

8.3 SSB in Supersymmetric Gauge Theories

Once there is gauge symmetry as well as supersymmetry, there are in general 3 possibilities with SSB:

1. Only gauge symmetry gets broken, but supersymmetry is not broken.
2. Only supersymmetry gets broken, but gauge symmetry is not broken.
3. Both gauge symmetry and supersymmetry get broken.

An example of the first possibility is treated in the problem sheet as the supersymmetric Higgs mechanism in SQED. The 2 latter possibilities are discussed in what follows.

Let us then extend our analysis of the supersymmetric vacuum and SSB to supersymmetric gauge theories, which contain both vector and chiral superfields. First, we should extract the extended scalar potential, which now has 2 additional contributions, beyond that from the superpotential of chiral superfields in eq. (5.35):

$$V_0(\phi_i, \bar{\phi}_i) \subset -\left(\mathcal{L}_{\rm SYM} + \mathcal{L}_{\rm SMC} + \mathcal{L}_{W+\overline{W}}\right) , \tag{8.32}$$

that is the addition to the scalar potential arises from the SYM Lagrangian in eq. (7.73), and from the minimal coupling Lagrangian in eq. (7.66). Thus, all in all, the scalar potential now reads:

$$V_0(\phi_i, \bar{\phi}_i) \equiv |\partial_i W|^2 + \bar{\phi} D \phi - \frac{1}{2g^2} \operatorname{tr}\left(D^2\right) , \tag{8.33}$$

where here the indices i sit in some representation r:

$$i \in \{1, \ldots, \dim r\} , \tag{8.34}$$

and D is the auxiliary field valued in the adjoint representation:

$$D \equiv D_A [T_A]_{ij} , \qquad i, j \in \{1, \ldots, \dim G\} . \tag{8.35}$$

We can solve for D via its Euler-Lagrange equations from the D-dependent terms in eqs. (7.73), (7.66):

$$\mathcal{L}_{\rm SYM} + \mathcal{L}_{\rm SMC} \supset \frac{c_{(r)}}{2g^2} D_A^2 - D_A \, \bar{\phi} \, T_A \phi , \tag{8.36}$$

where we also used the orthogonality condition of the generators in eq. (7.2). The solution of Euler-Lagrange equations yields:

$$D_A = \frac{g^2}{c_{(r)}} \, \bar{\phi} \, T_A \phi . \tag{8.37}$$

Substituting this back in eq. (8.33), we get for the scalar potential:

$$V_0 = |\partial_i W|^2 + \frac{g^2}{2c_{(r)}} \sum_A \left(\bar{\phi}\, T_A \phi\right)^2 \, . \tag{8.38}$$

Thus, in the presence of vector superfields the scalar potential is still a sum of squares, so the conditions for a supersymmetric vacuum read:

$$\partial_i W = 0 \, , \quad \forall i \qquad \text{and} \qquad \bar{\phi}\, T_A \phi = 0 \, , \quad \forall A \, . \tag{8.39}$$

Thus, these are now the supersymmetric vacuum equations, so that to the eqs. (8.5) from the superpotential, there is an additional requirement for the real operators $\tilde{D}_A$ to vanish:

$$\tilde{D}_A \equiv \bar{\phi}\, T_A \phi = 0 \, , \tag{8.40}$$

due to the minimal coupling of gauge fields to matter fields. Yet, it is easy to see that any two solutions of ϕ_i to the supersymmetric vacuum eqs. (8.39), which are related by constant gauge transformations, are physically equivalent:

$$\phi_i' \cong \phi_i \, , \quad \text{if} \quad \exists \omega_A \in \mathbb{C}^{\dim G} \, , \quad \phi_i' = \exp(i\omega_A T_A)\phi_i \, , \tag{8.41}$$

where ω_A constitute complex groups parameters.

In fact, the supersymmetric vacuum manifold of a supersymmetric gauge theory takes the general form:

$$\mathcal{M}_{\text{vac}} = \{\phi_i \in \mathbb{C}^n | \partial_i W = 0, \forall i\}/G \, , \tag{8.42}$$

where the quotient by the gauge group corresponds to the equivalence relation in eq. (8.41): It was proved that if the supersymmetric vacuum equations in eq. (8.5) from the superpotential have a solution, then thanks to gauge symmetry, there is always an equivalent solution that also satisfies the additional eqs. (8.40). Thus, the SSB of supersymmetric gauge theories seems to also be determined solely by the VEVs of the chiral auxiliary fields, F_i. This general statement is in fact entirely true for non-Abelian gauge theories, whereas for Abelian theories, as we shall discuss shortly in section 8.3.1, there exists a unique exception to it.

Consider now that supersymmetry is spontaneously broken, so that the vacuum has non-zero energy. Let us denote the VEVs associated with the auxiliary fields as follows:

$$f_i \equiv \partial_{\bar{i}} \overline{W} \, , \qquad d_A \equiv \bar{\phi}\, T_A \phi \, , \tag{8.43}$$

with the VEVs for the scalars, $\phi_i = \langle \phi_i \rangle$. Supersymmetry is broken, if some f_i or d_A are non-vanishing. Consider then any classical vacuum, where the scalar potential is minimized, so that applying eq. (8.11) on the extended scalar potential in eq. (8.38) yields the equations:

$$\partial_i \partial_j W f_j + \frac{g^2}{c(r)} \bar{\phi}_j [T_A]_{ji} d_A = 0 \,, \quad \forall i \,, \tag{8.44}$$

and their Hermitian conjugate. In addition, from the gauge invariance of the superpotential, which amounts to the constraint:

$$\frac{\delta W}{\delta \Omega} = \frac{\delta \overline{W}}{\delta \bar{\Omega}} = 0 \,, \tag{8.45}$$

with the gauge transformations of the chiral superfield multiplet in eq. (7.59), we also get:

$$\bar{f}_j [T_A]_{ji} \phi_i = 0 \,, \quad \forall A \,, \quad \Longleftrightarrow \quad \bar{\phi}_j [T_A]_{ji} f_i = 0 \,, \quad \forall A \,. \tag{8.46}$$

Now, let us extract the fermion mass terms of a general supersymmetric gauge theory, in particular from eqs. (5.36) and (7.66), which yields:

$$\mathcal{L}_{m_F} = \left(-\frac{1}{2} \partial_i \partial_j W \psi_i \psi_j - i\sqrt{2} \bar{\phi}_i \lambda_A [T_A]_{ij} \psi_j \right) + \text{H.C.} \quad , \tag{8.47}$$

that with the fermion mass-matrix defined as follows:

$$m_F \equiv \begin{pmatrix} \partial_i \partial_j W & i\sqrt{2} \bar{\phi}_k [T_B]_{ki} \\ i\sqrt{2} \bar{\phi}_k [T_A]_{kj} & 0 \end{pmatrix} \,, \tag{8.48}$$

can also be written as follows:

$$\mathcal{L}_{m_F} = -\frac{1}{2} \left(\psi_i, \lambda_A \right) m_F \begin{pmatrix} \psi_j \\ \lambda_B \end{pmatrix} + \text{H.C.} \quad . \tag{8.49}$$

Thus, eqs. (8.44), (8.46), can also be written as follows:

$$m_F \begin{pmatrix} f_j \\ \tilde{d}_B \end{pmatrix} = 0 \,, \qquad \tilde{d}_B \equiv -i \frac{g^2}{\sqrt{2} c_{(r)}} d_B \,, \tag{8.50}$$

so that any non-supersymmetric vacuum satisfies the following:

$$\begin{pmatrix} f_j \\ \tilde{d}_B \end{pmatrix} \neq 0 \,, \qquad m_F \begin{pmatrix} f_j \\ \tilde{d}_B \end{pmatrix} = 0 \,, \tag{8.51}$$

that is the fermion mass matrix has at least one eigenvector of eigenvalue 0. Therefore, we see that similar to SSB in chiral models, in particular as in eqs. (8.24), (8.25), the SSB of supersymmetry in general supersymmetric gauge theories also gives rise to a Goldstone fermion of spin 1/2:

$$\chi = f_i\psi_i + \tilde{d}_A\lambda_A \,, \tag{8.52}$$

that is the goldstino. This is a massless spinor, which corresponds to the broken fermionic supersymmetry. It is spanned by the chiral spinor fields and the gaugino fields, with the VEVs of the corresponding scalar auxiliary fields, F_i and D_A, respectively, fixing their coefficients.

8.3.1 SSB in Supersymmetric Abelian Gauge Theories

As noted after eq. (8.42), if the supersymmetric vacuum eqs. (8.5) from the superpotential have a solution, then the additional eqs. (8.40) for supersymmetric vacuum with gauge symmetry, can always be solved as well. Therefore it seems that SSB cannot occur, if eqs. (8.5) are satisfied, but eqs. (8.40) are not, namely if the VEVs of the chiral auxiliary fields, $\langle F_a\rangle$, vanish, but the VEVs of the vector auxiliary fields, $\langle D_A\rangle$, do not. Yet, Abelian gauge groups present another route for SSB to occur.

For an Abelian gauge group we can add to the Lagrangian the following term, which was introduced by Fayet and Iliopoulos (FI) (1974) [15]:

$$\mathcal{L}_{\rm FI} = 2\int d^2\theta d^2\bar{\theta}\,\kappa V = \kappa D \,, \tag{8.53}$$

where V in an Abelian vector superfield, and κ is some real coupling constant. This term clearly preserves supersymmetry being a D-term Lagrangian, and it is also gauge-invariant:

$$\begin{aligned}\int d^4x\, V'|_{\theta\theta\bar{\theta}\bar{\theta}} &= \int d^4x \left[V + \frac{i}{2}\left(\Phi - \bar{\Phi}\right)\right]\Bigg|_{\theta\theta\bar{\theta}\bar{\theta}} \\ &= \frac{1}{2}\int d^4x \left[D + \frac{i}{4}\partial^2\left(\phi - \bar{\phi}\right)\right] = \frac{1}{2}\int d^4x D = \int d^4x\, V|_{\theta\theta\bar{\theta}\bar{\theta}} \,,\end{aligned} \tag{8.54}$$

where in the third equality the total spacetime derivative term can be dropped via integration by parts.

For non-Abelian gauge groups V is a matrix, and such a term is not gauge-invariant, so a possible generalization to consider would be a trace. Yet, we get:

$$\int d^4x\, [\mathrm{tr}\,(V)]|_{\theta\theta\bar{\theta}\bar{\theta}} \sim \int d^4x\, \mathrm{tr}\,(D) = \int d^4x[D_A\, \mathrm{tr}\,(T_A)] = 0 \,, \tag{8.55}$$

where the last equality is due to the tracelessness of SU(N) generators. Thus a FI term is only relevant to a U(1) gauge group.

Let us then add the FI term to the SQED Lagrangian in eq. (7.58):

$$\begin{aligned}\mathcal{L}_{\text{SQED+FI}} = \frac{1}{4}\left(W^\alpha W_\alpha|_{\theta\theta} + \overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}\Big|_{\bar\theta\bar\theta}\right)\\ + \left[\bar\Phi_+ \exp\left(+2eV\right)\Phi_+ + \bar\Phi_- \exp\left(-2eV\right)\Phi_-\right]\Big|_{\theta\theta\bar\theta\bar\theta}\\ + m\left(\Phi_+\Phi_-|_{\theta\theta} + \bar\Phi_+\bar\Phi_-|_{\bar\theta\bar\theta}\right) + 2\kappa\, V|_{\theta\theta\bar\theta\bar\theta}\ .\end{aligned} \tag{8.56}$$

We can obtain the explicit scalar potential of this model by solving for the auxiliary fields F_+, F_-, and D, which yields:

$$\begin{aligned}V_0 &= \bar F_+ F_+ + \bar F_- F_- + \frac{1}{2}D^2\\ &= \frac{e^2}{8}\left(\bar\phi_+\phi_+ - \bar\phi_-\phi_-\right)^2 + \left(m^2 + \frac{1}{2}e\kappa\right)\bar\phi_+\phi_+ + \left(m^2 - \frac{1}{2}e\kappa\right)\bar\phi_-\phi_-\\ &\quad + \frac{1}{2}\kappa^2\,,\end{aligned} \tag{8.57}$$

where in the first equality we used eqs. (8.38), (8.37), and eq. (7.12) to switch from a non-Abelian to the Abelian case. There is no value of the scalar fields that makes $V_0 = 0$, so supersymmetry is spontaneously broken in this model.

We can then distinguish between 2 cases of SSB in this model, according to the relations among the coupling constants:

1. $\frac{1}{2}e\kappa \leqslant m^2$. In this case only supersymmetry is broken. D acquires a non-zero VEV, $\langle D\rangle = -\kappa$, and the photino λ is also the Goldstone fermion (or goldstino). The minimum of V_0 at the ground state is $V_0(\phi_+ = \phi_- = 0) = \frac{1}{2}\kappa^2$.
2. $\frac{1}{2}e\kappa > m^2$. In this case both supersymmetry and the U(1) gauge symmetry are spontaneously broken. The vector gauge field A_μ also acquires a mass, as the Goldstone boson, which depends on the (dynamical) chiral scalar with a non-zero VEV that arises, is being “eaten”.

One can analyse the second case in detail, using the definition:

$$C \equiv \sqrt{\frac{4}{e^2}\left(\frac{1}{2}e\kappa - m^2\right)}\,, \tag{8.58}$$

to find that the minimum of V_0 at the ground state reads:

$$V_0(\phi_+ = 0, \phi_- = C) = \frac{2m^2}{e^2}\left(e\kappa - m^2\right)\,, \tag{8.59}$$

and the model comprises:

- 2 spinors of mass $\sqrt{m^2 + \frac{1}{2}e^2C^2}$,
- 1 vector and 1 real scalar of mass $\sqrt{\frac{1}{2}e^2C^2}$,
- 1 complex scalar of mass $\sqrt{2m^2}$,
- 1 massless Goldstone spinor.

Note that the sum of squared masses, weighed by the number of DOFs, is identical for the bosonic and fermionic modes, on the LHS and RHS, respectively:

$$2 \times 2m^2 + 4 \times \tfrac{1}{2}e^2C^2 = 4\left(m^2 + \tfrac{1}{2}e^2C^2\right) . \tag{8.60}$$

This equality relationship also holds in the first U(1) gauge-symmetric case, and it just provides a more elaborate example for the supersymmetric mass sum rule from eq. (8.31), that we proved in chiral models.

To recap, as a generic rule, non-zero VEVs of auxiliary fields induce SSB of supersymmetry, while non-zero VEVs of dynamical scalar fields lead to the breaking of gauge symmetry.

Bibliography

[1] S. Weinberg, *The quantum theory of fields. Vol. 3: Supersymmetry*. Cambridge University Press, 6, 2013.

[2] J. Wess and J. Bagger, *Supersymmetry and supergravity*. Princeton University Press, Princeton, NJ, USA, 1992.

[3] M. E. Peskin and D. V. Schroeder, *An Introduction to quantum field theory*. Addison-Wesley, Reading, USA, 1995.

[4] S. Weinberg, *The Quantum theory of fields. Vol. 1: Foundations*. Cambridge University Press, 6, 2005, 10.1017/CBO9781139644167.

[5] S. Weinberg, *The quantum theory of fields. Vol. 2: Modern applications*. Cambridge University Press, 8, 2013, 10.1017/CBO9781139644174.

[6] D. Z. Freedman and A. Van Proeyen, *Supergravity*. Cambridge Univ. Press, Cambridge, UK, 5, 2012, 10.1017/CBO9781139026833.

[7] S. R. Coleman and J. Mandula, *All Possible Symmetries of the S Matrix*, *Phys. Rev.* **159** (1967) 1251.

[8] R. Haag, J. T. Lopuszanski and M. Sohnius, *All Possible Generators of Supersymmetries of the s Matrix*, *Nucl. Phys. B* **88** (1975) 257.

[9] J. Wess and B. Zumino, *Supergauge Transformations in Four-Dimensions*, *Nucl. Phys. B* **70** (1974) 39.

[10] J. Wess and B. Zumino, *A Lagrangian Model Invariant Under Supergauge Transformations*, *Phys. Lett. B* **49** (1974) 52.

[11] N. Seiberg, *Naturalness versus supersymmetric nonrenormalization theorems*, *Phys. Lett. B* **318** (1993) 469 [hep-ph/9309335].

[12] N. Seiberg, *The Power of holomorphy: Exact results in 4-D SUSY field theories*, in *Particles, Strings, and Cosmology (PASCOS 94)*, pp. 0357–369, 5, 1994, `hep-th/9408013`.

[13] J. Wess and B. Zumino, *Supergauge Invariant Extension of Quantum Electrodynamics*, *Nucl. Phys. B* **78** (1974) 1.

[14] L. O'Raifeartaigh, *Spontaneous Symmetry Breaking for Chiral Scalar Superfields*, *Nucl. Phys. B* **96** (1975) 331.

[15] P. Fayet and J. Iliopoulos, *Spontaneously Broken Supergauge Symmetries and Goldstone Spinors*, *Phys. Lett. B* **51** (1974) 461.