String Theory 1

Lecture #2

Chapter 1 Classical relativistic string

study relativistic classical string propagating in a fixed spacetime $\mathcal{M}$

1.1 Classical relativistic point particle

in a way that is generalisable to strings

1.2 Classical relativistic string: action principle

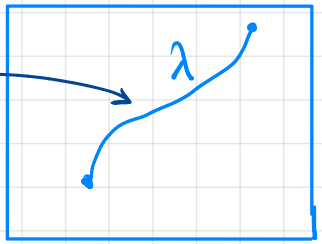
1.3 ...

1.1 Classical relativistic point particle

To motivate the formalism describing the string we first review the motion of a relativistic point-like particle.

Consider a particle of mass m traveling along a WL λ in spacetime M with metric $g_{\mu\nu}$.

D dimensional space time



$M, g_{\mu\nu}$

signature $(-1, +1, +1, \ldots)$

The equations of motion fix the trajectory of the particle to be the one with minimal proper length ($\int_\lambda ds$) ie the particle moves along a geodesic.

GR

Action principle:

$$S[\lambda] = -m \int_\lambda ds$$

dimensionless

$[m] = L^{-1}$

units $c = \hbar = 1$

$[L] = [\text{time}] = [\text{mass}]^{-1} = [\text{energy}]^{-1}$

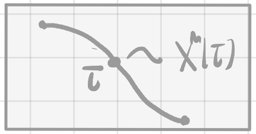
Polchinski exercise (1.1): m indeed the particle's mass (look at non-relativistic limit and see that this gives the appropriate KE & potential energy)

[GR: one can write the line element in terms of the metric]

We treat this action as a "0+1" dimensional field theory:

Let X^μ $\quad \mu = 0, \dots, D-1$ spacetime coordinates on $\mathcal{M}$

We can introduce a parametrisation of λ, $\tau \in \mathbb{R}$ and associate coordinates $X^\mu(\tau)$ to each point on λ. This is an embedding of λ into $\mathcal{M}$



$$\lambda : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathcal{M} \\ \tau & \longmapsto & X^\mu(\tau) \end{array}$$

The point on λ with param τ has coords $X^\mu(\tau)$ on $\mathcal{M}$

GR 1:

use the embedding and the metric on $g_{\mu\nu}$ on $\mathcal{M}$

$$ds = \underbrace{\sqrt{-g_{\mu\nu}\, dX^\mu\, dX^\nu}}_{\text{induced (pullback) metric on } \lambda} = \sqrt{-g_{\mu\nu} \frac{dX^\mu}{d\tau} \cdot \frac{dX^\nu}{d\tau}}\; d\tau$$

line element on λ
infinitesimal proper length of λ

$$S[X] = \int_\tau d\tau \sqrt{-g_{\mu\nu} \frac{dX^\mu}{d\tau} \cdot \frac{dX^\nu}{d\tau}}$$

Symmetries of the action:

- **1-dimensional reparametrisation invariance**

S is a function of λCM & we don't care about the choice of parametrisation!

$$\tau \longrightarrow \tilde{\tau}(\tau)$$ local diffeomorphisms on λ

to maintain inv. under reparams

$$X^\mu(\tau) \longrightarrow \tilde{X}^\mu(\tilde{\tau}) = X^\mu(\tau)$$ fields X^μ transform as WL scalars

($\frac{dX^\mu}{d\tau}$ WL vectors)

This is a gauge symmetry (redundancy of the description of the motion of the particle)

One can use the reparametrisation invariance to get rid of this redundancy (gauge fix)
For example we can choose $X^0 = t =$ time coord in spacetime. Then

$$S = -m \int dt \sqrt{\underbrace{-\dot{\underline{X}}\cdot\dot{\underline{X}} + 1}_{1-\underline{v}^2}} = S[\underline{X}], \quad \left(\dot{\underline{X}} = \frac{d\underline{X}}{dt}, \quad \underline{X} = (X^1, -, X^{D-1})\right)$$

$-m(1-\underline{v}^2)^{1/2}$
$= -m + \underbrace{\tfrac{1}{2}mv^2}_{KE} + \cdots$

Apparently: the motion is described by in terms of the X^μ; that is, seemingly there are D-degrees of freedom. However the motion of the particle should be entirely described by (D-1) spatial coords (plus initial conditions).

▶ The isometry group of $\mathcal{M}$ leaves invariant the line element

If $g_{\mu\nu} = \eta_{\mu\nu}$ (flat Minkowski metric) $\Rightarrow$ spacetime Poincaré invariance

rotations + translations

$$X^\mu(\tau) \longrightarrow \Lambda^\mu{}_\nu X^\nu(\tau) + b^\mu, \quad \Lambda \in SO(1,D-1), \quad b \in \mathbb{R}^{1,D-1}$$

isometry group of D-dim Minkowski space

[More generally, the space time isometry group is realised as internal symmetries of the WL field theory]

S is a fine classical action but there are two problems:

- it has a square-root $\Rightarrow$ difficult to quantize
  eg not quadratic in the time derivatives!
- what happens if $m = 0$?

To circumvent these problems, look for another "nicer" action which gives the same EOM for the massive particle and describes appropriately massless particles

Consider $\boxed{\text{instead}}$ the action

$$\tilde{S}[e,X] = \frac{1}{2}\int e(\tau)\left(e(\tau)^{-2} g_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau} - m^2\right)d\tau$$

where we introduce a new (auxiliary) field $e(\tau)$ on λ

$\boxed{\text{EOM}}$ for e are: $0 = \frac{\delta\tilde{S}}{\delta e(\tau)} \iff g_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau} + e(\tau)^2 m^2 = 0$

<u>For $m \neq 0$</u> this fixes $e(\tau)$ completely in terms of X^μ

& substituting this back into $\tilde{S}$ we get S

$\boxed{\tilde{S} \text{ \& } S \text{ are classically equivalent.}}$

Symmetries of $\tilde{S}$

- reparametrization of the worldline

$$\tau \longmapsto \tilde{\tau}(\tau)$$

$$X^\mu(\tau) \longmapsto \tilde{X}^\mu(\tilde{\tau}) = X^\mu(\tau)$$ scalars on λ

to maintain rep inv. $$e(\tau) \longmapsto \tilde{e}(\tilde{\tau}) = \frac{d\tau}{d\tilde{\tau}} e(\tau)$$ covector on λ

($e(\tau)d\tau$ is invariant)

- spacetime isometries (Poincaré for Minkowski) with $e(\tau)$ invariant

We can use reparametrisation invariance to gauge fix $e(\tau)$.

It is convenient to set $e(\tau)$ to be a constant.

let
$$e(\tau) = \begin{cases} 1/m & m \neq 0 \\ 1 & m = 0 \end{cases}$$

$$\tilde{S}_{\text{fixed}} = \begin{cases} \frac{1}{2} m \int \left(g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - 1 \right) d\tau & m \neq 0 \\ \frac{1}{2} \int g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \, d\tau & m = 0 \end{cases}$$

(affinely parametrized geodesic with this choice of gauge)

EOM for X

Euler Lagrange equations $\rightsquigarrow$ geodesic equation (GR1)

$$\frac{dX^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dX^\alpha}{d\tau} \frac{dX^\beta}{d\tau} = 0$$

Γ = Christoffel symbols associated to $g_{\mu\nu}$

τ = proper time on WL

EOM for $e(\tau)$: gives an extra constraint.

$m \neq 0$ $\quad g_{\mu\nu} \dfrac{dX^\mu}{d\tau}\dfrac{dX^\nu}{d\tau} + 1 = 0 \Rightarrow \dfrac{dX^\mu}{d\tau}$ is the TL 4-velocity

(TL geodesic)

$m = 0$ $\quad g_{\mu\nu} \dfrac{dX^\mu}{d\tau}\dfrac{dX^\nu}{d\tau} = 0 \Rightarrow \dfrac{dX^\mu}{d\tau}$ is the Null 4-velocity

(N geodesic)

Remark: we have fixed the gauge to eliminate redundancies in the description of the system but note that EOM of the extra degrees of freedom are still very important

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Conclusion:

through path integral quant.

- $\tilde{S}$ is a good starting point for quantization
(now $\tilde{S}$ is in fact quadratic in time derivatives)
- "0+1" field theory

Could add interactions, build up Feynman diagrams in a first-quantized theory.

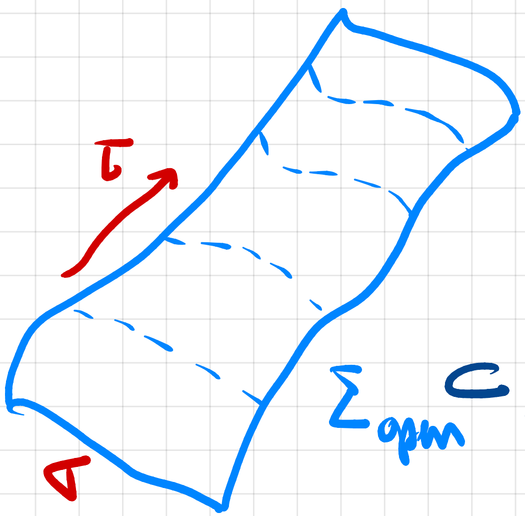
1.2 Classical relativistic string

Generalise to a string

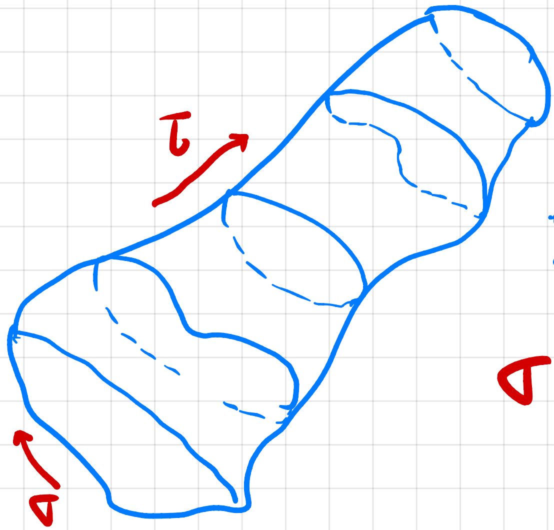
A string sweeps out a 2dim world sheet Σ in $\mathcal{M}$

(1dim object

"target space"



$\Sigma_{open} \subset \mathcal{M}$



$\Sigma_{closed} \subset \mathcal{M}$

σ is periodic

$\sigma \equiv \sigma + \ell$

$\sigma \in [0, \ell]$ ℓ = "length" measured in some arbitrary units on Σ

coords on the WS: $\xi^a = (\xi^0, \xi^1) = (\tau, \sigma)$, $a = 0, 1$

↳ parametrisation of WS

time coord

spatial coordinate

Embedding of $\Sigma \hookrightarrow \mathcal{M}$ ← target space

$$(\tau, \sigma) \longmapsto X^\mu(\tau, \sigma)$$

where for closed strings $X^\mu(\tau, \sigma + \ell) = X^\mu(\tau, \sigma)$ same point in space time

 $\sim X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + \ell)$

We now consider a "1+1" dimensional field theory on Σ with fields $X^\mu(\tau, \sigma)$.

Nambu-Goto action : natural generalisation of $S[\lambda]$

(analogue of $S[\lambda] = -m\int_\lambda ds$)

S_{NG} : action which describes the classical motion of a string along minimal area surfaces (principle)

$$S_{NG}[\Sigma] = \underbrace{-T}_{\text{"tension"}} \int_\Sigma dA$$

area of Σ

"tension" ~ mass / length $\quad [T] = L^{-2}$

area element (2 dim volume form) $\quad [dA] = L^2$

Euler Lagrange eqs : classical motion of the string along minimal area surfaces.

$$dA = \sqrt{-h}\, d\tau d\sigma = \sqrt{-(\partial_\tau X\cdot\partial_\tau X)(\partial_\sigma X\cdot\partial_\sigma X) + (\partial_\tau X\cdot\partial_\sigma X)^2}\; d\tau d\sigma$$

$$\begin{cases} h_{ab} = g_{\mu\nu}(X(\xi))\dfrac{\partial X^\mu}{\partial \xi^a}\dfrac{\partial X^\nu}{\partial \xi^b} \\ h = \det h_{ab} \end{cases}$$

$= $ induced worldsheet metric (pullback of $g_{\mu\nu}$ onto Σ by the embedding $\Sigma \hookrightarrow M$)

Notation $\quad u\cdot V = g_{\mu\nu} u^\mu V^\nu$ for space-time vectors

What is T ? T is interpreted as the string tension

ie mass of the string per unit length

ref: 1) D. Tong lecture notes 2) Polchinski problem 1.1

3) Becker + Becker + Schwarz exercise 2.7 with solution

Remark: $T = \frac{1}{2\pi\alpha'}$ α' = Regge slope

(historical reasons)

$[T] = L^{-2}$ $[\alpha'] = L^2$

$l_s = 2\pi\sqrt{\alpha'}$ string length scale

so $T = \frac{2\pi}{l_s^2} \Rightarrow$

$M_s = \frac{1}{\sqrt{\alpha'}}$ string mass scale

units $\hbar = c = 1$

$T = \frac{1}{2\pi} \frac{1}{(l_s / 2\pi)^2} = \frac{2\pi}{l_s^2}$

$[E] = [m] = [L]^{-1} = [\text{time}]^{-1}$

Symmetries of the NG-action (just as before for $S[\lambda]$)

- **2-dimensional reparametrisation invariance**

S is a function of $\Sigma \subset M$ and we don't care about the parametrization of Σ

can change our choice of parametrization of Σ

$$(\tau,\sigma) \longrightarrow (\tilde{\tau}(\tau,\sigma), \tilde{\sigma}(\tau,\sigma))$$ WS diffeomorphisms

to maintain S invariant

$$X^\mu(\tau,\sigma) \longrightarrow \tilde{X}^\mu(\tilde{\tau},\tilde{\sigma}) = X^\mu(\tau,\sigma)$$ fields X^μ transform like WS scalars

Again: reparametrisations are a gauge symmetry

- The isometry group of M leaves invariant the area element

↳ the target

If $g_{\mu\nu} = \eta_{\mu\nu}$ (flat Minkowski metric) $\Rightarrow$ spacetime Poincaré invariance

for example

$$X^\mu(\tau,\sigma) \longrightarrow \Lambda^\mu{}_\nu X^\nu(\tau,\sigma) + b^\mu, \text{ where } \Lambda \in SO(1,D-1),\ b \in \mathbb{R}^{1,D-1}$$

isometry group of Minkowski space

This is an "internal" symmetry ie a global symmetry wrt Σ

S gives a nice classical theory, it describes a 2dim field theory on Σ,

Euler-Lagrange eqs (which extremise the area of Σ)

$$\delta S = 0 \quad \Rightarrow \quad \partial_a\left(\sqrt{-h}\, h^{ab}(X)\, g_{\mu\nu}(X(\xi))\, \partial_b X^\nu\right) = 0$$

(where $\delta\sqrt{-h} = \frac{1}{2}\sqrt{-h}\, h^{ab}\, \delta h_{ab}$)

and study the classical dynamics of a string (PS 1)

Even in flat space $g_{\mu\nu} = \eta_{\mu\nu}$ this is hard! it is non linear

So: not clear how to quantise!

Next

Chapter 2

2.1 Classical relativistic point particle

2.2 Classical relativistic string → S_{NG}, S_P & symmetries

2.3 General classical solutions