String Theory 1

Lecture #3

Chapter 1 Classical relativistic string

today we continue
study relativistic classical string propagating in a fixed spacetime M

1.1 Classical relativistic point particle lecture 2

1.2 Classical relativistic string: action principle

1.3 Classical solutions

1.3.1 EOM & boundary conditions

1.3.2 Conserved charges associated to the symmetries of the action.

⋮

1.2 Classical relativistic string

Continued from lecture #2

Last lecture: Nambu-Goto action → Euler Lagrange eqs: classical motion of the string along minimal area surfaces

$$S_{NG}[\Sigma] = -T \int_\Sigma dA$$

$$dA = \sqrt{-h}\, d\tau\, d\sigma = \sqrt{-(\partial_\tau X \cdot \partial_\tau X)(\partial_\sigma X \cdot \partial_\sigma X) + (\partial_\tau X \cdot \partial_\sigma X)^2}\, d\tau\, d\sigma$$

$$h_{ab} = g_{\mu\nu}(X(\xi)) \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} = \text{induced world sheet metric}$$

(pullback of $g_{\mu\nu}$ onto Σ by the embedding $\Sigma \hookrightarrow M$)

$h = \det h_{ab}$

string tension $T = \frac{1}{2\pi\alpha'}$
(mass / unit length)

$[T] = L^{-2}$, $[\alpha'] = L^2$ — α' = Regge slope (historical reasons)

$\ell_s = 2\pi\sqrt{\alpha'}$, $M_s = 1/\sqrt{\alpha'}$
string length scale — string mass scale

symmetries
- 2-dimensional reparametrisation invariance
  $\xi^a \to \tilde{\xi}^a(\xi)$ — X^μ WS scalars
- The isometry group of M leaves invariant the area element

EOM: $\partial_a\left(\sqrt{-h}\, h^{ab}(X)\, g_{\mu\nu}(X(\xi))\, \partial_b X^\nu\right) = 0$ — non-linear in X

hard! even when M is flat

Not clear how to quantise S_{NG}

The Polyakov action:

Brink-DiVecchia-Howe
Deser-Zumino

h_{ab} (induced metric on $\Sigma \subset M$)

$$\boxed{S_P[\gamma_{ab}, X^\mu] = -\frac{T}{2}\int_\Sigma \sqrt{-\gamma}\,\gamma^{ab}\underbrace{\partial_a X^\mu \partial_b X^\nu g_{\mu\nu}}_{h_{ab}}\, d\tau\, d\sigma}$$

L Polyakov: used this to quantise the string using the PI-formalism

$\boxed{\xi^a = (\tau, \sigma)}$

$\gamma_{ab}(\xi)$ Lorentzian world-sheet metric **NEW** field on Σ

$\gamma = \det(\gamma_{ab})$

S_P: 2dim field theory on Σ with fields (γ, X)

EOM wrt δX^μ:

linear in ∂X when $g=\eta$

$$\partial_a \left(\sqrt{-\gamma}\, \gamma^{ab}\, g_{\mu\nu}(X)\, \partial_b X^\nu \right) = 0$$

Similar to that from S_{NG} but here γ is an independent field

EOM for the WS metric γ

GR1: (1 dim GR)

$$T_{ab} \equiv -\frac{2}{T} \frac{1}{\sqrt{-\gamma}} \frac{\delta S_P}{\delta \gamma^{ab}}$$

stress tensor or energy momentum tensor

$$\Rightarrow \quad T_{ab} = \underbrace{\partial_a X^\mu \partial_b X^\nu g_{\mu\nu}}_{h_{ab}} - \frac{1}{2}\gamma_{ab}\left(\gamma^{cd} \underbrace{\partial_c X^\mu \partial_d X^\nu g_{\mu\nu}}_{h_{cd}}\right) = 0$$

$$\Rightarrow \quad h_{ab} = \frac{1}{2}\left(\gamma^{cd} h_{cd}\right)\gamma_{ab}$$

$\gamma_{ab} \propto$ pullback metric h_{ab}

Using this in S_P one gets back S_{NG} :

in fact, the proportionality factor "drops out" of S_P

$$\frac{1}{2}\gamma^{ab}h_{cd}\sqrt{-\gamma} = \sqrt{-h}$$

We get then S_{NG} & same EoM

$\therefore$ S_P & S_{NG} are equivalent classically

same classical dynamics

Next : (A) Symmetries of S_P

(B) gauge fixing   use this to simplify

(A) Symmetries of the Polyakov action

WS pers-pective → gauge symmetries

▶ Word sheet reparametrisation $\xi^a \longmapsto \tilde{\xi}^a(\xi)$ diffeos of Σ

$$\gamma_{ab}(\xi) \longmapsto \tilde{\gamma}_{ab}(\tilde{\xi}) = \gamma_{cd}(\xi) \frac{\partial \tilde{\xi}^c}{\partial \xi^a} \frac{\partial \tilde{\xi}^d}{\partial \xi^b}$$ symmetric 2 tensor on Σ

&

$$X^\mu(\xi) \mapsto \tilde{X}^\mu(\tilde{\xi}) = X^\mu(\xi)$$ (WS scalars)

(WS) local diffeomorphism invariance

Moreover: We have a conservation equation

∇ = Levi Civita for γ_{ab}

$$\boxed{\nabla_a T^{ab} = 0}$$ when EOM are satisfied ("on shell")

WS perspective → global symmetries

▶ spacetime isometries (Poincaré invariance when M = Minkowski) and γ does not transform

So far: these where already in S_{NG}

special to 2dims

▶ **Weyl invariance** ie local scale symmetry acting on the 2dim metric on Σ

$$\gamma_{ab} \longmapsto e^{2\omega(\xi)}\gamma_{ab}, \quad X^\mu \text{ invariant}$$

$$[\sqrt{\gamma} \mapsto e^{2\omega}\sqrt{\gamma}\ ;\ \gamma^{ab} \mapsto e^{-2\omega}\gamma^{ab}]$$

Weyl invariance is also a gauge symmetry

[Weyl invariance very important in quantisation: theory anomalous unless $D=26$]

Why is Weyl invariance special in 2dims (special to the string)?

Consider instead a p-dimensional extended object with (p+1)dim WV

$$\text{factor } \gamma^{ab}\sqrt{-\gamma} \xrightarrow[\text{Weyl transf.}]{} \underbrace{e^{-2\omega}\gamma^{ab} e^{(p+1)\omega}}_{e^{(p-1)\omega}}\sqrt{-\gamma}$$

$\boxed{\text{not invariant unless } p=1 \text{ is a string}}$

- Lack of Weyl inv for higher dim p-branes makes it harder to understand non-perturbative physics in strings (D-branes ...) & M-theory

Tracelessness of T_{ab} :

$\boxed{\text{EOM for } \gamma \quad T_{ab}=0 \quad 3 \text{ constraints?}}$

There is an important consequence of Weyl invariance

$$\boxed{T_{ab} \text{ is traceless}}$$

regardless of the EOM

Recall

$$T_{ab} = \underbrace{\partial_a X^\mu \partial_b X^\nu g_{\mu\nu}}_{h_{ab}} - \frac{1}{2}\gamma_{ab}\left(\gamma^{cd}\underbrace{\partial_c X^\mu \partial_d X^\nu g_{\mu\nu}}_{h_{cd}}\right)$$

$$\mathrm{Tr}\,T = T_{ab}\gamma^{ab} = \gamma^{ab}h_{ab} - \frac{1}{2}\cdot 2\cdot\gamma^{bc}h_{bc} \overset{\text{trivially}}{=} 0$$

automatically

$$h_{ab} = \frac{1}{2}(\gamma^{cd}h_{cd})\gamma_{ab}$$

$$\Rightarrow \gamma^{ab}h_{ab} = \frac{1}{2}(\gamma^{cd}\gamma_{cd})\,\mathrm{tr}\,\gamma$$

$$\Rightarrow \mathrm{tr}\,\gamma = 2$$

so $\mathrm{Tr}\,T = 0$ is <u>not</u> a constraint

$\Rightarrow$ $T_{ab} = 0$ only two EOM

Why is T_{ab} traceless? consequence of Weyl inv

Recall $\delta S = \frac{\delta S}{\delta \gamma^{ab}} \delta \gamma^{ab} \propto \sqrt{-\gamma}\, T^{ab}\, \underline{\delta \gamma_{ab}}$

Consider an infinitesimal Weyl transformation

$$\gamma_{ab} \longmapsto e^{2\omega(\xi)} \gamma_{ab} = (1 + 2\omega(\xi) + \cdots)\, \gamma_{ab}$$

so $\delta \gamma_{ab} = 2\omega(\xi)\, \gamma_{ab}$

Weyl inv.

$$\Rightarrow \quad \delta_w S \propto 2\sqrt{-\gamma}\, \omega(\xi)\, T^{ab} \gamma_{ab} = 0 \quad \text{for } \underline{\text{any}}\ \omega$$

δS wrt Weyl sym

$\therefore \quad \boxed{T_{ab}\gamma^{ab} = T^a{}_a = 0}$

regardless of EOM

* (this does not require EOM!) *

S_P is 2 dim field theory describing D, 2dim scalar fields $X^\mu(\xi)$ coupled to the WS metric γ_{ab} ie 2dim gravity coupled to scalars

This begs the question { How general is S_P?

Can one add terms to the action which are

- compatible with (power counting) renormalizability (at most 2 derivs)

and

- consistent with the symmetries of the action

Two possible terms (for the closed string and no other fields)

* $$S_{HE} = \frac{\lambda_2}{4\pi} \int_\Sigma \sqrt{-\gamma}\, \underline{R^{(2)}(\gamma)}\, d\tau d\sigma$$ ⇝ Hilbert-Einstein terms for 2dim gravity

WS Ricci scalar

Check out PS 1

Integrand is (locally) a total derivative

⇒ does not affect the classical equations of motion

S_{HE} is topological (it depends only on the global topology of Σ)

in fact $S_{HE} = \lambda_2 \chi(\Sigma)$ (related to a coupling constant!)

(open string: Σ has boundaries and there is an extra term)

$\boxed{\text{Ignore for now}}$ but it is an important term in string perturbation theory!

when topologies of WS are important

* $$S_{CT} = \lambda \int_\Sigma \underbrace{\sqrt{-\gamma}\, d\tau d\sigma}$$ ⇝ cosmological constant term on Σ

area element inv under Reparams but not Weyl invariant

⇒ inconsistent EOM (BBS exercise) ⇒ $\lambda = 0$

(any term $\sim \int \sqrt{-\gamma}\, V(X)\, d\tau d\sigma$ not Weyl invariant)

(B) Gauge fixing the Polyakov action

As is usual in theories with gauge symmetries one can "choose" a gauge to simplify the action.

reparametrizations: $\gamma_{ab} \to e^{2\omega(\tau,\sigma)} \eta_{ab}$ conformal gauge

$\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

3 independent degrees freedom

use a reparametrization (which involves 2 functions of coords) to gauge away 2 components of γ

Weyl: $e^{2\omega(\tau,\sigma)} \eta_{ab} \to \eta_{ab}$ unit gauge

↳ use a Weyl transformation to gauge away the remaining degree of freedom

$\boxed{\text{Remark}}$: locally one can prove that one can always choose this gauge $\gamma_{ab} = \eta_{ab}$.

However we do not know if this can be done globally on Σ !

There are in fact topological obstructions which are better understood in Euclidean signature.

To deal with Lorentzian signatures one does a "Wick rotation" to Euclidean signature

Global criterion for the choice of conf gauge: (BLT)

Consider the gauge condition $\gamma_{ab} = e^{2\Lambda}\hat{\gamma}_{ab}$

(In conf gauge: $\tilde{\gamma}_{ab} = \eta_{ab}$)

Change in metric under Weyl & reparams

$$\delta\gamma_{ab} = -(\nabla_a \xi_b + \nabla_b \xi_a) + 2\Lambda\gamma_{ab} \equiv -(P\xi)_{ab} + 2\tilde{\Lambda}\gamma_{ab}$$

$$(P\xi)_{ab} = \nabla_a \xi_b + \nabla_b \xi_a - (\nabla_c \xi^c)\gamma_{ab}$$ traceless, sym

and $$\tilde{\Lambda} = \Lambda - \frac{1}{2}(\nabla_c \xi^c)$$

Trace of $\delta\gamma_{ab} = 4\tilde{\Lambda} \Rightarrow$ Can always set it to zero by choosing $\Lambda = \frac{1}{2}\nabla_c \xi^c$

etc!

$\frac{1}{2}(p+1)(p+2-2)$

$p+1$ dim (p-brane) $\gamma = \frac{1}{2}(p+1)(p+2)$ comp

reparam of WV allows to fix $p+1$ d.o.f $\Rightarrow \frac{1}{2}p(p+1)$ leftover d.o.f

conf gauge is unique to 2 dims!

Polyakov action in conformal gauge

indeed it simplifies drastically

$$\boxed{S_P^{CG}[X^\mu] = -\frac{T}{2}\int\left(-\partial_\tau X\cdot\partial_\tau X + \partial_\sigma X\cdot\partial_\sigma X\right)d\tau d\sigma = -\frac{T}{2}\int \partial_a X\cdot\partial_b X\,\eta^{ab}\,d\tau d\sigma}$$

$\Rightarrow$ theory of D massless scalar fields in flat (1+1)-dim space (though one term with the wrong sign)

EOM (for X^μ): $\quad \boxed{\partial_a(g_{\mu\nu}\partial^a X^\nu) = 0}$

simplifies!

(for $g_{\mu\nu} = \eta_{\mu\nu}$ $\quad \partial_a\partial^a X^\mu = 0 \quad$ 2dim wave eq)

EOM for γ: recall $T_{ab} = \partial_a X \cdot \partial_b X - \frac{1}{2}\gamma_{ab}\gamma^{cd}\partial_c X \cdot \partial_d X$

$T_{ab} = 0 \quad \longleftrightarrow \quad$ constraints after gauge fixing

In the conformal gauge

$$T_{ab} = \partial_a X \cdot \partial_b X - \frac{1}{2}\eta_{ab}\,\partial_c X \cdot \partial^c X = 0$$

$$T_{\tau\tau} = T_{\sigma\sigma} = \frac{1}{2}\left(\partial_\tau X \cdot \partial_\tau X + \partial_\sigma X \cdot \partial_\sigma X\right)$$

$$T_{\tau\sigma} = \partial_\tau X \cdot \partial_\sigma X$$

automatically

tracelessness of T_{ab}: $\quad T^a{}_a = \eta^{ab} T_{ab} = -T_{\tau\tau} + T_{\sigma\sigma} \overset{\checkmark}{=} 0$

should hold irrespective of these constraints

2 constraints (instead of 3)

In summary:

$$S_P^{CG}[X] = -\frac{T}{2}\int_\Sigma \partial_a X\cdot\partial_b X\,\eta^{ab}\,d\tau\,d\sigma$$

gauge fixed Polyakov action $\gamma_{ab} = \eta_{ab}$

EOM $\quad \partial_a(g_{\mu\nu}\partial^a X^\nu) = 0$

[When $M = \mathbb{R}^{1,D-1}$ $\quad g_{\mu\nu} = \eta_{\mu\nu}$

$\partial_a\partial^a X^\mu = 0$ wave eq!]

2 constraints

$$T_{\tau\tau} = T_{\sigma\sigma} = \frac{1}{2}\left(\partial_\tau X\cdot\partial_\tau X + \partial_\sigma X\cdot\partial_\sigma X\right) = 0$$

$$T_{\tau\sigma} = \partial_\tau X\cdot\partial_\sigma X = 0$$

Conservation of T_{ab}: $\quad \nabla_a T^{ab} = 0$

1.3 Classical Solutions of S_P

$\mathcal{M} = \mathbb{R}^{1,D-1}$ with

$\boxed{g_{\mu\nu} = \eta_{\mu\nu}}$

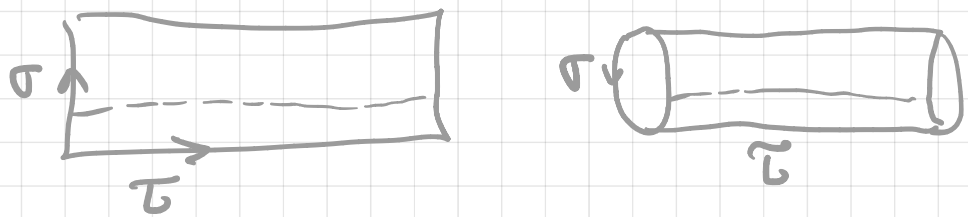
We are interested in solving the equations of motion for the fields X^μ which in the conformal gauge: $\partial_a \partial^a X^\mu = 0$ 2 dim wave eq

together with constraints coming from T_{ab}

For a single string propagating without sources we describe the string by considering

$\tau \to$ time coordinate on Σ

$-\infty \leq \tau \leq \infty$



$\sigma \to$ spatial coordinate on Σ

strings with finite spatial length $\sigma \in [0, \ell]$

1.3.1 Equations of motion and boundary conditions

Writing the action as $S[X] = \int_\Sigma d\tau\, d\sigma\, \mathcal{L}[X^\mu, \partial_a X^\mu]$

a standard computation gives
in classical field theory

$$\delta S = \int_\Sigma d\tau\, d\sigma \left\{ \frac{\partial \mathcal{L}}{\partial X^\mu} \delta X^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_a X^\mu)} \delta \partial_a X^\mu \right\}$$

Π^a_μ : conjugate momentum

$$= \int_\Sigma d\tau\, d\sigma \left\{ \underbrace{\partial_a \left(\frac{\partial \mathcal{L}}{\partial(\partial_a X^\mu)} \delta X^\mu \right)}_{\text{total derivative}} + \underbrace{\left[\frac{\partial \mathcal{L}}{\partial X^\mu} - \partial_a \left(\frac{\partial \mathcal{L}}{\partial(\partial_a X^\mu)} \right) \right]}_{\rightarrow \text{ Euler-Lagrange (EOM)}} \delta X^\mu \right\}$$

$\delta S = 0$:

- vanishing of the second term $\forall\, \delta X$ gives the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial X^\mu} - \partial_a \Pi^a_\mu = 0$$

- 1st term must vanish too!

For the Polyakov action: $S_P^{CG}[X^\mu] = -\frac{T}{2}\int_\Sigma d\tau\, d\sigma\, \partial_a X \cdot \partial^a X$

- Euler-Lagrange equations (EOM)

$$0 = \partial_a \left(-\frac{T}{2} \cdot 2\eta^{ab}\partial_b X^\mu \right)$$

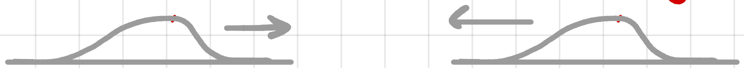
(note $\frac{\partial \mathcal{L}}{\partial X^\mu} = 0$)

$$\eta^{ab}\partial_a\partial_b X^\mu = -\partial_\tau^2 X^\mu + \partial_\sigma^2 X^\mu = 0$$

two dim wave eq for waves travelling at $c=1$

General solution: $X^\mu(\sigma,\tau) = X_R^\mu(\tau-\sigma) + X_L^\mu(\tau+\sigma)$
(prelims)

R-moving L-moving wave fronts



use light-cone coords: $\xi^\pm = \tau \pm \sigma$

$$\partial_\pm = \frac{\partial}{\partial \xi^\pm} = \frac{1}{2}(\partial_\tau \pm \partial_\sigma) \; ; \quad d\tau\, d\sigma = d\xi^+ d\xi^- \frac{\partial(\tau,\sigma)}{\partial(\xi^+,\xi^-)} = \frac{1}{2} d\xi^+ d\xi^-$$

$$\gamma_{++} = \gamma_{--} = \gamma^{++} = \gamma^{--} = 0 \; ; \quad \gamma_{+-} = \gamma_{-+} = -\frac{1}{2} \; ; \quad \gamma^{+-} = \gamma^{-+} = -2$$

$\gamma_{ab} = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \gamma^{ab} = -2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

↳ next total derivate term in δS