String Theory 1

Lecture #4

Chapter 1

Classical relativistic string

↳ study relativistic classical string propagating in a fixed spacetime M

1.1 Classical relativistic point particle

1.2 Classical relativistic string: action principle

1.3 Classical solutions

1.3.1 EOM & boundary conditions

1.3.2 Conserved charges associated to the symmetries of the action

1.3.3 Solutions of EOM + bound. conds

1.3.4 Satisfying the constraints

1.3.5 The Witt-algebra & conformal symmetries

1.3 Classical solutions continued

1.3.1 EOM and boundary conditions

$\gamma_{ab} = \eta_{ab} \rightarrow$ CG

$$S_P[X] = -\frac{T}{2}\int_\Sigma \partial_a X \cdot \partial_b X\, \eta^{ab}\, d\tau\, d\sigma$$

gauge fixed Polyakov action

target space: $M = \mathbb{M}_D$ D-dim Minkowski

EOM $\quad \partial_a(\partial^a X^\nu) = 0 \quad : \quad X^\mu(\xi) = X^\mu_L(\xi^+) + X^\mu_R(\xi^-) \qquad \xi^\pm = \tau \pm \sigma$

- impose conditions coming from the boundary term in $\delta S = 0$
- constraints from EOM for γ

$$\begin{cases} T_{\tau\tau} = T_{\sigma\sigma} = \frac{1}{2}(\partial_\tau X \cdot \partial_\tau X + \partial_\sigma X \cdot \partial_\sigma X) = 0 \\ T_{\tau\sigma} = \partial_\tau X \cdot \partial_\sigma X = 0 \end{cases}$$

- Conservation of T_{ab} : $\qquad \partial_a T^{ab} = 0$

total derivative term in δS must vanish too

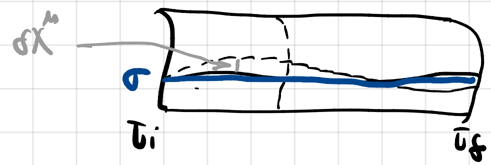
$$0 = \int_\Sigma d\tau\, d\sigma\, \partial_a \left(\frac{\partial \mathcal{L}}{\partial(\partial_a X^\mu)} \delta X^\mu \right)$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_a X^\mu)} = -T (\partial_a X^\nu)\, \eta_{\mu\nu}$$

$$= -T \int_{\tau_i}^{\tau_f} d\tau \int_0^\ell d\sigma \left\{ \frac{\partial}{\partial \tau} \left(\eta_{\mu\nu}\, \partial_\tau X^\mu\, \delta X^\nu \right) + \frac{\partial}{\partial \sigma} \left(\eta_{\mu\nu}\, \partial_\sigma X^\mu\, \delta X^\nu \right) \right\}$$

first term (int wrt τ) : $\quad -T \int_0^\ell d\sigma\, \eta_{\mu\nu} \left(\partial_\tau X^\mu\, \delta X^\nu \right) \Big|_{\tau = \tau_i}^{\tau = \tau_f} = 0 \quad$ because

$$\boxed{\delta X^\mu (\tau_i, \sigma) = 0 \qquad \delta X^\mu (\tau_f, \sigma) = 0}$$



ie string is kept fixed at initial & final positions

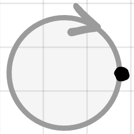
ie variations of the WS with fixed initial (at $\tau = \tau_i, \tau_f$) conditions

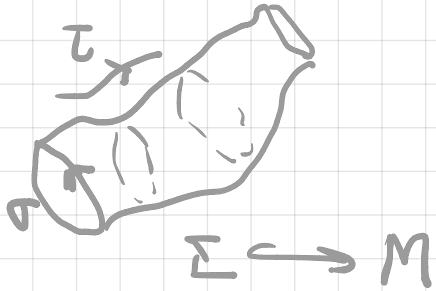
(cannobergo : particle $\delta X(\tau_i) = 0$  $\delta X(\tau_f) = 0$

var of trajectory with fixed initial & final positions)

So the second term must vanish too:

$$0 = -T\int_{\tau_i}^{\tau_f} d\tau \left(\eta_{\mu\nu}\,\partial_\sigma X^\mu\, \delta X^\nu\right)\Big|_{\sigma=0}^{\sigma=\ell}$$

<u>closed strings</u> periodicity conditions  $\sigma \sim \sigma+\ell$



ie $X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+\ell)$

so boundary term vanishes

Moreover: solutions of the EOM must be periodic in σ with period ℓ

open strings: boundary conditions on the string endpoints

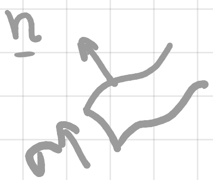
$$0 = -T \int_{\tau_i}^{\tau_f} d\tau \, \left(\partial_\sigma X \cdot \delta X \right) \Big|_{\sigma=0}^{\sigma=\ell} \rightsquigarrow \underline{\partial_\sigma X_\mu \, \delta X^\mu = 0 \quad \text{at} \quad \sigma = 0, \ell}$$

- **Neumann (NN)** no constraints on δX^μ at $\sigma = 0, \ell$

  so endpoints move freely in $\mathcal{M}$

  & $\partial_\sigma X^\mu (\tau, \ell) = 0$ & $\partial_\sigma X^\mu (\tau, 0) = 0$

  "no momentum flowing off the string"



momentum $\perp$ to $\partial\Sigma$

- Dirichlet (DD) $\quad \delta X^\mu = 0$ at $\sigma = 0, \ell$ $\quad$ ends of string fixed in M

ie $\quad X^\mu(\tau, \ell) = x^\mu_\ell(\tau)$, $\quad X^\mu(\tau, 0) = x^\mu_0(\tau)$

This involves a choice of spacetime vectors $\Rightarrow$ break Poincaré invariance

- One can have mixed boundary conditions: for example

  - Neumann (NN) on $p+1$ words

    Dirichlet (DD) on $D-(p+1)$ words

The ends of the string are free to move only on a subspace $\mathcal{Q} \subset M$, $\dim \mathcal{Q} = p+1$.

This subspace is called a Dp-brane with x^μ_0 & x^μ_ℓ interpreted as the position of the brane.

(Very important! needed for internal consistency of the non-perturbative theory; more later)

- One can also have (ND) boundary conditions.

1.3.2 Solutions of EOM + bound. conds. $< \begin{smallmatrix} CS \\ OS \end{smallmatrix}$ NN

general solution of the wave eq $X^\mu(\tau,\sigma) = X^\mu_R(\xi^-) + X^\mu_L(\xi^+)$

see Zwiebach Chapter 7 for detailed solutions

Closed strings : $X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+\ell)$, $\quad (\xi^\pm \to \xi^\pm \pm \ell)$

Expand in Fourier modes:

separately periodic up to a zero mode

$$X^\mu_L(\xi^+) = \frac{1}{2}x^\mu + \frac{\pi}{\ell}\alpha' p^\mu \xi^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \frac{1}{n} \tilde{\alpha}^\mu_n \, e^{-\frac{2\pi i}{\ell} n \xi_+}$$

$$X^\mu_R(\xi^-) = \frac{1}{2}x^\mu + \frac{\pi}{\ell}\alpha' p^\mu \xi^- + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \frac{1}{n} \alpha^\mu_n \, e^{-\frac{2\pi i}{\ell} n \xi^-}$$

where x^μ, p^μ, $\tilde{\alpha}^\mu_n$ and α^μ_n are the Fourier coeffts.

X^μ is real-valued : $\quad x^\mu \in \mathbb{R}$, $p^\mu \in \mathbb{R}$, $\tilde{\alpha}^\mu_{-n} = (\tilde{\alpha}^\mu_n)^*$, $\alpha^\mu_{-n} = (\alpha^\mu_n)^*$

$\alpha^\mu_0 = \tilde{\alpha}^\mu_0 = \sqrt{\frac{\alpha'}{2}}\, p^\mu$ from periodicity $\sigma \to \sigma + \ell$

Useful for later:

$$\partial_+ X^\mu(\xi^+) = \partial_+ X_L^\mu = \frac{2\pi}{\ell}\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \tilde{\alpha}_n^\mu \, e^{-\frac{2\pi i}{\ell} n \xi_+}$$

$$\partial_- X^\mu(\xi^-) = \partial_- X_R^\mu = \frac{2\pi}{\ell}\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \alpha_n^\mu \, e^{-\frac{2\pi i}{\ell} n \xi_-}$$

where $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}\, p^\mu$

Prefactors for convenient physical interpretations as we will see below

Open strings with Neumann (NN) boundary conditions

$$\partial_\sigma X^\mu(\tau, \ell) = 0 \qquad \partial_\sigma X^\mu(\tau, 0) = 0$$

Due to the boundary conditions X_L^μ & X_R^μ are no longer independent ($\tilde{\alpha}_n^\mu = \alpha_n^\mu$)

$$X^\mu(\tau,\sigma) = \underbrace{x^\mu + \frac{2\pi\alpha'}{\ell} p^\mu \tau}_{\text{average position}} + i\sqrt{2\alpha'} \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \frac{1}{n} \alpha_n^\mu e^{-\frac{i\pi n}{\ell}\tau} \cos\left(\frac{n\pi\sigma}{\ell}\right)$$

$$x^\mu(\tau) = \frac{1}{\ell}\int_0^\ell d\sigma\, X^\mu(\tau,\sigma)$$

X^μ real-valued : $\alpha_{-n}^\mu = (\alpha_n^\mu)^*$

$$\partial_\pm X^\mu = \frac{\pi}{\ell}\sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^\mu e^{-\frac{i\pi n}{\ell}\xi^\pm}, \qquad \alpha_0^\mu = \sqrt{2\alpha'}\, p^\mu$$

See lecture notes for open strings with DD & ND boundary conds

1.3.3 Conserved charges

Recall Noether's theorem: for each symmetry in the action there is a corresponding conserved current. We also have Noether charges

- (global) symmetries corresponding to the isometries of $\mathcal{M}$: Poincaré invariance

$$X^\mu \longmapsto \underbrace{\Lambda^\mu{}_\nu}_{\text{Lorentz transfs}} X^\nu + V^\mu$$

$\uparrow$ translations

conserved current $\Pi^\mu_a = \dfrac{\delta \mathcal{L}}{\partial \partial_a X^\mu}$ momentum conjugate

$(\partial_a \Pi^a_\mu)\eta^{ab} = 0$

(see PS 1)

- <u>translations</u> $X^\mu(\xi) \mapsto X^\mu(\xi) + V^\mu$

current $q^a_\mu = -T\sqrt{-\gamma}\,\gamma^{ab}\partial_a X_\mu = -T\eta^{ab}\partial_b X_\mu$

conservation $\partial_a q^a{}_\mu = 0$

conservation of the energy momentum current

charges: $\int_0^\ell (q^\mu)^\tau d\sigma = T \int_0^\ell \partial_\tau X^\mu \equiv p^\mu$ center of mass momentum

(spatial integral of the τ-component of each current)

<u>Interpretation of the coefficients</u>

$$\int_0^\ell d\sigma\,(q^\mu)^\tau = T \int_0^\ell \partial_\tau X^\mu = \begin{cases} p^\mu & \text{closed string \& open string with (NN) bound. conds} \\ 0 & \text{open string with (DD) bound. cond} \end{cases}$$

$\partial_\tau X^\mu = \frac{2\pi\alpha'}{\ell} p^\mu$ + terms that vanish upon σ-integration

p^μ: center of mass momentum of the string

- **Lorentz transformations**: $X^\mu \longmapsto \Lambda^\mu{}_\nu X^\nu$

current $\quad J^a_{\mu\nu} = -T\eta^{ab}(X_\mu \partial_b X_\nu - X_\nu \partial_b X_\mu) = X_\mu q^a_\nu - X_\nu q^a_\mu$

conservation $\quad \partial_a J^a_{\mu\nu} = 0$

→ conservation of angular momentum current

charges: $\frac{1}{\ell}\int_0^\ell (J^{\mu\nu})^\tau d\sigma$

Interpretation of the coefficients

$$M^{\mu\nu} = \int_0^\ell d\sigma\, (J^{\mu\nu})^\tau = \begin{cases} \ell^{\mu\nu} + E^{\mu\nu} + \tilde{E}^{\mu\nu} & \text{closed strings} \\ \ell^{\mu\nu} + E^{\mu\nu} & \text{(NN) open string} \end{cases}$$

$J^{\mu\nu}_a = -T(X^\mu \partial_a X^\nu - X^\nu \partial_a X^\mu)$

$\ell^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$ center of mass contribution

two mode of spacetime angular momentum

$$E^{\mu\nu} = -i\sum_{n\neq 0}\frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n)$$

$$\tilde{E}^{\mu\nu} = -i\sum_{n\neq 0}\frac{1}{n}(\tilde{\alpha}^\mu_{-n}\tilde{\alpha}^\nu_n - \tilde{\alpha}^\nu_{-n}\tilde{\alpha}^\mu_n)$$

contribution of L & R moving waves to the spacetime angular momentum

- WS - symmetries : WS diffeomorphisms (next)

conserved current T_{ab} , $\eta^{ab}\partial_a T_{bc} = 0$

1.3.4 Conserved currents for WS diffeomorphisms

Work in the light-cone coordinates

- conservation: $\eta^{ab}\partial_a T_{bc} = 0 \implies \partial_+ T_{--} + \partial_- T_{+-} = 0$

$$\partial_- T_{++} + \partial_+ T_{-+} = 0$$

tracelessness $\eta^{ab} T_{ab} = 0 \implies T_{+-} + T_{-+} = 0$

$\implies T_{+-} = T_{-+}$ ✓ symmetric tensor

$\underline{T_{+-} = 0}$ automatic

$$\implies \underline{\partial_+ T_{--} = 0 \qquad \partial_- T_{++} = 0}$$ These are extremely powerful!

* 2dim Lorentzian version of holomorphicity (antiholomorphicity). These give us our $\boxed{\text{infinite}}$ set of conserved charges! *

- Finally enforce $\underline{T_{++} = 0 \qquad T_{--} = 0}$

Ⓐ Closed strings

let $f(\xi^-)$ be an arbitrary function and consider

$$Q_f = \int_0^\ell d\sigma \, f(\xi^-) \, T_{--}(\xi^-) \qquad \partial_+ T_{--} = 0$$

$$\Rightarrow \frac{\partial}{\partial \tau} Q_f = \int_0^\ell d\sigma \, (\cancel{2\partial_+} - \partial_\sigma)(f(\xi^-) T_{--}(\xi^-))$$

$\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$

so $\partial_\tau = 2\partial_+ - \partial_\sigma$

$$= - \int_0^\ell d\sigma \, \partial_\sigma (f(\xi^-) T_{--}(\xi^-)) = - (f(\xi^-) T_{--}(\xi^-)) \Big|_{\sigma=0,\, \tau \text{ fixed}}^{\sigma=\ell,\, \tau \text{ fixed}}$$

$= 0$ if $f(\xi^-)$ is periodic

That is: the current $f(\xi^-) T_{--}(\xi^-)$ is also conserved !

Since f is arbitrary $\Rightarrow$ there is an infinite set of conserved currents

Similarly: T_{++} is conserved and so is $g T_{++}$, $g = g(\xi^+)$ periodic

A complete set of periodic functions in σ is given by

$$\left\{ f_m(\xi^-) = e^{\frac{2\pi i}{\ell} m \xi^-} \, , \quad m \in \mathbb{Z} \, . \right\}$$

Define then an infinite set of charges :

$$L_m = \frac{T\ell}{2\pi} \int_0^\ell d\sigma \, e^{\frac{2\pi i}{\ell} m \xi^-} \, T_{--}(\xi^-) = \frac{T}{2\ell} \int_0^\ell d\sigma \, e^{\frac{2\pi i}{\ell} m \xi^-} \, \partial_- X_R \, \partial_- X_R$$

$$\left(T_{--} = \partial_- X \cdot \partial_- X = \partial_- X_R \cdot \partial_- X_R \right)$$

Fourier modes of T_{--}

Using the mode expansion for X^μ $\quad \left(\partial_- X_R^\mu = \frac{2\pi}{\ell} \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \alpha_n^\mu \, e^{-\frac{2\pi i}{\ell} n \xi^-} \right)$

take $\tau = 0$ WLOG as
L_m are conserved

$$L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{m-n} \cdot \alpha_n \qquad \text{with} \qquad \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} \, p^\mu$$

Note that $\quad L_{-m} = (L_m)^* \quad$ because T_{--} is real

Similarly: for $T_{++}(\xi^+)$, $g_m(\xi^+) = e^{\frac{2\pi i}{\ell} m \xi^+}$

$$\tilde{L}_m = \frac{T\ell}{2\pi} \int_0^{\ell} d\sigma \; e^{\frac{2\pi i}{\ell} m \xi^+} \; T_{++}(\xi^+)$$

Fourier modes of T_{++}

$$= \frac{1}{2} \sum_{n \in \mathbb{Z}} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n ,$$

$\tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$

to be continued...