

String Theory 1

Lecture #5

Chapter 1

Classical relativistic string

↳ study relativistic classical string propagating in a fixed spacetime M

✓ 1.1 Classical relativistic point particle

✓ 1.2 Classical relativistic string: action principle

1.3 Classical solutions

✓ 1.3.1 EOM & boundary conditions

✓ 1.3.2 Canonical charges associated to the symmetries of the action

✓ 1.3.3 Solutions of EOM + bound. cond.

1.3.4 Conserved currents for WS diffeomorphisms & constraints

1.3.5 The Witt-algebra & conformal symmetries

1.3.4 Conserved currents for WS diffeomorphism & constraints $T_{++} = T_{--} = 0$

Last week:

0 dim QFT
of D free
fields X^M

$$S_P[X] = -\frac{T}{2} \int_{\Sigma} \partial_a X \cdot \partial_b X \eta^{ab} d\bar{u} d\sigma$$

gauge fixed
Polyakov
action ($\delta_{ab} = \eta_{ab}$)

target space: $M = M_0$ 0-dim Minkowski

EOM $\partial_a (\partial^a X^\nu) = 0 : X^M(\xi) = X^M_L(\xi^+) + X^M_R(\xi^-) \quad \xi^\pm = \tau \pm \sigma$

▶ studied conditions coming from the boundary term in $\delta S = 0$
& constructed solns for CS, OS(N,N)

▶ Conservation of T_{ab} : $\partial_+ T_{--} = 0 \quad \partial_- T_{++} = 0$

* These give us our infinite set of conserved charges! *

▶ constraints from
EOM pr δ

$$T_{++} = T_{--} = 0$$

Last lecture: (A) Closed strings

$$\partial_+ T_{--} = 0 \Rightarrow Q_f = \int_0^{\ell} d\sigma f(\xi^-) T_{--}(\xi^-)$$

\downarrow arbitrary

are conserved
 $f(\xi^-)$ periodic $\sigma \rightarrow \sigma + \ell$

let $\left\{ f_m(\xi^-) = e^{\frac{2\pi i}{\ell} m \xi^-}, m \in \mathbb{Z} \right\}$ a complete set of periodic functions in σ

$$L_m = \frac{T\ell}{2\pi} \int_0^{\ell} d\sigma e^{\frac{2\pi i}{\ell} m \xi^-} T_{--}(\xi^-) = \frac{T}{2\alpha'} \int_0^{\ell} d\sigma e^{\frac{2\pi i}{\ell} m \xi^-} \partial_- X_\mu \partial_- X_\mu$$

$(T_{--} = \partial_- X \cdot \partial_- X = \partial_- X_\mu \cdot \partial_- X_\mu)$

Fourier modes of T_{--}

using the mode expansion for X^μ $\left(\partial_- X_\mu = \frac{2\pi}{\ell} \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \alpha_n^\mu e^{-\frac{2\pi i}{\ell} n \xi^-} \right)$

take $\tau=0$ wlog as L_m are conserved

$$L_m = \frac{1}{\alpha'} \sum_{n \in \mathbb{Z}} \alpha_{m-n} \cdot \alpha_n \quad \text{with} \quad \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$$

and $L_{-m} = (L_m)^*$ because T_{--} is real

similarly for T_{++}

We have then

$$L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{m-n} \cdot \alpha_n$$

$$\tilde{L}_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n$$

We still need to impose constraints from the stress tensor.

$$T_{++} = 0 \quad \& \quad T_{--} = 0$$

Recall that L_m & \tilde{L}_m are the Fourier components of T_{--} & T_{++} respectively. Then setting

$$L_m = 0, \quad \tilde{L}_m = 0 \quad \forall m \in \mathbb{Z}$$

imposes the constraints $T_{++} = 0$ & $T_{--} = 0$

* The vanishing of these charges is equivalent to quadratic constraints on the oscillators *

$$\{ \alpha^M, p^N, \alpha_n^M, \tilde{\alpha}_n^M \}$$

Consider these constraints for L_0 & \tilde{L}_0 (particularly interesting)

$$L_0 = \frac{1}{2} \sum_{n \neq 0} \alpha_{-n} \cdot \alpha_n = \frac{1}{2} \alpha_0 \cdot \alpha_0 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \quad \alpha_0^M = \sqrt{\frac{\alpha'}{\alpha}} p^M$$

$$= \frac{\alpha'}{4} p^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = 0 \quad (p^2 = p \cdot p = p^m p_m)$$

$$\tilde{L}_0 = \frac{\alpha'}{4} p^2 + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n = 0$$

(Recall p^M = spacetime centre of mass momentum)

► We have a mass shell condition: $M^2 = -p^2$

$$-p^2 = M^2 = \frac{2}{\alpha'} \sum_{n=1}^{\infty} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n)$$

rest mass² of a string
in a given state of oscillation

contributions of osc. modes to the
effective mass of the string in spacetime

► Moreover: $\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n = -\frac{\alpha'}{4} p^2$

level matching
condition

(relates L/R modes)

(do not forget we still need to enforce $L_m = 0, \tilde{L}_m = 0 \quad \forall m \neq 0$)

ⓑ **Open strings** with **(NN)** boundary conditions on all X^{μ}

Let $f(\xi^+)$ & $g(\xi^-)$ be arbitrary functions. Then

$$\partial_+ (g(\xi^-) T_{--}(\xi^-)) = 0 \quad \& \quad \partial_- (f(\xi^+) T_{++}(\xi^+)) = 0$$

Let

$$Q_{f,g} = \int_0^l d\sigma (f(\xi^+) T_{++} + g(\xi^-) T_{--})$$

and seek conditions on f & g st Q is conserved:

$$\begin{aligned} \partial_\tau Q_{f,g} &= \int_0^l d\sigma \left(\underbrace{\partial_\sigma (f(\xi^+) T_{++})}_{\partial_\tau = 2\partial_- + \partial_\sigma} - \underbrace{\partial_\sigma (g(\xi^-) T_{--})}_{\partial_\tau = 2\partial_+ - \partial_\sigma} \right) \\ &= (f(\xi^+) T_{++} - g(\xi^-) T_{--}) \Big|_0^l \end{aligned}$$

At $\sigma=0, l$: $\partial_+ X^\mu = \partial_- X^\nu$ so $T_{++} = T_{--}$

$$\partial_\pm X^\mu = \frac{\pi \alpha'}{e} \sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^\mu e^{-i\frac{\pi}{l} n \xi^\pm} \quad \xi^\pm = \tau \pm \sigma$$

\Rightarrow Q is conserved if $f(\xi^+) = g(\xi^-)$ at $\sigma = 0, l$

(i) $\sigma = 0 \Rightarrow f(\tau) = g(\tau)$ i.e. f & g are the same function

(ii) $\sigma = l \Rightarrow f(\tau+l) = g(\tau-l) = f(\tau-l)$
 $\therefore f(\tau) = f(\tau+2l)$

f periodic function with period $2l$

Let $f(\xi^+) = e^{\pi i m \xi^+ / l}$ $g(\xi^-) = e^{\pi i m \xi^- / l}$

and define

$$L_m = \frac{T\pi}{l} \int_0^l d\sigma (e^{\pi i m \sigma / l} T_{++} + e^{\pi i m \sigma / l} T_{--}) \stackrel{\bar{i}=0}{=} \frac{T\pi}{l} \int_0^l d\sigma (e^{\pi i m \sigma / l} T_{++}(\sigma) + e^{-\pi i m \sigma / l} T_{--}(\sigma))$$

\Rightarrow $L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{m-n} \cdot \alpha_n$ with $\alpha_0^m = \sqrt{2\alpha'} p^m$

L_0 gives open-string mass-shell condition

$$L_0 = 0 : M^2 = -p^2 = \frac{1}{\alpha'} \sum_{n=1} \alpha_{-n} \cdot \alpha_n$$

rest mass of a string
due to the oscillators

Summary

We have constructed explicitly the space of solutions of the eqs of motion i.e., the phase space.

This is an infinite dimensional affine space with coordinates

$$\{x^M, p^M, \alpha_n^M, \tilde{\alpha}_n^M\}$$

subject to quadratic constraints

$$\{L_n = 0, \tilde{L}_n = 0, \forall n \in \mathbb{Z}\}$$

} for the closed string;
for the open string neglect $\tilde{\alpha}, \tilde{L}$

where L_n (& \tilde{L}_n) constitute an infinite set of conserved charges corresponding to the Fourier modes of T_{ab}

$$L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{m-n} \cdot \alpha_n, \quad \tilde{L}_m = \dots$$

L_m (& \tilde{L}_m) are the Noether charges corresponding to reparametrisation invariance

1.3.5 The Witt-algebra & conformal symmetries

This subsection:

- ▶ pass to the hamiltonian formalism and compute the Poisson brackets for the classical observables
(and this sets up the stage for the quantisation procedure)
- ▶ L_n & \tilde{L}_n satisfy an algebra, the algebra of the generators of conformal transformations

Ⓐ In a Hamiltonian formulation:

the space of solutions is given by the canonical fields

$$X^M(\tau, \sigma)$$

and conjugate momenta

$$\pi^M(\tau, \sigma) = \frac{\partial \mathcal{L}}{\partial(\partial_\tau X^M(\tau, \sigma))} = T \partial_\tau X^M(\tau, \sigma)$$

where the Lagrangian is $\mathcal{L} = \frac{T}{\alpha} [\partial_\tau X \cdot \partial_\tau X - \partial_\sigma X \cdot \partial_\sigma X]$
(unit gauge)

The Hamiltonian is then

$$H = \int_0^{\ell} d\sigma (\partial_\tau X(\tau, \sigma) \cdot \pi(\tau, \sigma) - \mathcal{L}) = T \int_0^{\ell} d\sigma (\partial_+ X \cdot \partial_+ X + \partial_- X \cdot \partial_- X)$$
$$= \begin{cases} \frac{2\pi}{\ell} (L_0 + \tilde{L}_0) & \text{closed strings} \\ \frac{\pi}{\ell} L_0 & \text{open strings} \end{cases}$$

In this formalism observables are functionals

$$F(X(\bar{t}, \sigma), \Pi(\bar{t}, \sigma))$$

Phase space is a Poisson manifold i.e. a manifold together with Poisson brackets

$$\{F, G\}_{PB} = -\{G, F\}_{PB}$$

symplectic
pairing

For fields $F(\tau, \sigma)$, $G(\tau, \sigma')$, it is defined as

$$\{F(\tau, \sigma), G(\tau, \sigma')\}_{PB} \equiv \int d\tilde{\sigma} \left\{ \frac{\partial F(\tau, \sigma)}{\partial \pi^m(\tau, \tilde{\sigma})} \frac{\partial G(\tau, \sigma')}{\partial X^m(\tau, \tilde{\sigma})} - \frac{\partial G(\tau, \sigma')}{\partial \pi^m(\tau, \tilde{\sigma})} \frac{\partial F(\tau, \sigma)}{\partial X^m(\tau, \tilde{\sigma})} \right\}$$

This leads to the canonical equal time P.B.s

$$\{\pi^m(\tau, \sigma), X^\nu(\tau, \sigma')\}_{PB} = \eta^{m\nu} \delta(\sigma - \sigma')$$

$$\{X^m(\sigma), X^\nu(\sigma')\}_{PB} = 0, \quad \{\pi^m(\sigma), \pi^\nu(\sigma')\}_{PB} = 0.$$

From these we can compute the Poisson brackets of the oscillator modes by extracting the Fourier components of X^M :

$$\{ \alpha_m^M, \alpha_n^N \}_{PB} = i m \delta_{m+n, 0} \eta^{MN}$$

$$\{ P^M, X^N \}_{PB} = \eta^{MN}$$

$$\{ \tilde{\alpha}_m^M, \tilde{\alpha}_n^N \}_{PB} = i m \delta_{m+n, 0} \eta^{MN}, \quad \{ \alpha_m^M, \tilde{\alpha}_n^N \} = 0$$

Similarly, for the open string we have only one set of oscillators

Closed strings: extract Fourier modes and then compute P.B.s

$$n \neq 0 \quad \frac{1}{\ell} \int_0^\ell e^{-\frac{2\pi i n \sigma}{\ell}} X^M(0, \sigma) d\sigma = i \sqrt{\frac{\alpha'}{2}} \frac{1}{n} (\alpha_n^M - \tilde{\alpha}_{-n}^M)$$

$$n \neq 0 \quad \int_0^\ell e^{-\frac{2\pi i n \sigma}{\ell}} \Pi^M(0, \sigma) d\sigma = \frac{1}{\sqrt{2\alpha'}} (\alpha_n^M + \tilde{\alpha}_{-n}^M)$$

zero mode

$$\frac{1}{\ell} \int_0^\ell X^M(0, \sigma) d\sigma = X^M$$

zero mode

$$\int_0^\ell \Pi^M(0, \sigma) d\sigma = P^M$$

We now use this to compute the P.B.s for the constraints, i.e. for the Fourier modes of T_{ab} :

We find

$$\left. \begin{aligned} \{L_m, X^\mu\}_{PB} &= -\frac{\ell}{2\pi} e^{\frac{2\pi i}{\ell} m \xi^-} \partial_- X^\mu \\ \{\tilde{L}_m, X^\mu\}_{PB} &= -\frac{\ell}{2\pi} e^{\frac{2\pi i}{\ell} m \xi^+} \partial_+ X^\mu \end{aligned} \right\}$$

Noether charges L_m
generate the
symmetry
 $\mathbb{R}^{1,1} \propto \delta X^\mu$

and

$$\begin{aligned} \{L_m, L_n\}_{PB} &= i(m-n)L_{m+n} \\ \{\tilde{L}_m, \tilde{L}_n\}_{PB} &= i(m-n)\tilde{L}_{m+n} \end{aligned}$$

Witt algebra

L_m & \tilde{L}_n form a Lie algebra (with Lie bracket $\{, \}_{PB}$ + Jacobi id)

Next: This is the algebra of infinitesimal conformal transformations on the world-sheet.

In the Hamiltonian picture symmetry transformations parametrized by ϵ are generated by canonical charges Q_ϵ via the P.B.

$$\delta_\epsilon F = \{Q_\epsilon, F\}_{PB}$$

↗
in infinitesimal change
of an observable F under
the transformation

Transformations arising from diffeomorphisms on space-time are parametrized by a vector field

$$\delta F = \mathcal{L}_\xi F$$

\mathcal{L}_ξ Lie derivative
"flows of F "

③ Conformal transformations

Let Σ be (Riemannian or Lorentzian) manifold with metric γ .

A conformal transformation of Σ is a diffeomorphism $\xi \mapsto \tilde{\xi}(\xi)$ that preserves the metric up to rescaling

$$\gamma_{ab}(\xi) \mapsto \tilde{\gamma}_{ab}(\tilde{\xi}) = e^{2\Lambda(\xi)} \gamma_{ab}(\xi)$$

(a special case is an isometry for which $\Lambda = 0$)

The infinitesimal conformal transformations can be described explicitly:
we can compute the generators of such transformations on Σ

let

$$\xi^a \mapsto \tilde{\xi}^a = \xi^a + \epsilon^a(\xi)$$

be a general **infinitesimal** diffeomorphism. Then

$$\gamma_{ab} \mapsto \gamma_{ab} - \underbrace{\nabla_a \epsilon_b - \nabla_b \epsilon_a}_{-d_\xi \gamma_{ab}}$$

$$\tilde{\gamma}_{ab}(\tilde{\xi}) = \tilde{\partial}_a \tilde{\xi}^c \tilde{\partial}_b \tilde{\xi}^d \gamma_{cd}(\xi)$$

This corresponds to a conformal transformation if ϵ^a satisfies

$$-\nabla_a \epsilon_b - \nabla_b \epsilon_a = 2\Lambda \gamma_{ab} = (\nabla_c \epsilon^c) \gamma_{ab} \quad \text{conformal killing equation}$$

$$\tilde{\gamma}_{ab}(\tilde{\xi}) = \frac{\partial \tilde{\xi}^c}{\partial \xi^a} \frac{\partial \tilde{\xi}^d}{\partial \xi^b} \gamma_{cd} = e^{2\Lambda(\xi)} \gamma_{ab}(\xi)$$

A solution of this equation is called a conformal killing vector

($\Lambda=0$: Killing equation, ϵ a Killing vector)

In the unit gauge & in the light-cone coordinates

(Recall $\eta_{+-} = \eta_{-+} = -\frac{1}{2}$ $\eta^{+-} = \eta^{-+} = -2$)

$$(++) : \partial_+ \epsilon_+ = 0 \Rightarrow \partial_+ \epsilon^- = 0 \Rightarrow \epsilon^- = \epsilon^-(\Sigma^-)$$

$$(--): \partial_- \epsilon_- = 0 \Rightarrow \partial_- \epsilon^+ = 0 \Rightarrow \epsilon^+ = \epsilon^+(\Sigma^+)$$

$$(+-): \Lambda = \partial_+ \epsilon_- + \partial_- \epsilon_+ = -\frac{1}{2} (\partial_a \epsilon^a) = -\frac{1}{2} (2\partial_- \epsilon_+ - 2\partial_+ \epsilon_-)$$

trivially true so no further restrictions on ϵ^\pm

$\therefore \epsilon^- = \epsilon^-(\Sigma^-)$ & $\epsilon^+ = \epsilon^+(\Sigma^+)$ generate
infinitesimal conformal transformations

We can then pick a complete set for ϵ^\pm in terms of

$$e^{\frac{w_i}{r} \Sigma^\pm}$$

One can think of the infinitesimal WS reparametrizations

$$\delta \xi^\pm = \epsilon^\pm(\xi^\pm)$$

as being generated by $V^\pm \propto \epsilon^\pm(\xi^\pm) \partial_\pm$ ← local basis for the tangent space

[Recall correspondence between tangent vector fields, say $\epsilon^\pm \partial_\pm$, and the 1 parameter group of diffeomorphisms $\xi^a \mapsto \xi^a + \epsilon^a$]

GR 1

We can pick a basis by using a complete set in terms of $e^{\frac{in}{\ell} \xi^\pm}$ for $\epsilon^\pm(\xi^\pm)$ which then gives a complete set of operators

$$V_n = -\frac{\ell}{2\pi} e^{\frac{in}{\ell} \xi^-} \partial_- \quad \tilde{V}_n = \frac{\ell}{2\pi} e^{\frac{in}{\ell} \xi^+} \partial_+$$

$$\langle L_m, X^m \rangle_{PB} = -\frac{1}{2\pi} e^{\frac{im}{\ell} \xi^-} \partial_- X^m$$

$$\langle \tilde{L}_m, X^m \rangle_{PB} = -\frac{1}{2\pi} e^{\frac{im}{\ell} \xi^+} \partial_+ X^m$$

These operators have commutation relations

$$[V_m, V_n] = i(n-m) V_{n+m} \quad \text{and similar for } \tilde{V}_m \text{'s}$$

This Lie algebra gives precisely the **Witt algebra**

Then L_m (& \tilde{L}_m 's) generate conformal space transformations on the phase space

$$\{L_m, X^\mu\}_{PB} = -\frac{1}{2\pi} e^{\frac{2\pi i}{\alpha'} m \xi^-} \partial_- X^\mu, \quad \{\tilde{L}_m, X^\mu\}_{PB} = -\frac{1}{2\pi} e^{\frac{2\pi i}{\alpha'} m \xi^+} \partial_+ X^\mu$$

This means that after fixing the gauge to the unit gauge the world sheet theory still has residual gauge symmetries (the conformal symmetries)

Constraints $L_m = 0$ (& $\tilde{L}_m = 0$)

mean that observables do not vary under conformal transformations

In principle one can use the residual gauge symmetries to do some further gauge fixing.

However there is no way to do this and simultaneously preserve space-time Lorentz covariance

Remark

Only in 2 dim the conformal algebra is infinite dimensional
 $\text{witt} \oplus \widetilde{\text{witt}}$

In $d > 2$, the conformal algebra is
 $\text{so}(2, d) \cong \underbrace{\text{so}(1, d-1)}_{\text{special conf transformations}}$

This, in $d=2$:
 $\text{so}(2, 2) \cong \underbrace{\mathfrak{sl}(2, \mathbb{R})}_{\{V_0, V_{\pm 1}\}} \times \underbrace{\mathfrak{sl}(2, \mathbb{R})}_{\{\widetilde{V}_0, \widetilde{V}_{\pm 1}\}}$

which is the "global" part of $\text{witt} \oplus \widetilde{\text{witt}}$

Remark:
 $V_0 + \widetilde{V}_0 = \partial_\tau$
 $V_0 - \widetilde{V}_0 = \partial_\sigma$

The appearance of the conformal symmetry suggests that the 2dim field theory on the WS of the string is in fact a 2dim conformal field theory.

We will look into this later

(See CFT course in Trinity Univ, or Polchinski vol 1 chapter 2)



Next: quantisation.



0



Chapter 2

Old covariant quantisation

There are several approaches (and the relation between them is not trivial)

1 Covariant BRST quantisation

modern path
integral quantisation

$$Z = \int \frac{[DX^\mu][D\psi]}{\text{Vol}(\text{Diff} \times \text{Weyl})} e^{\frac{i}{\hbar} S_p[X^\mu, \psi]}$$

This is the **best** quantum treatment of gauge theories.

(uses Faddeev-Popov-deWitt gauge fixing
& identifies BRST symmetries & currents
cancellation of Weyl anomaly requires $D=26$)

→ Right thing to do but requires more experience (AQFT) and takes longer.

2 light-cone quantisation

Fix all gauge symmetries in the classical theory
(so Virasoro constraints are implemented classically)

But then the classical theory is not Poincaré invariant.

Hard work to see that anomaly cancels

(subtle! eg $\sum_{n=1}^{\infty} n = -\frac{1}{12} = S(-1)$)

3 Old covariant quantisation

Start with the classical system in the conformal gauge and **then**
quantise (promoting X^μ & \tilde{T}^μ to operators, ...)

One imposes the constraints $T_{\pm\pm} = 0$ on the quantum Hilbert space

→ manifestly covariant

→ one needs $D=26$ to cancel anomaly in Virasoro algebra

↳ next continue OCP.