String Theory 1

Lecture #7

Chapter 2 Old covariant quantisation

2.1 Introduction

2.2 Hilbert space (without constraints)

2.3 Constraints, Normal ordering and the Virasoro algebra

2.4 Constraints and physical states

2.5 Mass-shell and level matching conditions

2.6 Level 1 states & dealing with ghosts

2.7 Level 2 states & dealing with ghosts

2.8 Null states and $D=26$

2.9 Physical states of the closed string

Last lecture: OCQ of the relativistic string

- promote $\begin{cases} \text{fields eg } X^\mu \longrightarrow \text{operators} \\ \{\cdot,\cdot\}_{PB} \longrightarrow i[\cdot,\cdot] \end{cases}$

oscillator modes satisfy an algebra

$$[\hat{p}^\mu, \hat{x}^\nu] = -i\eta^{\mu\nu}$$

$$[\alpha^\mu_m, \alpha^\nu_n] = m\,\delta_{m+n}\,\eta^{\mu\nu} \qquad (\alpha^\mu_n)^\dagger = \alpha^\mu_{-n}$$

similar for $\tilde{\alpha}$ in the case of CS

- Hilbert space (before imposing the constraints)

$$\mathcal{H}_{open} = L^2(\mathbb{R}^{1,D-1}) \otimes \mathcal{H}^{Fock}_{open} = \mathrm{Span}\Big\{\prod_{i=1}^{k} \alpha^{\mu_i}_{-n_i} |k,0\rangle\Big\}_{n_i\geq 1} = \bigoplus_{N=1}^{\infty} \mathcal{H}^{Fock}[N]$$

N-evalue $\sum_{i=0}^{k} n_i$

$$N \equiv \sum_{k>0} \alpha_{-k}\cdot\alpha_k$$

$$[N, \alpha^\mu_n] = -n\,\alpha^\mu_n$$

"counting op"

states fall into reps of $SO(1,D-1)$

manifestly covariant

$$\mathcal{H}_{closed} = L^2(\mathbb{R}^{1,D-1}) \otimes \mathcal{H}^{Fock}_L \otimes \mathcal{H}^{Fock}_R = \mathrm{Span}\Big\{\prod_{i=1}^{k}\prod_{j=1}^{\ell} \alpha^{\mu_i}_{-n_i}\,\tilde{\alpha}^{\nu_j}_{-m_j}|k,0\rangle\Big\}_{n_i,m_j\geq 1}$$

$\longrightarrow N, \tilde{N}$

Constraints

$m \neq 0 \qquad L_m = \frac{1}{2} \sum_{k \in \mathbb{Z}} \alpha_{m-k} \cdot \alpha_k \qquad L_m^\dagger = L_{-m}$

$m = 0 \qquad L_0 = \frac{1}{2} \sum_{n \in \mathbb{Z}} \underbrace{: \alpha_{-n} \cdot \alpha_n :}_{\text{NOP}} = \frac{1}{2} \alpha_0^2 + N$

Virasoro generators

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{D}{12}(m^2-1) m \, \delta_{m+n,0}$$

Virasoro algebra

Physical states $\qquad \mathcal{H}_{phys} \subset \mathcal{H}$

$|\psi\rangle \in \mathcal{H}_{phys}$ iff $\quad L_m |\psi\rangle = 0 \quad \forall \; m \geq 1$

$(L_0 - a)|\psi\rangle = 0 \quad$ for a fixed $a \in \mathbb{R}$

open string $\qquad \mathcal{H}_{phys} = \mathrm{Ker}(L_2) \cap \mathrm{Ker}(L_1) \cap \mathrm{Ker}(L_0 - a)$

L_1 & L_2 generate all L_m $m \geq 3$

closed string $\qquad \mathcal{H}_{phys} = \mathcal{H}_{phys} \otimes \tilde{\mathcal{H}}_{phys}$

Summary (so far)

- conditions $\quad (L_0 - a)|\psi\rangle = 0 \,, \quad (\tilde{L}_0 - a)|\psi\rangle = 0$

Closed string

mass shell condition $\quad \boxed{\alpha' M^2 = 2(N + \tilde{N} - 2a)}$

level matching condition $\quad \boxed{N = \tilde{N}}$

Open string

mass shell condition $\quad \boxed{\alpha' M^2 = N - a}$

- Ground state

$|k, 0\rangle$

$(N = 0)$

$a < 0$	massive	ground state
$a = 0$	massless	ground state
$a > 0$	tachyonic	ground state (!)

↳ continue discussing the construction of $\mathcal{H}_{phys} \subset \mathcal{H}$

2.6 Level 1 states & dealing with ghosts

Recall: earlier we encountered a problem with negative norm states (ghosts) at level one of the open string (eg: $\alpha_{-1}^{0}|0;K\rangle$).
We want to see if this issue remains after applying the constraints.

Consider a general level one open string state

$$\zeta\cdot\alpha_{-1}|0;K\rangle = |\zeta;K\rangle \qquad \zeta\in\mathbb{R}^{1,D-1}$$

polarization vector
(D-degrees of freedom)

& impose the physical state conditions for L_0 & L_{+1}.

The L_n conditions for $n\geq 2$ are satisfied automatically once the ones for L_0 & L_{+1} are imposed

$$L_m\,\zeta\cdot\alpha_{-1}|0;K\rangle = \zeta_\mu[L_m,\alpha_{-1}^{\mu}]|0;K\rangle = \zeta\cdot\alpha_{m-1}|0;K\rangle$$

$$= 0 \qquad m\geq 2$$

- mass-shell condition $\quad -\alpha' K^2 = \alpha' M^2 = 1 - a$

- L_{+1} condition: $\quad \leftarrow L_{+1} |0;K\rangle = 0$

$$L_{+1}(\zeta \cdot \alpha_{-1})|0;K\rangle = \eta_{\mu\nu} \zeta^\mu [L_{+1}, \alpha_{-1}^\nu] |0;K\rangle$$

$$= \eta_{\mu\nu} \zeta^\mu \underline{\alpha_0^\nu} |0;K\rangle = \sqrt{2\alpha'}\, \zeta \cdot K\, |0;K\rangle$$

$[L_m, \alpha_n^\mu] = -n\, \alpha^\mu_{m+n}$ $\qquad$ $\alpha_0^\mu = \sqrt{2\alpha'}\, K^\mu$

$$L_{+1}|\zeta;K\rangle = 0 \iff \zeta \cdot K = 0$$

transverse polarisation
hence ζ has $D-1$ independent components

- Norm of a general level 1 state

$$\langle \zeta;K|\zeta';K'\rangle = \langle 0;K|(\zeta \cdot \alpha_{+1})(\zeta' \cdot \alpha_{-1})|0;K'\rangle$$

$$= \zeta_\mu \zeta'_\nu \langle 0;K|[\alpha_{+1}^\mu, \alpha_{-1}^\nu]|0;K'\rangle = \zeta \cdot \zeta'\, \delta(K-K')$$

For $\zeta = \zeta'$ require $\zeta^2 \geq 0$ to avoid ghosts

space like or null

Level 1 summary

$\alpha' K^2 = a-1$: so K is $\begin{cases} \text{spacelike} & \text{if } a>1 \\ \text{lightlike} & \text{if } a=1 \\ \text{timelike} & \text{if } a<1 \end{cases}$

$\zeta \cdot K = 0$: transverse polarization ζ

$\zeta^2 \geq 0$: polarization is Null or space like to avoid ghosts

For $a>1$, K is spacelike (L_0 condition)

Then the constraint $\xi \cdot K = 0$ (L_1 condition)

is satisfied by timelike polarisations ξ, $\xi^2 < 0$.

That is, we would get negative norm states (ghosts).

So for $a>1$, the Virasoro constraints are not enough to eliminate them.

Thus we reject $a>1$ and require $a \leq 1$

Critical theory: Consider the case $a=1$ (threshold case)

$$a=1 \quad \Longrightarrow \quad K^2=0$$

$$\Longrightarrow \quad |\zeta;K\rangle = \zeta\cdot\alpha_{-1}|0;K\rangle$$ is a massless state

Additionally, there is a zero norm state

$\zeta=K$, $K^2=0$

$$|K;K\rangle = (K\cdot\alpha_{-1})|0;K\rangle$$ w/ longitudinal polarisation

This state is orthogonal to $\boxed{\text{all}}$ physical states $|\zeta',K\rangle$

$$\langle K;K|\zeta',K'\rangle = (K\cdot\zeta')\,\delta(K-K') = 0$$

($K\cdot\zeta'=0$ for physical states)

Hence: the longitudinal polarisation decouples leaving

$D-2$ physical polarisations like a photon!

However: for $a=1$ ground state is a tachyon

<u>Remark</u>: the decoupling of the longitudinal degree of freedom is due to the fact that it corresponds to a state that is "pure gauge":

$$L_{-1}|0;K\rangle = \sqrt{2\alpha'}\, K\cdot\alpha_{-1}|0;K\rangle = \sqrt{2\alpha'}\,|K;K\rangle$$

$$L_{-1}|0;K\rangle = \left(\frac{1}{2}\sum_{k=-\infty}\alpha_{-k-1}\cdot\alpha_k\right)|0,K\rangle = \frac{1}{2}\Big(\underset{k=0}{\alpha_{-1}\cdot\alpha_0} + \underset{k=-1}{\alpha_0\cdot\alpha_{-1}} + \sum_{k=2}^{\infty}\alpha_{k-1}\cdot\alpha_{-k}\Big)|0;K\rangle$$

$$= \alpha_{-1}\cdot\alpha_0|0;K\rangle = \sqrt{2\alpha'}\,K\cdot\alpha_{-1}|0;K\rangle$$

ie $|K;K\rangle$ is created by the action of L_{-1} which is a generator of a conformal transformation. In this sense we say that $|K;K\rangle$ is pure gauge state

↪ Virasoro constraints for $a=1$ precisely restrict the level 1 states to consistently describe space-time photons

$\underline{a < 1}$ $\quad K$ is timelike

$\xi \cdot K = 0 \Rightarrow \xi$ spacelike

(D-1 degrees of freedom)

$\xi^2 > 0$ +ve norm states

massive vector boson in D-dims.

Summary: Physical states:

$$(L_0 - a)|\psi\rangle = 0$$

$$L_m|\psi\rangle = 0, \quad m \geq 1$$

(only need to check $m = 1, 2$)

open strings

- Ground state $|k, 0\rangle$

$a < 0$	massive	ground state
$a = 0$	massless	ground state
$a > 0$	tachyonic	ground state (!)

- $N = 1$ $\qquad \xi \cdot \alpha_{-1}|0;k\rangle = |\xi; k\rangle \qquad \xi \in \mathbb{R}^{1,D-1}$ polarization vector (D-degrees of freedom)

$\alpha' k^2 = a - 1$: so k is $\begin{cases} \cancel{\text{spacelike if } a > 1} \\ \text{light like if } a = 1 \\ \text{timelike if } a < 1 \end{cases}$

ghosts! so $a \leq 1$

$\xi \cdot k = 0$: transverse polarization

$\xi^2 \geq 0$: polarization is Null or space like to avoid ghosts

In particular

Critical theory: Consider the case $a=1$ (threshold case)

$N=0$: $|k,0\rangle$ tachyon [$\alpha' M^2 = -1$ (OS)]

$N=1$: $|\xi;k\rangle = \xi\cdot\alpha_{-1}|0;k\rangle$ is a massless state ($k^2=0$)

photon: $D-2$ degrees of freedom as the longitudinal state $|k;k\rangle$ decouples

[orthogonal to all physical states $|\xi';k'\rangle$
$\langle k,k|\xi';k'\rangle = 0$
& it is pure gauge $L_{-1}|0;k\rangle = \sqrt{2\alpha'}\,|k,k\rangle$]

$\hookrightarrow$ Virasoro constraints for $a=1$ precisely restrict the level 1 states to consistently describe space-time photons

What is the role of L_{-m} $m \geq 1$?

The remaining L_{-m} $(m \geq 1)$ lead to redundancies in the spectrum corresponding to the associated gauge symmetries.

$\hookrightarrow$ next: discuss these redundancies

2.7 level 2 states & dealing with ghosts

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<u>Level 2</u> state for $a=1$ (threshold value as for $a>1$ there are ghosts in the physics)

Consider the level 2 state $|\phi, K\rangle$ with momentum K

$$|\phi,K\rangle = [c_1\, \alpha_{-1}\cdot\alpha_{-1} + c_2\, \alpha_{-2}\cdot K + c_3 (\alpha_{-1}\cdot K)(\alpha_{-1}\cdot K)]|0;K\rangle$$

onshell mass condition $\quad -\alpha' K^2 = 1 \quad \leftarrow \quad \alpha' M^2 = N - a$

$L_1 |\phi, K\rangle = 0 \quad$ iff $\quad c_1 + c_2 - 2c_3 = 0$

$L_2 |\phi, K\rangle = 0 \quad$ iff $\quad D c_1 - 4c_2 - 2c_3 = 0$

so $\quad |\phi, K\rangle = c_1 [\alpha_{-1}\cdot\alpha_{-1} + \frac{1}{5}(D-1)\,\alpha_{-2}\cdot K + \frac{1}{10}(D+4)(\alpha_{-1}\cdot K)(\alpha_{-1}\cdot K)]|0,K\rangle$

is a physical state for any D

Norm $\langle \phi; k | \phi; k' \rangle = -\frac{2}{25} |c_1|^2 (D-1)(D-26) \, \delta(k-k')$

$\boxed{\text{no}}$ ghosts $\Rightarrow$ $1 \leq D \leq 26$ ($\exists$ ghosts for $D > 26$)

"null" (physical·zero norm) states when $D=26$ (or $D=1$)

so when $a=1$ & $D=26$ there are more "null" states
(proliferation of null states @ $D=26$ sign of large gauge (conf.) symmetry)

1972 Brower; Goddard, Thorn No ghost theorem

↳ $\boxed{\text{For } a=1 \text{ and } D=26 \text{ the phys has no ghosts}}$

Note: there are $\boxed{\text{no}}$ ghosts for $a \leq 1$ & $1 \leq D \leq 25$
(this is a corollary of the no ghost theorem for the critical string)

BUT these theories are inconsistent at the level of string loops (need to look at one loop interactions)

2.8 Null states and $D=26$

Definition:

A state $|\psi\rangle$ is called spurious if

- it is orthogonal to all physical states and
- obeys $(L_0 - a)|\psi\rangle = 0$

A null state is a spurious state which is also physical.

A null state has zero norm as it is orthogonal to itself.

An example of a null state is $|k; k\rangle$ at level 1 & $a=1$, and this state is also pure gauge.

Null states are physical states that decouple from the dynamics.

These states are "quotiented out", that is

two physical states are equivalent if they differ by a null state

$$|\psi\rangle_{phys} \sim |\psi\rangle_{phys} + |\varphi\rangle_{null}$$

so we define

$$\mathcal{H}_{red} = \mathcal{H}_{phys} / \mathcal{H}_{null}$$

physically $\boxed{\text{distinct}}$ states

We expect: physical states of the form $L_{-m}|\varphi\rangle$ $\forall m > 0$ are null!

spurious states generated by the residual symmetries (conformal transformations)

Consider the state

$$|\psi\rangle = \sum_{m>0} L_{-m} |\psi_m\rangle \quad \text{st} \quad L_0 |\psi_m\rangle = (a-m) |\psi_m\rangle$$

Then $\langle \varphi | \psi \rangle = 0 \quad \forall \ |\varphi\rangle \in \mathcal{H}_{\text{Fock}}$

$$(L_0 - a) |\psi\rangle = 0$$

$$\sum_{m>0} (L_0 - a) L_{-m} |\psi_m\rangle = \sum_{m>0} \big(\underbrace{[L_0, L_{-m}]}_{m L_{-m}} + L_{-m} L_0 - a L_{-m} \big) |\psi_m\rangle = \sum_{m>0} L_{-m} \left(m - a + L_0 \right) |\psi_m\rangle$$

So $|\psi\rangle$ is spurious.

One can in fact prove that any spurious state is of this form, ie all spurious states are "pure gauge". (GSW p83)

[pure gauge states ~ states generated by residual symmetries (conformal symmetries!)]

this is expected in gauge theories

One can prove moreover that all spurious states are of the form

$$|\psi\rangle = L_{-1}|\chi_1\rangle + L_{-2}|\chi_2\rangle$$

with $L_0|\chi_1\rangle = (a-1)|\chi_1\rangle$, $L_0|\tilde{\chi}_2\rangle = (a-2)|\chi_2\rangle$

[for $m \geq 3$: can replace L_{-m} by commutators

$$[L_{-p}, L_{-q}] = (-p+q)\, L_{-p-q}, \quad 1 < p, q < m$$

(no central term as $-p-q \neq 0$)

eg

$$[L_{-1}, L_{-2}] = L_{-3}$$

$$L_{-3}|\chi_3\rangle = [L_{-1}, L_{-2}]|\chi_3\rangle = L_{-1}(L_{-2}|\chi_3\rangle) - L_{-2}(L_{-1}|\chi_3\rangle)$$
$$= L_{-1}|\chi_1'\rangle + L_{-2}|\chi_2'\rangle$$

etc. ...]

see GSW p 83

$\boxed{\text{Null states}}$: (ie spurious states which are physical)

▶ Consider first: $|\psi\rangle = \underbrace{L_{-1}}_{\text{clearly spurious}} |\chi\rangle$, $L_0|\chi\rangle = (a-1)|\chi\rangle$

with $L_m|\chi\rangle = 0 \quad \forall\ m>0$.

- $(L_0 - a)|\psi\rangle = 0$ ✓ by construction

- $L_1|\psi\rangle = [L_1, L_{-1}]|\chi\rangle = 2L_0|\chi\rangle = 2(a-1)|\chi\rangle$

  $= 0$ iff $\underline{a=1}$

- $L_2|\psi\rangle = [L_2, L_{-1}]|\chi\rangle = 3L_1|\chi\rangle = 0$

So, for $\underline{a=1}$ we get an infinite set of null states

$|\psi\rangle = L_{-1}|\chi\rangle$, $L_m|\chi\rangle = 0 \quad \forall m>0$, $L_0|\chi\rangle = 0$

(generalizing the case $\sqrt{2\alpha'}\,|k;k\rangle = L_{-1}|0;k\rangle$).

► Now consider another example

$$|\psi\rangle = (L_{-2} + \gamma L_{-1}^2)|\chi_2\rangle, \quad L_m|\chi_2\rangle = 0 \ \ \forall\, m>0, \quad L_0|\chi_2\rangle = (a-2)|\chi_2\rangle$$

$$L_1|\psi\rangle = [L_1, L_{-2} + \gamma L_{-1}^2]|\chi_2\rangle = \Big(3L_{-1} + \gamma\cdot \underbrace{2(L_0L_{-1} + L_{-1}L_0)}_{[L_1, L_{-1}^2] = \underbrace{[L_1,L_{-1}]}_{2L_0}L_{-1} + L_{-1}\underbrace{[L_1,L_{-1}]}_{2L_0}}\Big)|\chi_2\rangle$$

$$= \Big(3L_{-1} + 2\gamma \underbrace{[L_0, L_{-1}]}_{L_{-1}} + 4\gamma L_{-1}(a-2)\Big)|\chi_2\rangle$$

$$= \big(3 + 2\gamma + 4\gamma(a-2)\big)L_{-1}|\chi_2\rangle$$

$$L_1|\psi\rangle = 0 \iff \gamma = \frac{3}{2(3-2a)} \qquad \left(\text{for } a=1 \text{ we have } \gamma = \frac{3}{2}\right)$$

$$L_{+2}|\psi\rangle = L_{+2}(L_{-2} + \gamma L_{-1}^2)|\chi\rangle = [L_2, L_{-2} + \gamma L_{-1}^2]|\chi\rangle$$

$$= \Big(4L_0 + \frac{D}{12}\cdot 6 + \gamma\big(\underbrace{[L_2, L_{-1}]}_{3L_1}L_{-1} + \cancel{L_{-1}[L_2, L_{-1}]}\big)\Big)|\chi\rangle$$

$$= \Big(4L_0 + \frac{1}{2}D + 3\gamma[L_1, L_{-1}]\Big)|\chi\rangle = \Big(4L_0 + \frac{1}{2}D + 3\gamma\cdot 2L_0\Big)|\chi\rangle$$

$$= \frac{1}{2}\big(4(2+3\gamma)(a-2) + D\big)|\chi\rangle$$

$$L_2|\psi\rangle = 0 \iff D = 4(2+3\gamma)(2-a)$$

So the spurious state $|\psi\rangle$ is null iff

$$\gamma = \frac{3}{2(3-2a)} \qquad D = 4(2+3\gamma)(2-a)$$

critical bosonic string:

$\underline{a=1}$: $|\psi\rangle$ null if $\gamma = \frac{3}{2}$ & $\underline{D=26}$

critical dimension

- all spurious states are null at higher levels as long as $D=26$
  ie states associated to residual gauge (conformal) symmetries generated by L_{-m} $(m>0)$ <u>decouple</u>

Summary

Open strings: by studying the low level spectrum we found

- $a > 1$ $\quad D > 26$: there are ghosts in the phys
- $a = 1$ $\quad D = 26$ critical string

  we found infinite families of null states
- $a \leq 1$ $\quad D \leq 25$ (subcritical case)

no inconsistencies at tree-level (no ghosts)
but inconsistent at the level of string loops (need to look at one loop interactions)

OCQ: $a=1$ $D=26$ needs 1-loop interactions to prove (no proof at tree level)

(no ghosts, many null states)

LCQ: · $a=1$ $D=26$ follows by requiring Lorentz spacetime invariance

· manifestly ghost free

BRST quantization: $a=1$, $D=26$ required for quantum gauge (conformal) invariance.

Too bad we have no time to go over the modern (BRST) quant.!

$$\boxed{\text{From now on} \quad a=1 \quad D=26}$$

Next · closed string spectrum

2.9 Physical states of the closed string

(at least low level $N=0,1$)

Recall

states are of the form $\prod_{i=1}^{K} \alpha_{-n_i}^{\mu_i} \prod_{j=1}^{K} \tilde{\alpha}_{-m_j}^{\nu_j} |0,\tilde{0};K\rangle, \quad n_i, m_j \geq 1)$

$\hookrightarrow$ two oscillator vacua

Physical state conditions

$$\begin{cases} (L_0 - \tilde{L}_0)|\phi\rangle = 0 & \Leftrightarrow \quad N = \tilde{N} \\ (L_0 + \tilde{L}_0 - 2a)|\phi\rangle = 0 & \Leftrightarrow -\alpha' K^2 = -4a + 2(N+\tilde{N}) \overset{a=1}{\Leftrightarrow} -\alpha' K^2 = 4(N-1) \\ L_m|\phi\rangle = 0 \;\&\; \tilde{L}_m|\phi\rangle \;\; \forall m \geq 1 & (\text{sufficient to prove this for } m=1,2) \end{cases}$$

ground state $(N=\tilde{N}=0)$: $|0,\tilde{0};K\rangle$ with $-\alpha' K^2 = -4$ (tachyon)

Level 1 states: $N = \tilde{N} = 1$

General state $|\Omega, K\rangle \equiv \underbrace{\Omega_{\mu\nu}}_{\text{spacetime two-tensor}} \alpha^{\mu}_{-1} \tilde{\alpha}^{\nu}_{-1} |0, \tilde{0}; K\rangle$

mass shell condition: $-\alpha' K^2 = 4(1-a) = 0 \implies$ massless state for the critical string

Impose Virasoro constraints: only need to impose $L_{+1}|\Omega, K\rangle = 0$
$\tilde{L}_{+1}|\Omega, K\rangle = 0$

$[L_m, \alpha^{\mu}_n] = -n\, \alpha^{\mu}_{m+n}$

$$L_{+1}|\Omega, K\rangle = \Omega_{\mu\nu} L_1 \alpha^{\mu}_{-1} \tilde{\alpha}^{\nu}_{-1} |0; K\rangle = \Omega_{\mu\nu} (\alpha^{\mu}_0 \tilde{\alpha}^{\nu}_{-1}) |0; K\rangle$$

$$\underset{\alpha^{\mu}_0 = \sqrt{\frac{\alpha'}{2}}\hat{p}^{\mu}}{=} \sqrt{\frac{\alpha'}{2}} K^{\mu} \Omega_{\mu\nu} \tilde{\alpha}^{\nu}_{-1} |0; K\rangle = 0 \iff K^{\mu} \Omega_{\mu\nu} = 0$$

similarly

$$\tilde{L}_{+1}|\Omega, K\rangle = \sqrt{\frac{\alpha'}{2}} K^{\nu} \Omega_{\mu\nu} \alpha^{\mu}_{-1} |0; K\rangle = 0 \iff K^{\nu} \Omega_{\mu\nu} = 0$$

So far $\quad |\Omega, K\rangle \equiv \Omega_{\mu\nu}\, \alpha_{-1}^{\mu}\, \tilde{\alpha}_{-1}^{\nu}\, |0; K\rangle$ $\quad$ massless state

$\hookrightarrow$ with $\quad K^{\mu}\Omega_{\mu\nu} = 0 \,, \quad K^{\nu}\Omega_{\mu\nu} = 0$

Null states: which degrees of freedom of $|\Omega; K\rangle$ are null?

$L_{-1}|\chi\rangle \,, \quad L_0|\chi\rangle = 0 \quad : \quad |\chi\rangle = \xi\cdot\tilde{\alpha}_{-1}|0,K\rangle \quad (N_{\chi}=0,\ \tilde{N}_{\chi}=1)$ $\quad$ recall $L_0 = N + \frac{1}{2}\alpha_0^2$

$\tilde{L}_{-1}|\tilde{\chi}\rangle \,, \quad \tilde{L}_0|\tilde{\chi}\rangle = 0 \quad : \quad |\tilde{\chi}\rangle = \xi'\cdot\alpha_{-1}|0,K\rangle \quad (N_{\tilde{\chi}}=1,\ \tilde{N}_{\tilde{\chi}}=0)$

$L_{-1}|\chi\rangle, \tilde{L}_{-1}|\tilde{\chi}\rangle$ are spurious by construction

$$L_{-1}|\chi\rangle = \xi_{\mu}\, L_{-1}\, \tilde{\alpha}_{-1}^{\mu}\, |0,K\rangle = \xi\cdot\tilde{\alpha}_{-1}\, \frac{1}{2}(2\,\alpha_{-1}\cdot\alpha_0)\,|0;K\rangle = \sqrt{\tfrac{\alpha'}{2}}\,(K\cdot\alpha_{-1})(\xi\cdot\tilde{\alpha}_{-1})|0;K\rangle$$

$L_{-1} = \frac{1}{2}\sum_{k=-\infty}^{\infty} \alpha_{-1-k}\cdot\alpha_k \,,$

so spurious states when $\quad \Omega_{\mu\nu} = \sqrt{\tfrac{\alpha'}{2}}\, K_{\mu}\,\xi_{\nu}$

Similarly $\quad \tilde{L}_{-1}|\tilde{\chi}\rangle = \sqrt{\tfrac{\alpha'}{2}}\,(\xi'\cdot\alpha_{-1})(K\cdot\tilde{\alpha}_{-1})|0;K\rangle \quad$ so $\quad \Omega_{\mu\nu} = \sqrt{\tfrac{\alpha'}{2}}\,\xi'_{\mu}\,K_{\nu}$

These are also physical (hence null) when $\quad K^{\mu}\Omega_{\mu\nu} = 0 \ \ \& \ \ K^{\nu}\Omega_{\mu\nu} = 0$

ie $\quad K\cdot\xi = 0 \quad \& \quad K\cdot\xi' = 0$

Thm $|\Omega, K\rangle \equiv \Omega_{\mu\nu}\, \alpha_{-1}^{\mu}\, \tilde{\alpha}_{-1}^{\nu}\, |0; K\rangle$ massless state

$\hookrightarrow$ with $K^{\mu}\Omega_{\mu\nu} = 0$, $K^{\nu}\Omega_{\mu\nu} = 0$

and two states $|\Omega, K\rangle$ & $|\hat{\Omega}, K\rangle$ are equivalent if

$$|\hat{\Omega}; K\rangle = |\Omega; K\rangle + |\Omega_{null}; K\rangle ; \quad (\Omega_{null})_{\mu\nu} = \underbrace{K_{\mu}\xi_{\nu} + K_{\nu}\xi'_{\mu}}_{K\cdot\xi = 0,\ K\cdot\xi' = 0}$$

To understand this better decompose the state into space-time Lorentz irreps

$$\Omega_{\mu\nu} = \underbrace{\gamma_{\mu\nu}}_{\text{traceless symmetric}} + \underbrace{\varphi\,\eta_{\mu\nu}}_{\text{trace}} + \underbrace{b_{\mu\nu}}_{\text{antisymmetric}}$$

$$D^2 = \tfrac{1}{2}D(D+1) - 1 + 1 + \tfrac{1}{2}D(D-1)$$

Physical states associated to $\gamma_{\mu\nu}$

$K^\mu \gamma_{\mu\nu} = 0$ ($\&$ $K^\nu \gamma_{\mu\nu} = 0$) $\quad$ $\gamma_{\mu\nu}$ transverse, symmetric, traceless

$\left(\frac{1}{2}D(D+1)-1\right) - D = \frac{1}{2}D(D-1)-1$

$S' = S$ for symmetry

with gauge invariance: $\left\{ \gamma_{\mu\nu} \longrightarrow \gamma_{\mu\nu} + S_\mu K_\nu + S_\nu K_\mu \right.$

with $S \cdot K = 0$ $\quad$ ($\Rightarrow \hat{\gamma}_{\mu\nu}$ is traceless)

degrees of freedom: $\frac{1}{2}D(D+1) - 1 - D - (D-1) = \frac{1}{2}(D-2)(D-1) - 1$

sym 2-tensor $\quad$ trace $\quad$ $K^\mu \gamma_{\mu\nu} = 0$ $\quad$ S s.t. $S \cdot K = 0$

dim of irrep of $SO(D-2)$ for massless transverse-polarised spin 2 particle

<u>Spacetime interpretation</u>: right degrees of freedom expected of a graviton in the traceless harmonic gauge $\quad$ GR

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \gamma_{\mu\nu}(x)$$

infinitesimal diffeomorphisms parametrized by S which in momentum space become

$$\gamma_{\mu\nu}(x) \sim \gamma_{\mu\nu} + \boxed{\partial_\mu S_\nu(x) + \partial_\nu S_\mu(x)}$$

$\gamma_{\mu\nu}(k) \to \gamma_{\mu\nu}(k) + S_\mu k_\nu + S_\nu k_\mu$ with $S \cdot k = 0$

Physical states associated to $b_{\mu\nu}$ — Ramond-Kalb field

$K^\mu b_{\mu\nu} = 0$ & $K^\nu b_{\mu\nu} = 0$ — $b_{\mu\nu}$ transverse, antisymmetric

$S' = -S$ for antisymmetry

with gauge invariance: $b_{\mu\nu} \to b_{\mu\nu} + \xi_\mu K_\nu - \xi_\nu K_\mu$, $\xi \cdot K = 0$

Note that ξ has redundancy $\xi_\mu \sim \xi_\mu + K_\mu \lambda$

degrees of freedom $\underbrace{\frac{1}{2}D(D-1)}_{\text{antisymmetric 2tensor}} - \underbrace{(D-1)}_{K^\mu b_{\mu\nu}=0} - \underbrace{(D-2)}_{\xi \text{ s.t. } \xi\cdot K=0,\ \xi\sim\xi+K\phi} = \underbrace{\frac{1}{2}(D-2)(D-3)}_{\text{dim of } SO(D-2) \text{ irrep corresponding to a massless 2-form}}$

In spacetime this is interpreted as a 2-form gauge field

$$b = \frac{1}{2} b_{\mu\nu}(x)\, dx^\mu \wedge dx^\nu \sim b + d\xi \,, \quad \xi \sim \xi + d\lambda$$

one form

↳ next lecture ⇝ · scalar state

· §3 Interactions