String Theory 1

Lecture #8

In summary: critical strings $(a=1, D=26)$ at low levels

NN

- ground state (both open & closed strings): tachyon

NN

- open string at level 1:

$$|\xi;k\rangle = \xi\cdot\alpha_{-1}|0;k\rangle \quad \text{is a massless state}$$

$$\xi\cdot k = 0\,, \quad |\xi;k\rangle \sim |\xi;k\rangle + |k;k\rangle$$

$$\xi_\mu \sim \xi_\mu + \partial_\mu\lambda$$

photon

longitudinal d.o.f decouples

$$L_{-1}|\chi\rangle = |k,k\rangle$$

Next

- closed string level $N=\tilde{N}=1$

2.9 Physical states of the closed string

(at least low level $N=0,1$)

Recall

states are of the form $\prod_{i=1}^{K} \alpha_{-n_i}^{\mu_i} \prod_{j=1}^{K} \tilde{\alpha}_{-m_j}^{\nu_j} |0,\tilde{0};K\rangle,\quad n_i,m_j \geq 1)$

↳ two oscillator vacua

Physical state conditions

$$\begin{cases} (L_0-\tilde{L}_0)|\phi\rangle=0 \quad \Leftrightarrow \quad N=\tilde{N} \\ (L_0+\tilde{L}_0-2a)|\phi\rangle=0 \quad \Leftrightarrow -\alpha' K^2=-4a+2(N+\tilde{N}) \overset{a=1}{\Leftrightarrow} -\alpha' K^2=4(N-1) \\ L_m|\phi\rangle=0 \ \ \& \ \ \tilde{L}_m|\phi\rangle \ \forall m\geq 1 \end{cases}$$

(sufficient to prove this for $\underline{m=1,2}$)

ground state $(N=\tilde{N}=0)$: $|0,\tilde{0};K\rangle$ with $-\alpha' K^2=-4$ (tachyon)

$\alpha_m^{\mu}|0,\tilde{0};K\rangle=0,\quad \tilde{\alpha}_m^{\mu}|0,\tilde{0};K\rangle \quad m\geq 1$

Level 1 states: $N = \tilde{N} = 1$

$N = \tilde{N}$ by the level matching condition

General state $|\Omega, K\rangle \equiv \underbrace{\Omega_{\mu\nu}}_{\text{spacetime two-tensor}} \alpha_{-1}^{\mu} \tilde{\alpha}_{-1}^{\nu} |0; K\rangle$

mass shell condition: $-\alpha' K^2 = 4(1-a) = 0 \implies$ massless state for the critical string

Impose Virasoro constraints: only need to impose $L_{+1}|\Omega, K\rangle = 0$, $\tilde{L}_{+1}|\Omega, K\rangle = 0$

$[L_m, \alpha_n^{\mu}] = -n\, \alpha_{m+n}^{\mu}$

$$L_{+1}|\Omega, K\rangle = \Omega_{\mu\nu} L_1 \alpha_{-1}^{\mu} \tilde{\alpha}_{-1}^{\nu} |0; K\rangle = \Omega_{\mu\nu} (\alpha_0^{\mu} \tilde{\alpha}_{-1}^{\nu}) |0; K\rangle$$

$$= \sqrt{\frac{\alpha'}{2}} K^{\mu} \Omega_{\mu\nu} \tilde{\alpha}_{-1}^{\nu} |0; K\rangle = 0 \iff \underline{K^{\mu} \Omega_{\mu\nu} = 0}$$

$\alpha_0^{\mu} = \sqrt{\frac{\alpha'}{2}}\, \hat{p}^{\mu}$

similarly

$$\tilde{L}_{+1}|\Omega, K\rangle = \sqrt{\frac{\alpha'}{2}} K^{\nu} \Omega_{\mu\nu} \alpha_{-1}^{\mu} |0; K\rangle = 0 \iff \underline{K^{\nu} \Omega_{\mu\nu} = 0}$$

So far $\quad |\Omega, K\rangle \equiv \Omega_{\mu\nu}\, \alpha_{-1}^{\mu}\, \tilde{\alpha}_{-1}^{\nu}\, |0; K\rangle$ massless state

$\hookrightarrow$ with $K^{\mu}\Omega_{\mu\nu} = 0$, $K^{\nu}\Omega_{\mu\nu} = 0$

Null states: which degrees of freedom of $|\Omega; K\rangle$ are null? recall $L_0 = N + \frac{1}{2}\alpha_0^2$

$L_{-1}|\chi\rangle$, $L_0|\chi\rangle = 0$: $|\chi\rangle = \xi \cdot \tilde{\alpha}_{-1}|0, K\rangle$ $(N_\chi = 0, \tilde{N}_\chi = 1)$ $\quad \tilde{L}_0|\chi\rangle = |\chi\rangle$

$\tilde{L}_{-1}|\tilde{\chi}\rangle$, $\tilde{L}_0|\tilde{\chi}\rangle = 0$: $|\tilde{\chi}\rangle = \xi' \cdot \alpha_{-1}|0, K\rangle$ $(N_{\tilde{\chi}} = 1, \tilde{N}_{\tilde{\chi}} = 0)$

$L_{-1}|\chi\rangle, \tilde{L}_{-1}|\tilde{\chi}\rangle$ are spurious by construction

$$L_{-1}|\chi\rangle = \xi_\mu L_{-1} \tilde{\alpha}_{-1}^{\mu} |0, K\rangle = \xi \cdot \tilde{\alpha}_{-1} \frac{1}{2}(2\, \alpha_{-1}\cdot\alpha_0)|0; K\rangle = \sqrt{\tfrac{\alpha'}{2}}\, (K\cdot\alpha_{-1})(\xi\cdot\tilde{\alpha}_{-1})|0; K\rangle$$

$L_{-1} = \frac{1}{2}\sum_{k=-\infty}^{\infty} \alpha_{-1-k}\cdot\alpha_k$

so $|\Omega, K\rangle$ spurious when Ω is such that $\Omega_{\mu\nu} = \sqrt{\frac{\alpha'}{2}}\, K_\mu \xi_\nu$

Similarly $\quad \tilde{L}_{-1}|\tilde{\chi}\rangle = \sqrt{\frac{\alpha'}{2}}\, (\xi'\cdot\alpha_{-1})(K\cdot\tilde{\alpha}_{-1})|0; K\rangle \quad$ so $\Omega_{\mu\nu} = \sqrt{\frac{\alpha'}{2}}\, \xi'_\mu K_\nu$

These are also physical (hence null) when $K^\mu \Omega_{\mu\nu} = 0$ & $K^\nu \Omega_{\mu\nu} = 0$

ie $\quad K\cdot\xi = 0 \quad$ & $\quad K\cdot\xi' = 0$

Thm $|\Omega, k\rangle \equiv \Omega_{\mu\nu}\, \alpha_{-1}^{\mu}\, \tilde{\alpha}_{-1}^{\nu}\, |0; k\rangle$ massless state

$\hookrightarrow$ with $k^{\mu}\Omega_{\mu\nu} = 0$, $k^{\nu}\Omega_{\mu\nu} = 0$

and two states $|\Omega, k\rangle$ & $|\hat{\Omega}, k\rangle$ are equivalent if

$|\hat{\Omega}; k\rangle = |\Omega; k\rangle + |\Omega_{null}; k\rangle$; $(\Omega_{null})_{\mu\nu} = \underline{k_{\mu}\xi_{\nu} + k_{\nu}\xi'_{\mu}}$

$k\cdot\xi = 0$, $k\cdot\xi' = 0$

To understand this better decompose the state into space-time Lorentz irreps

$$\Omega_{\mu\nu} = \underset{\text{traceless symmetric}}{\gamma_{\mu\nu}} + \underset{\text{trace}}{\varphi\,\eta_{\mu\nu}} + \underset{\text{antisymmetric}}{b_{\mu\nu}}$$

$$D^2 = \tfrac{1}{2}D(D+1) - 1 + 1 + \tfrac{1}{2}D(D-1)$$

Physical states associated to $\gamma_{\mu\nu}$

$K^\mu \gamma_{\mu\nu} = 0$ ($\&$ $K^\nu \gamma_{\mu\nu} = 0$) $\quad$ $\gamma_{\mu\nu}$ transverse, symmetric, traceless

$\left(\frac{1}{2}D(D+1)-1\right) - D = \frac{1}{2}D(D-1)-1$

with gauge invariance: $\left\{ \gamma_{\mu\nu} \longrightarrow \gamma_{\mu\nu} + \xi_\mu K_\nu + \xi_\nu K_\mu = \hat{\gamma}_{\mu\nu} \right.$

$\xi' = \xi$ for symmetry

with $\xi \cdot K = 0$ ($\Rightarrow \hat{\gamma}_{\mu\nu}$ is traceless)

degrees of freedom: $\frac{1}{2}D(D+1) - 1 - D - (D-1) = \frac{1}{2}(D-2)(D-1) - 1$

sym 2-tensor; trace; $K^\mu \gamma_{\mu\nu} = 0$; ξ st $\xi \cdot K = 0$

dim of irrep of $SO(D-2)$ for massless transverse-polarised spin 2 particle

Spacetime interpretation: right degrees of freedom expected of a graviton in the traceless harmonic gauge GR

$S_{HE} \to$ free theory for γ

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \gamma_{\mu\nu}(x)$$

$$\gamma_{\mu\nu}(x) \sim \gamma_{\mu\nu} + \boxed{\partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x)}$$

infinitesimal diffeomorphisms parametrised by ξ which in momentum space become

$\gamma_{\mu\nu}(k) \to \gamma_{\mu\nu}(k) + \xi_\mu k_\nu + \xi_\nu k_\mu$ with $\xi \cdot k = 0$

Physical states associated to $b_{\mu\nu}$ — Ramond-Kalb field

$K^\mu b_{\mu\nu} = 0$ & $K^\nu b_{\mu\nu} = 0$ — $b_{\mu\nu}$ transverse, antisymmetric

$S' = -S$ for antisymmetry

with gauge invariance: $b_{\mu\nu} \to b_{\mu\nu} + \xi_\mu K_\nu - \xi_\nu K_\mu$, $\xi \cdot K = 0$

Note that ξ has redundancy $\xi_\mu \sim \xi_\mu + K_\mu \lambda$

degrees of freedom $\underbrace{\frac{1}{2} D(D-1)}_{\text{antisymmetric 2 tensor}} - \underbrace{(D-1)}_{K^\mu b_{\mu\nu}=0} - \underbrace{(D-2)}_{\xi \text{ s.t } \xi\cdot K=0,\ \xi\sim\xi+K\lambda} = \underbrace{\frac{1}{2}(D-2)(D-3)}_{\text{dim of } SO(D-2) \text{ irrep corresponding to a massless 2-form}}$

In spacetime this is interpreted as a 2-form gauge field

$$b = \frac{1}{2} b_{\mu\nu}(x)\, dx^\mu \wedge dx^\nu \sim b + d\xi \,, \quad \underbrace{\xi}_{\text{one form}} \sim \xi + d\lambda$$

Physical state associated to the scalar   subtle !

We have the state $\frac{1}{2}\varphi\,\alpha_{-1}\cdot\tilde{\alpha}_{-1}|0,K\rangle$ $\Omega_{\mu\nu} = \frac{1}{2}\varphi\,\eta_{\mu\nu}$

Virasoro constraints: $K^\mu \Omega_{\mu\nu} = 0 \Rightarrow \varphi K_\nu = 0$

$\Rightarrow K^\mu = 0$ (eliminates all non-zero momentum modes !)

which implies that the field is a constant so no degrees of freedom.

This is not right: $L_1(\alpha_{-1}\cdot\tilde{\alpha}_{-1}|0;K\rangle) \neq 0$ (and similarly for $\tilde{L}_1$)

so this is $\boxed{\text{not}}$ a physical state!

$\boxed{\text{We can}}$ however construct a level 1 physical state which is a spacetime scalar.

Given two vectors S & $\tilde{S}$, define the level 1 state

$$|\varphi_{S,\tilde{S}},K\rangle = \varphi\left[(S\cdot\alpha_{-1})(\alpha_0\cdot\tilde{\alpha}_{-1}) + (\alpha_0\cdot\alpha_{-1})(\tilde{S}\cdot\tilde{\alpha}_{-1}) + \alpha_{-1}\cdot\tilde{\alpha}_{-1}\right]|0;K\rangle$$

Now impose the Virasoro constraints.

$[L_m, \alpha^\mu_n] = -n\,\alpha^\mu_{m+n}$

$$L_1|\varphi_{s,\tilde{s}};K\rangle = \varphi\, L_1\, \alpha^\mu_{-1}\left[\zeta_\mu(\alpha_0\cdot\tilde{\alpha}_{-1}) + \alpha_{0\mu}(\tilde{\zeta}\cdot\tilde{\alpha}_{-1}) + \alpha_{-1\mu}\right]|0;K\rangle$$

$$= \varphi\,(\alpha^\mu_0 + \cancel{\alpha^\mu_{-1}L_1})\left[\zeta_\mu(\alpha_0\cdot\tilde{\alpha}_{-1}) + \alpha_{0\mu}(\tilde{\zeta}\cdot\tilde{\alpha}_{-1}) + \tilde{\alpha}_{-1\mu}\right]|0;K\rangle$$

$$= \varphi\left[\zeta\cdot\alpha_0\,(\alpha_0\cdot\tilde{\alpha}_{-1}) + \frac{\alpha'}{2}\cancelto{0}{K^2}(\tilde{\zeta}\cdot\tilde{\alpha}_{-1}) + \alpha_0\cdot\tilde{\alpha}_{-1\mu}\right]|0;K\rangle$$

$$= \varphi\left[\underline{\zeta\cdot\alpha_0 + 1}\right]\alpha_0\cdot\tilde{\alpha}_{-1}|0;K\rangle$$

$$= 0 \qquad \text{iff} \qquad \underline{\zeta\cdot K = -\sqrt{\frac{2}{\alpha'}}}$$

Similarly: $$\tilde{L}_1|\varphi_{s,\tilde{s}};K\rangle = \varphi\left[\tilde{\zeta}\cdot\alpha_0 + 1\right]\alpha_0\cdot\alpha_{-1}|0;K\rangle$$

$$= 0 \qquad \text{iff} \qquad \tilde{\zeta}\cdot K = -\sqrt{\frac{2}{\alpha'}}$$

So $|\varphi_{s,\tilde{s}};K\rangle$ is physical iff $\zeta\cdot K = \tilde{\zeta}\cdot K = -\sqrt{\frac{2}{\alpha'}}$

Despite the fact that it seems to depend on ζ & $\tilde{\zeta}$ this state corresponds to a scalar.

To see this consider the dependence on ζ (and $\tilde{\zeta}$)

$$|\varphi_{s,\tilde{\zeta}};K\rangle - |\varphi_{s',\tilde{\zeta}};K\rangle = \varphi\,(\zeta-\zeta')\cdot\alpha_{-1}\,(\alpha_0\cdot\tilde{\alpha}_{-1})|0;K\rangle$$

OTOH
$$\tilde{L}_{-1}\big(\underbrace{(\zeta-\zeta')\cdot\alpha_{-1}|0;K\rangle}\big) = \big((\zeta-\zeta')\cdot\alpha_{-1}\big)(\tilde{\alpha}_{-1}\cdot\alpha_0)|0;K\rangle$$

$$\tilde{L}_{-1} = \frac{1}{2}\sum_k \tilde{\alpha}_{-1-k}\cdot\tilde{\alpha}_k$$

$$|\varphi_{s,\tilde{\zeta}};K\rangle - |\varphi_{s',\tilde{\zeta}};K\rangle = \varphi\,\tilde{L}_{-1}\big((\zeta-\zeta')\cdot\alpha_{-1}|0;K\rangle\big)$$ pure gauge

and similarly for $\tilde{\zeta}$ & $\tilde{\zeta}'$. Then

$$\left\{\begin{array}{l} |\varphi_{s,\tilde{\zeta}};K\rangle - |\varphi_{s',\tilde{\zeta}};K\rangle \\ |\varphi_{s,\tilde{\zeta}};K\rangle - |\varphi_{s,\tilde{\zeta}'};K\rangle \end{array}\right.$$ are spurious

They are also null
$$\left\langle\begin{array}{ll} L_1: & (\zeta-\zeta')\cdot K = 0 \quad \left(\text{as } \zeta\cdot K = -\sqrt{\tfrac{2}{K}} = \zeta'\cdot K\right) \\ \tilde{L}_1: & (\tilde{\zeta}-\tilde{\zeta}')\cdot K = 0 \quad \left(\text{as } \tilde{\zeta}\cdot K = -\sqrt{\tfrac{2}{\alpha'}}\ \tilde{\zeta}'\cdot K\right) \end{array}\right.$$

Hence we identify a state with ς ($\tilde{\varsigma}$) and ς' ($\tilde{\varsigma}'$)

$$|\varphi_{s,\tilde{\varsigma}};K\rangle \sim |\varphi_{s',\tilde{\varsigma}};K\rangle + \varphi\,\tilde{L}_{-1}\left((\varsigma-\varsigma')\cdot\alpha_{-1}\right)|0;K\rangle$$

$$|\varphi_{s,\tilde{\varsigma}};K\rangle \sim |\varphi_{s,\tilde{\varsigma}'};K\rangle + \varphi\,L_{-1}\left((\tilde{\varsigma}-\tilde{\varsigma}')\cdot\tilde{\alpha}_{-1}\right)|0;K\rangle$$

ie indep of choice of ς,ς' up to gauge equivalence

This scalar φ is called the dilaton:

it plays a very important role in the context of string interactions.

<u>Final remark</u> : We have gone over the OCQ of the Polyakov action which describes a free field theory in 1+1 dims

In this scheme we discussed how to construct the Hilbert space of physical states. The consistency of the quantum spectrum (no ghosts) requires

$$a = 1 \quad \& \quad D = 26$$

(together with some input from the interacting string to rule out $0 \leq a \leq 1$ & $1 \leq D \leq 25$)

Next

- interactions & vertex operators

Chapter 3 Interactions

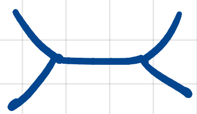
3.1 Generalities

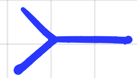
QFT:

- to understand interactions one adds non-linear terms to the action

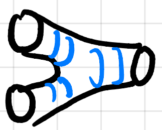
  ↳ · doesn't work for the string because anything you try to add breaks gauge invariance.

- scattering amplitudes → Feynman diagrams

  eg  etc.

  interactions encoded at vertices say  3point int.

  ↳ in string theory this is replaced by, for instance

   or  : no such vertices!

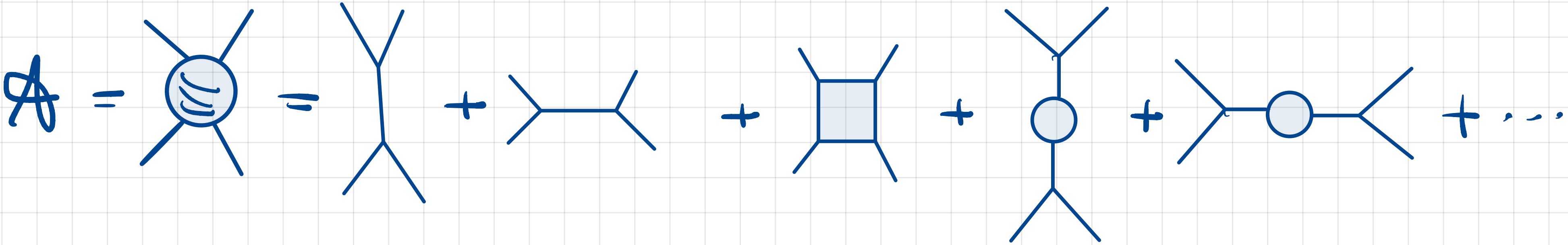
QFT:

- to understand interactions one adds non-linear terms to the action

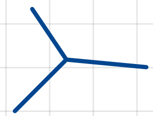
  (↳ · doesn't work for the string because anything you try to add breaks gauge invariance.)

(asymptotic) series for scattering amplitudes: sum over all possible Feynman diagrams with fixed external legs (& integrate over the positions of all vertices)

QFT Feynman diagrams



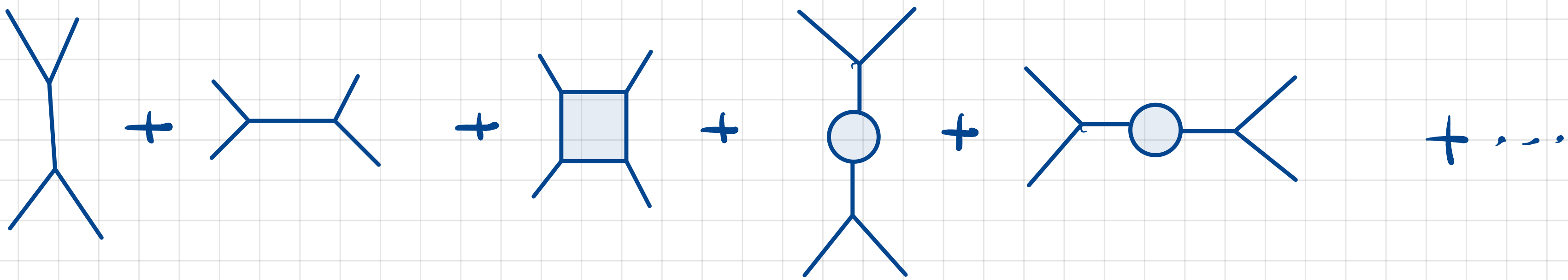
· interaction encoded at vertices



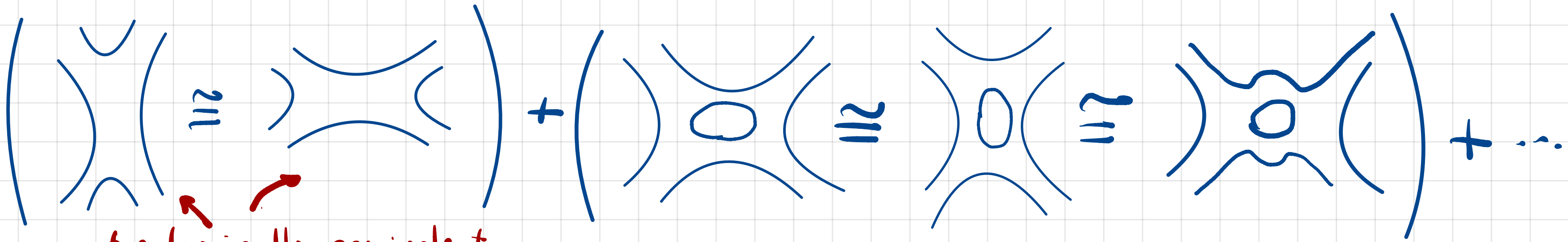
3-point function

(read this off a Lagrangian)

QFT
Feynman diagrams



string theory



topologically equivalent

sums over diagrams of strings branching and joining

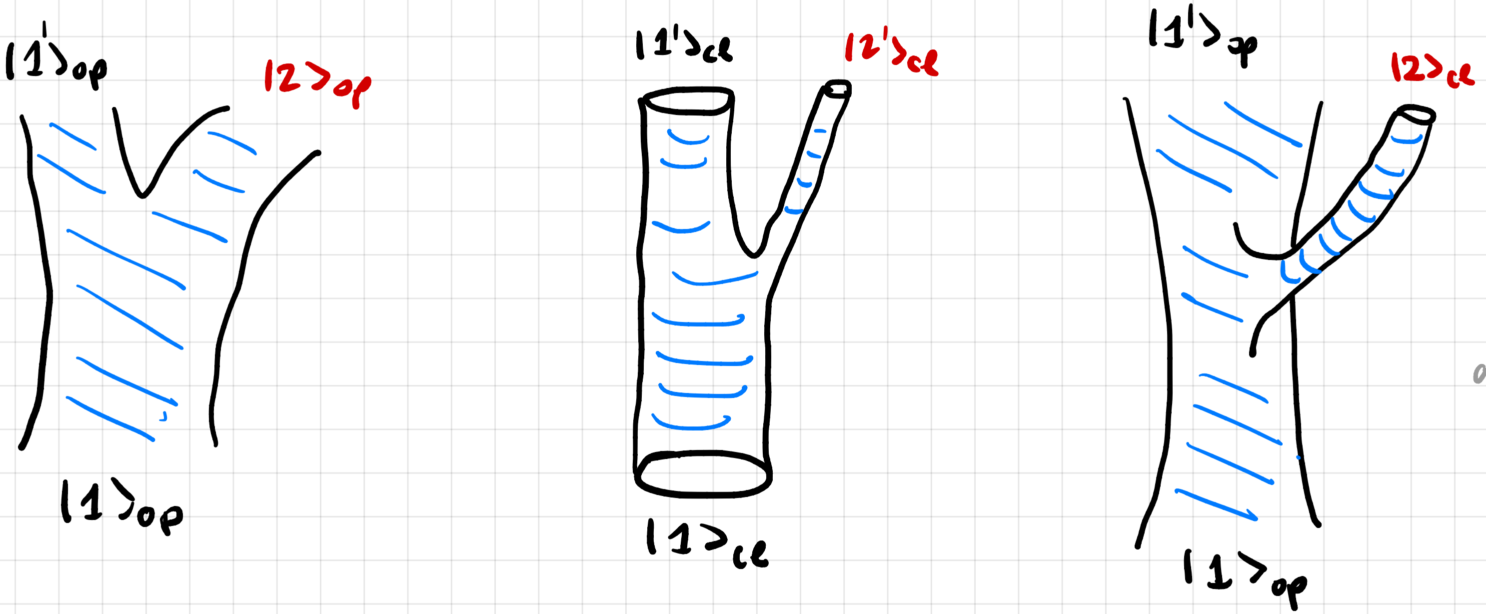
- each diagram locally looks like a 2dim manifold
- different particle diagrams → topologically equiv string diagrams
- solid black lines
  - OS: boundaries of WS
  - CS: only ends of the WS are boundaries

In string theory:

We want to compute for example the amplitude of a given configuration of quantised strings at an initial time to evolve into a new configuration at a later time

$\mathcal{A}$: conf of q-strings $(\Gamma_i) \longrightarrow$ conf of q-strings (Γ_f)

This means we need to describe the branching and joining of quantised strings



open string $|1\rangle_{op}$ which splits into two $|1'\rangle_{op}$ & $|2\rangle_{op}$

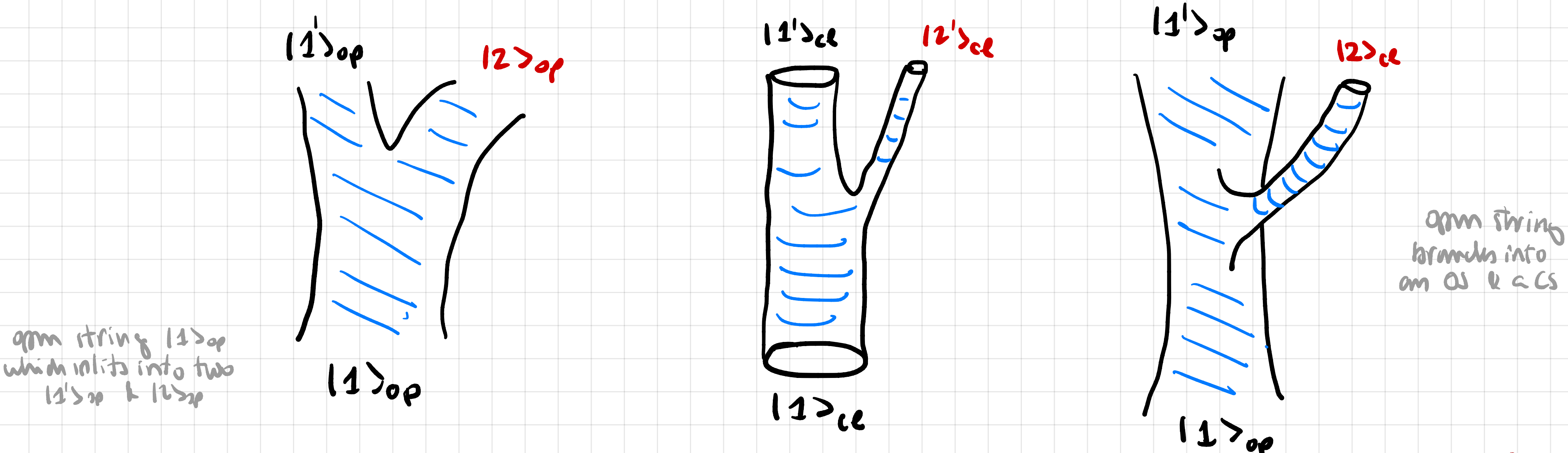
open string branches into an OS & a CS

In string theory :

We want to compute for example the amplitude of a given configuration of quantised strings at an initial time to evolve into a new configuration at a later time

A: conf of q-strings $(T_i) \longrightarrow$ conf of q-strings (T_f)

This means we need to describe the branching and joining of quantised strings



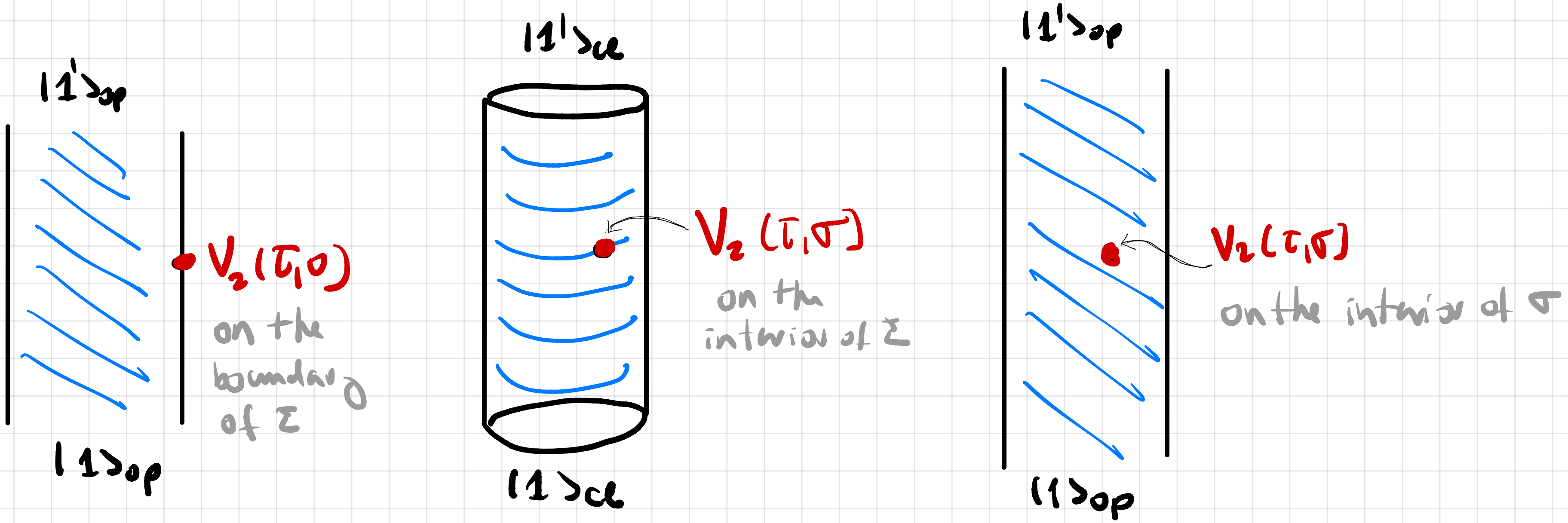
Problem : it is not known how to do this .... !

all we have at hand is the phys/the null physical states of a string

So, we need to work with the quantised formalism we developed.

Suppose $|a\rangle$ is a physical state which is emitted / absorbed.

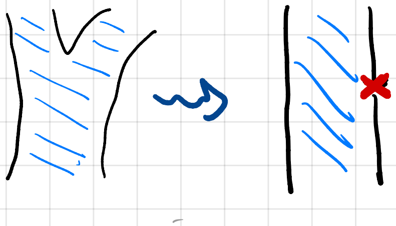
We describe the emission\absorption of a quantum state (say $|a\rangle$) from a fixed string WS by the action of a local operator or Vertex operator.



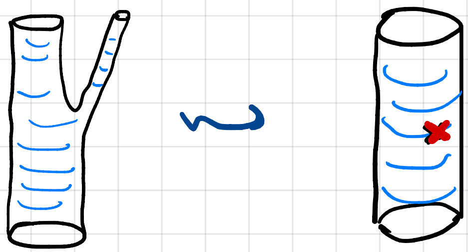
How ?

Tools : we will describe this process by

- Wick rotation of the world sheet
  (Lorentzian signature → Euclidean signature)
- conformal transformation



or



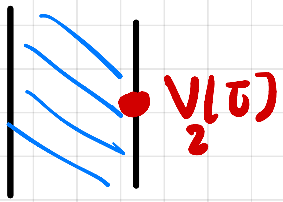
3.2 Vertex operators : Introduction

Require: $\boxed{\text{two}}$ key requirements

① time evolution on the worldsheet is a gauge transformation ⇒ position of the vertex operator should not be meaningful.

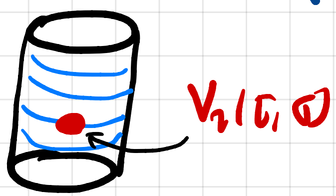
open string vertex operator $\xrightarrow{\text{what matters is}}$ $\int d\tau \underbrace{V_2(\tau)}_{\downarrow}$

inserted on the boundary



closed string vertex operator $\rightarrow$ $\int d\tau\, d\sigma \underbrace{V_2(\tau, \sigma)}_{\downarrow}$

inserted on the interior



② absorption & emission vertex operators to map

$$\mathcal{H}_{phys} \longrightarrow \mathcal{H}_{phys}$$

and

$$\mathcal{H}_{null} \longrightarrow \mathcal{H}_{null}$$

Let's see how this works for $\boxed{\text{the open strings first}}$

Consider the action of a vertex operator on a physical state $|\phi_{phys}\rangle$, that is

(local) boundary operator at (say) $\sigma = 0$

$$\int d\tau \; V(\tau, 0) \; |\phi_{phys}\rangle$$

$\mathcal{H}_{phys} \xrightarrow{V} \mathcal{H}_{phys}$ Applying the physical state conditions:

$$m \geq 0 : \quad (L_m - a\,\delta_{m,0}) \left[\int d\bar{\tau}\; V(\bar{\tau},0) \,|\phi_{phys}\rangle \right] = \int d\bar{\tau}\; [L_m, V(\bar{\tau},0)]\,|\phi_{phys}\rangle$$

$$= 0 \quad \text{if} \quad \boxed{[L_m, V(\bar{\tau},0)] = \partial_{\bar{\tau}}(\underset{\smile}{\text{local}}\ \text{operator}) \qquad m \geq 0}$$

$\int d\bar{\tau}\, \partial_{\bar{\tau}}(\cdots) = \underbrace{(\ )_f - (\ )_i}$

local operator at past & future infinity vanish asymptotically

$\mathcal{H}_{null} \xrightarrow{V} \mathcal{H}_{null}$

$$m \geq 1 : \underbrace{\int d\bar{\tau}\; V(\bar{\tau},0)\, \overbrace{[L_{-m}\,|\phi\rangle]}^{\text{null}}} = \int d\bar{\tau} \left\{ \underbrace{[V(\bar{\tau},0), L_{-m}]\,|\phi\rangle} + \underbrace{L_{-m}\, V(\bar{\tau},0)\,|\phi\rangle}_{\text{null}} \right\}$$

need this to vanish up to a total derivative

$\langle \psi_{phys} | L_{-m} V(\bar{\tau},0) | \phi\rangle = 0$

This is null if $\boxed{[V(\bar{\tau},0), L_{-m}] = \partial_{\bar{\tau}}(\text{local op}) \;, \quad m \geq 1}$

We have seen that if $[L_m, V(\tau, 0)] = \partial_\tau(\text{local op}) \ \forall m$

then $\mathcal{H}_{\text{phys}} \xleftrightarrow{V} \mathcal{H}_{\text{phys}}$, $\mathcal{H}_{\text{null}} \xleftrightarrow{V} \mathcal{H}_{\text{null}}$

The converse is also true: suppose $V(\tau)$ is an operator st $\mathcal{H}_{\text{phys}} \xrightarrow{V} \mathcal{H}_{\text{phys}}$, $\mathcal{H}_{\text{null}} \xrightarrow{V} \mathcal{H}_{\text{null}}$.

Let $|\phi\rangle$ be a physical (or null) state. Then as $V(\tau)|\phi\rangle$ is physical (or null), we have

$$0 = \int d\tau \, (L_m - \delta_{m,0}) V(\tau) |\phi\rangle = \int d\tau \, [L_m - \delta_{m,0}, V(\tau)] |\phi\rangle$$

$$= \int d\tau \, [L_m, V(\tau)] |\phi\rangle \qquad \forall m$$

Therefore $[L_m, V(\tau)] = \partial_\tau(\text{local operator})$ //

What happens to VOs under conformal transformation?

conformal transformations of the open string are of the form $\tau \to \tilde{\tau}(\tau)$. We want

$\int V(\tau,0)\,d\tau$ to be invariant under $\tau \to \tilde{\tau}$ (as emitted / absorbed state must be inv under $\tau \to \tilde{\tau}$)

$$\int V(\tau,0)\,d\tau \longrightarrow \int \tilde{V}(\tilde{\tau},0)\,d\tilde{\tau} = \int V(\tau,0)\,d\tau$$

$$= \int V(\tau,0)\,\frac{d\tau}{d\tilde{\tau}}\,d\tilde{\tau}$$

so require $\quad V(\tau,0) \longrightarrow \tilde{V}(\tilde{\tau},0) = V(\tau,0)\,\dfrac{d\tau}{d\tilde{\tau}}$ under C-transfs

Definition: an operator $A(\tau)$ is a primary operator of weight h if under the transformation $\tau \to \tilde{\tau}(\tau)$ it transforms as

$$\boxed{A(\tau) \longrightarrow \tilde{A}(\tilde{\tau}) = A(\tau)\left(\frac{d\tau}{d\tilde{\tau}}\right)^h}$$

For an operator with $h=1$

$$\int \tilde{A}(\tilde{\tau})\, d\tilde{\tau} = \int A(\tau) \frac{d\tau}{d\tilde{\tau}} d\tilde{\tau} = \int A(\tau)\, d\tau$$

ie the integrated operator is conformally invariant.

For infinitesimal transformations $\tau \longrightarrow \tilde{\tau} = \tau + \epsilon(\tau)$

we have

$$A(\tau) \longrightarrow \tilde{A}(\tilde{\tau}) = A(\tau)\left(1 - h\frac{d\epsilon}{d\tau}\right) \qquad \left(\frac{d\tau}{d\tilde{\tau}}\right)^h$$

OTOH (LHS)

$$\tilde{A}(\tilde{\tau}) = \tilde{A}(\tau+\epsilon) \underset{\text{taylor}}{=} \tilde{A}(\tau) + \epsilon\,\partial_\tau A(\tau) + \mathcal{O}(\epsilon^2)$$

Then we find the variation of A

$$\delta A(\tau) = \tilde{A}(\tau) - A(\tau) = -\epsilon\,\partial_\tau A - h(\partial_\tau \epsilon) A$$

$$= -\partial_\tau(\epsilon A) - (h-1)\,\partial_\tau \epsilon\, A$$

This is a total derivative when $h = 1$

Under a conformal transformation (From now on set $\ell = \begin{cases} 2\pi & CS \\ \pi & OS \end{cases}$)

$$\tau \longrightarrow \tilde{\tau} = \tau + \epsilon(\tau) \quad \text{with} \quad \epsilon = -e^{im\tau}$$

ϕ varies as

$$\delta\phi(\tau) = e^{im\tau}\left(-i\,\partial_\tau \phi + h\,m\,\phi\right) \rightarrow [L_m, \phi(\tau)]$$

Recall
$\{L_m, \phi(\tau)\}_{PB} = \delta_{\epsilon_m}\phi(\tau)$
PS 1

So the action of the Virasoro operators on the ops ϕ is

$$\boxed{[L_m, \phi(\tau)] = e^{im\tau}(-i\partial_\tau + mh)\,\phi(\tau)}$$

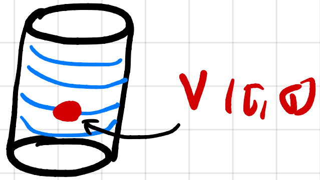
Equivalently, this is the condition for ϕ to have conformal weight h.

For $h=1$: $[L_m, \phi(\tau)] = \partial_\tau\left(-i\,e^{im\tau}\phi(\tau)\right)$ ✓ TD

The problem is to identify primaries of weight 1 which correspond to the emission/absorption of physical states in the string Hilbert space. Then we use this to compute string amplitudes!

$\boxed{\text{closed strings}}$: analogous

closed string vertex operator $\rightarrow$ $\int d\tau\, d\sigma \underbrace{V(\tau,\sigma)}_{\text{inserted on the interior}}$



A primary operator of dimension $(h, \tilde{h})$ is an operator transforming under conformal transfs of the WS according to

$$\phi(\xi^+, \xi^-) \longrightarrow \tilde{\phi}(\tilde{\xi}^+, \tilde{\xi}^-) = \left(\frac{d\xi^+}{d\tilde{\xi}^+}\right)^h \left(\frac{d\xi^-}{d\tilde{\xi}^-}\right)^{\tilde{h}} \phi(\xi^+, \xi^-)$$

The corresponding infinitesimal transformations are

$$\delta\phi(\tau, \sigma) = -\partial_+(\tilde{\epsilon}\phi) - (\tilde{h}-1)(\partial_+\tilde{\epsilon})\phi - \partial_-(\epsilon\phi) - (h-1)(\partial_-\epsilon)\phi$$

This is a total derivative if $h = \tilde{h} = 1$

For $\epsilon_n^{\pm} = \frac{i}{2} e^{2im\xi^{\pm}}$ this gives the action of L_m:

$$[L_m, \phi(\xi^{\pm})] = \frac{1}{2} e^{2im\xi^+}(-i\partial_+ + 2mh)\phi(\xi^{\pm})$$

$$[\tilde{L}_m, \phi(\xi^{\pm})] = \frac{1}{2} e^{2im\xi^-}(-i\partial_- + 2m\tilde{h})\phi(\xi^{\pm})$$

↳ Next

- state ↔ vertex correspondence

3.3 Vertex operators for the open string

on the WS

on the boundary of Σ

Consider the boundary scalar operator $X^\mu(\tau, 0)$:

$$X^\mu(\tau,0) = x^\mu + 2\alpha' p^\mu + i\sqrt{2\alpha'}\sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-in\tau}$$

Conformal transformations:

$$[L_m, X^\mu(\tau,0)] = -ie^{im\tau} \frac{d}{d\tau}(X^\mu(\tau,0))$$

$$[L_m, \alpha_n^\mu] = -n\,\alpha_{m+n}^\mu$$

$\Rightarrow$ $h=0$ (indeed a WS scalar)

so $X^\mu(\tau)$ is not a VO

$$\left([L_m, A(\tau)] = e^{im\tau}(-i\partial_\tau + mh)A(\tau)\right)$$