String Theory 1

Lecture #10

3 Interactions

3.1 Generalities

3.2 Vertex operators: introduction

3.3 Vertex operators: open string

3.4 The state vertex correspondence open strings

3.5 Vertex operator: closed string

3.6 3-point interactions

3.7 4-point tachyon amplitude

3.8 Comments on the general picture

Last lecture

state-vertex correspondence $|\psi\rangle \leftrightarrow V_\psi$

open string:

incomming state

$$|\psi\rangle = \lim_{z\to 0} \frac{1}{z} V_\psi(-i\log z)\,|0;0\rangle$$

$\bar{\tau} = -it$

$z = e^{i\bar{\tau}} = e^{t}$

outgoing state

$$\langle\phi| = \lim_{z\to\infty} z\,\langle 0;0|V_\psi(-i\log z)$$

closed string $z = e^{i\xi^{+}}$ $\quad \bar{z} = e^{i\xi^{-}}$ $\quad z\bar{z} = e^{2t}$ $\quad \frac{z}{\bar{z}} = e^{2i\sigma}$

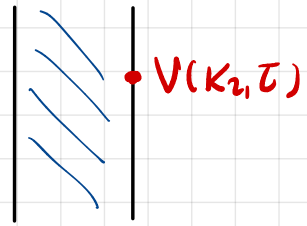
$$|\psi\rangle = \lim_{z\to 0} (z\bar{z})^{-1}\, V_\psi(-it\,,\sigma)\,|0;0\rangle$$

3.6 Three point scattering

tree level

3-point open string interaction (open string)

$\uparrow \langle 0; -k_3|$ outgoing os tachyon



absorption of os tachyon at τ $(\sigma = 0)$

$$= A_3^{op}(k_1, k_2, k_3)$$

3-tachyon interaction

$\uparrow |0; k_1\rangle$ incoming open string tachyon

divide out by an infinite volume of the gauge group

$$= g_o \int_{-\infty}^{\infty} d\tau \, \langle 0; -k_3 | V_T(\tau; k_2) | 0; k_1 \rangle \Big/ \mathrm{Vol(conf.)}$$

open string coupling constant

value of τ is immaterial

expect this because τ-translations correspond to a residual gauge invariance

$$\mathcal{A}_3^{\infty}(k_1,k_2,k_3) = g_0 \int d\tau \, \underbrace{\langle 0;-k_3| e^{i\tau L_0}}_{\langle 0;-k_3|e^{i\tau}} V_T(0;k_2) \underbrace{e^{-i\tau L_0} |0;k_1\rangle}_{e^{-i\tau}|0;k_1\rangle} \Big/ \mathrm{Vol(conf.)}$$

$\leftarrow L_0|\psi\rangle_{phys} = |\psi\rangle_{phys}$

$$= g_0 \, \langle 0;-k_3| \underbrace{V_T(0;k_2)}_{\underbrace{e^{k\cdot\sum_{n=1}^{\infty}\frac{\alpha_{-m}}{m}}}_{\underbrace{1+\text{creation operators}}_{1}} : e^{ik\cdot x(0)} : \underbrace{e^{-k\sum_{n=1}^{\infty}\frac{\alpha_n}{n}}}_{\underbrace{1+\text{annihilation operators}}_{1}}} |0;k_1\rangle \left(\int d\tau \Big/ \mathrm{Vol(conf.)} \right)$$

$$= g_0 \, \langle 0;-k_3| e^{ik_2\cdot x(0)} |0;k_1\rangle \left(\int d\tau \Big/ \mathrm{Vol(conf.)} \right)$$

$$\mathcal{A}_3^{\varphi}(k_1,k_2,k_3) = g_0 \langle 0;-k_3 | 0; k_1+k_2 \rangle \left(\int d\tau \middle/ \mathrm{Vol(conf.)} \right)$$

$$= g_0\, \delta(k_1+k_2+k_3) \left[\int_{-\infty}^{\infty} d\tau \middle/ \mathrm{Vol(conf.)} \right]$$

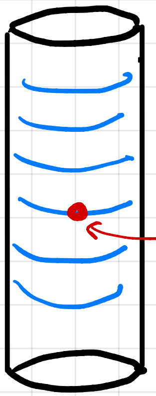
alternatively: gauge fix at $\tau = 0$
(more later: use Fadeev-Popov procedure)

$$\boxed{\mathcal{A}_3^{\varphi}(k_1,k_2,k_3) = g_0\, \delta(k_1+k_2+k_3)}$$

expected result from Feynman rules for a 3-point vertex interaction

tree point tree level closed string diagrams
(tachyon absorbing a tachyon)

$\langle 0; -k_3|$



$|0; k_1\rangle$

$V_T^{cl}(k_2, \xi^{+})$ $= A_3^{cl}(k_1, k_2, k_3)$

$$= g_{cl} \int d^2\xi^{\pm} \, \langle 0; -k_3| \, V_T^{cl}(k_2, \xi^{\pm}) \, |0; k_1\rangle \Big/ \mathrm{Vol(conf.)}$$

closed string coupling

Recall

$$V_T^{cl}(\xi^\pm, k) = e^{2i\xi^- L_0 + 2i\xi^+ \tilde{L}_0} V_T^{cl}(0,k)\, e^{-2i\xi^- L_0 - 2i\xi^+ \tilde{L}_0}$$

Hamiltonian: $H \sim (L_0 + \tilde{L}_0) \sim \partial_\tau$

$L_0 - \tilde{L}_0 \sim \partial_\sigma$

$$A_3^{cl}(k_1, k_2, k_3)$$

$L_0 |\text{phys}\rangle = |\text{phys}\rangle$

$$= g_{cl} \int d^2\xi^\pm \underbrace{\langle 0; -k_3 | e^{2i\xi^- L_0 + 2i\xi^+ \tilde{L}_0}}_{\langle 0;-k_3| e^{2i\xi^- + 2i\xi^+}} V_T^{cl}(0,k)\, \underbrace{e^{-2i\xi^- L_0 - 2i\xi^+ \tilde{L}_0} |0; k_1\rangle}_{e^{-2i\xi^- - 2i\xi^+} |0;k_1\rangle} \Big/ \text{Vol(conf.)}$$

$$= g_{cl} \int d^2\xi^\pm \langle 0; -k_3 | \underbrace{V_T^{cl}(k_2, 0)}_{(1+\text{creation}) : e^{ik_2 \cdot x} : (1+\text{annihilation})} |0; k_1\rangle \Big/ \text{Vol(conf.)}$$

$$A_3^{cl}(k_1,k_2,k_3) = g_{cl} \int d^2\xi^{\pm} \langle 0; -k_3 | 0; k_1+k_2 \rangle \Big/ \mathrm{Vol(conf.)}$$

$$= g_{cl}\, \delta(k_1+k_2+k_3) \left(\int d^2\xi^{\pm} \Big/ \mathrm{Vol(conf.)} \right)$$

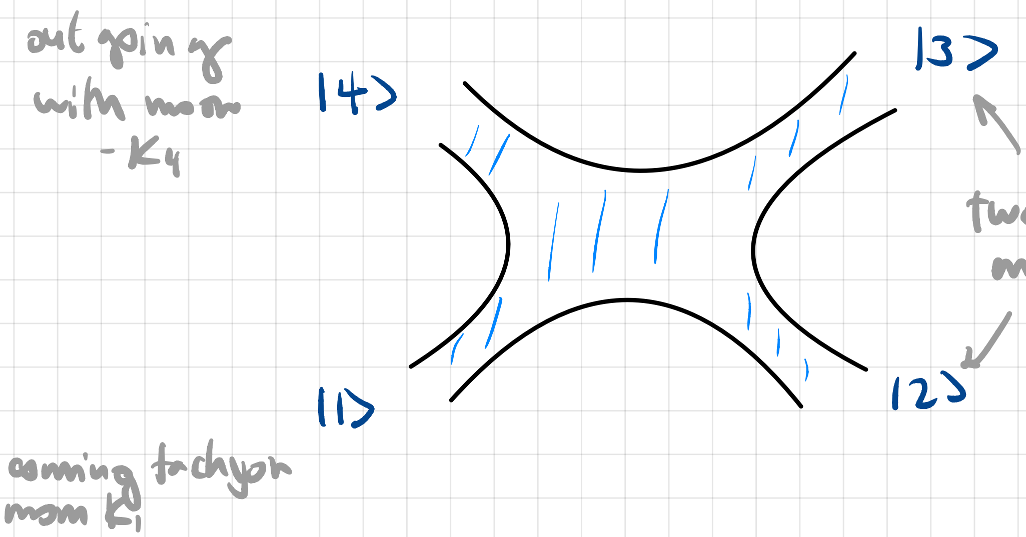
$$\boxed{A_3^{cl}(k_1,k_2,k_3) = g_{cl}\, \delta(k_1+k_2+k_3)}$$

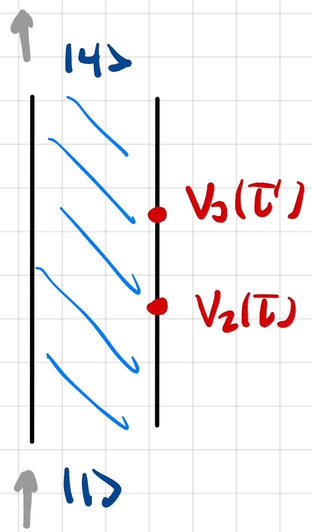
PS 3 3 point amplitudes with tachyons & level-one states

3.7 The Veneziano amplitude

or 4-point tachyon amplitude

[This section: features of amplitudes than can be generalized to n-point amplitudes in open & closed string amplitudes]





$$A_4(k_1,k_2,k_3,k_4) = g_0^2 \int_{\tau'>\tau} d\tau\, d\tau' \langle 0;-k_4|\, V_3(\tau') V_2(\tau)\, |0;k_1\rangle \Big/ \mathrm{Vol(conf)}$$

joining two 3-point diagrams

Use the residual gauge freedom to fix $\tau' = 0$

$$A_4(k_1,k_2,k_3,k_4) = g_0^2 \int_{-\infty}^{0} d\tau \, \langle 0; -k_4 | \, V_3(0) \, V_2(\tau) \, | 0; k_1 \rangle$$

(so no need to divide by Vol(conf))

$$= g_0^2 \int_{-\infty}^{0} d\tau \, \langle 0, -k_4 | \, V_3(0) \, e^{i\tau L_0} \, V_2(0) \, \underbrace{e^{-i\tau L_0} | 0; k_1 \rangle}_{e^{-i\tau} | 0; k_1 \rangle}$$

$$= g_0^2 \int_{-\infty}^{0} d\tau \, \langle 0; -k_4 | \, V_3(0) \, e^{i\tau(L_0 - 1)} \, V_2(0) \, | 0; k_1 \rangle$$

$$= g_0^2 \, \langle 0; -k_4 | \, V_3(0) \, \underbrace{\left(\int_{-\infty}^{0} d\tau \, e^{i\tau(L_0 - 1)} \right)}_{\text{how do we deal with this?}} V_2(0) \, | 0; k_1 \rangle$$

$\rightarrow$ propagator!

$$\Phi_4(k_1,k_2,k_3,k_4) = g_0^2 \langle 0;-k_4|V_3(0)\left(\int_{-\infty}^{0} d\tau\, e^{i\tau(L_0-1)}\right)V_2(0)|0;k_1\rangle$$

$$\left[\ \overset{?}{=}\ g_0^2 \langle 0;-k_4|V_3(0)\left(\frac{-i+\text{osc. modes}}{L_0-1}\right)V_2(0)|0;k_1\rangle\ \right]$$

poles

To obtain a convergent diagram, we consider instead

$$\int_{-\infty}^{0} d\tau\, e^{i\tau(L_0-1-i\epsilon)}$$

and we "rotate" the contour: $t = i\tau$ (Wick rotation)

$$\int_{-\infty}^{0} d\tau\, e^{i\tau(L_0-1-i\epsilon)} \longrightarrow \int_{-\infty}^{0} dt\, e^{t(L_0-1)}$$

$$\Phi_4(k_1,k_2,k_3,k_4) = g_0^2 \langle 0;-k_4 | V_3(0) \left(\int_{-\infty}^{0} d\bar{\tau}\, e^{i\bar{\tau}(L_0-1)} \right) V_2(0) | 0;k \rangle$$

$$= g_0^2 \langle 0;-k_4 | V_3(0) \left(\int_{-\infty}^{0} d\bar{\tau}\, e^{i\bar{\tau}(L_0-1-i\epsilon)} \right) V_2(0) | 0;k_1 \rangle$$

"rotate" contour $\bar{\tau} = -it$

$$= g_0^2 \langle 0;-k_4 | V_3(0) \left(\int_{-\infty}^{0} dt\, e^{t(L_0-1)} \right) V_2(0) | 0;k_1 \rangle$$

pull out integral

$$= g_0^2 \int_{-\infty}^{0} dt\, \langle 0;-k_4 | V_3(0) \underbrace{e^{t(L_0-1)} V_2(0) | 0;k_1 \rangle}_{\text{back track (but now with } t \text{ instead of } i\bar{\tau})}$$

$$= g_0^2 \int_{-\infty}^{0} dt\, \langle 0;-k_4 | V_3(0) \underbrace{e^{tL_0} V_2(0) e^{-tL_0}}_{\text{Vertex operator at } -it \text{ ie } V_2(-it)} | 0;k_1 \rangle$$

$$\mathcal{A}_4(k_1,k_2,k_3,k_4) = g_0^2 \int_{-\infty}^{0} dt \, \langle 0;-k_4 | V_3(0) \, V_2(-it) \, | 0; k_1 \rangle$$

- Compare with original expression

$$\mathcal{A}_4(k_1,k_2,k_3,k_4) = g_0^2 \int_{-\infty}^{0} d\tau \, \langle 0;-k_4 | \, V_3(0) \, V_2(\tau) \, | 0; k_1 \rangle$$

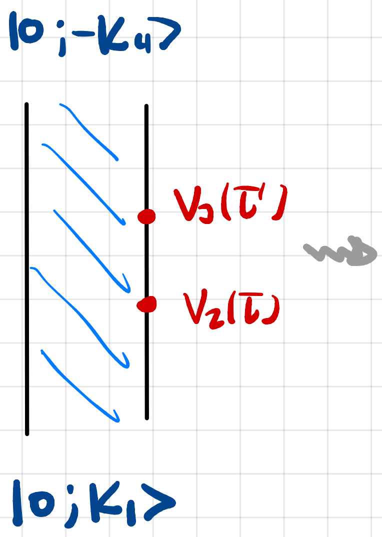
Now all operators are in Euclidean worldsheet time

(We have manipulated amplitude st we work instead in Euclidean WS time)

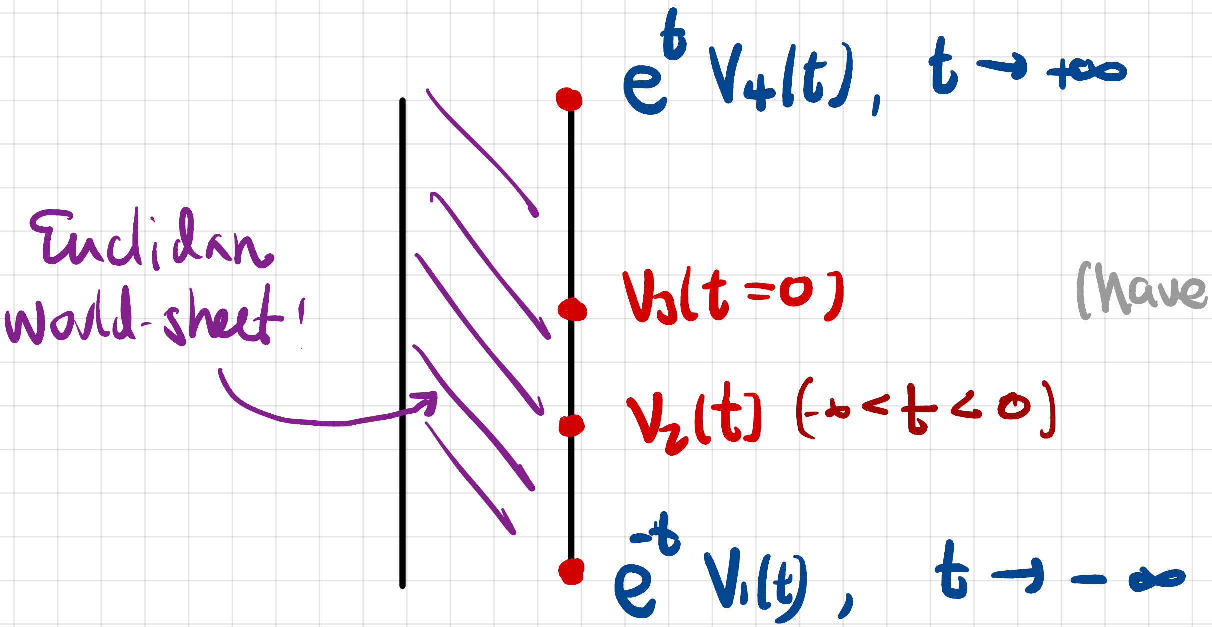
↳ Next: We will rework this in Euclidean WS

The amplitude now has an interpretation in Euclidean worldsheet time, so we discover a Euclidean interpretation of the amplitude

$$\mathcal{A}_4(k_1,k_2,k_3,k_4) = g_o^2 \int_{-\infty}^{0} dt \, \langle 0;-k_4 | V_3(0) \, V_2(-it) | 0;k_1 \rangle$$



amplitude in a Lorentzian world-sheet



(have taken $\tau' = 0$)

$\hookrightarrow$ $|0;k\rangle = \lim_{t\to-\infty} e^{-t} V_T(t) |0;0\rangle$

physical state obtained by inserting the appropriate VO in past infinity

Consider now a Euclidean conformal map

$$z = e^{t+i\sigma} \quad , \quad \bar{z} = e^{t-i\sigma}$$

new coordinates on EWS

What happens to the strip?

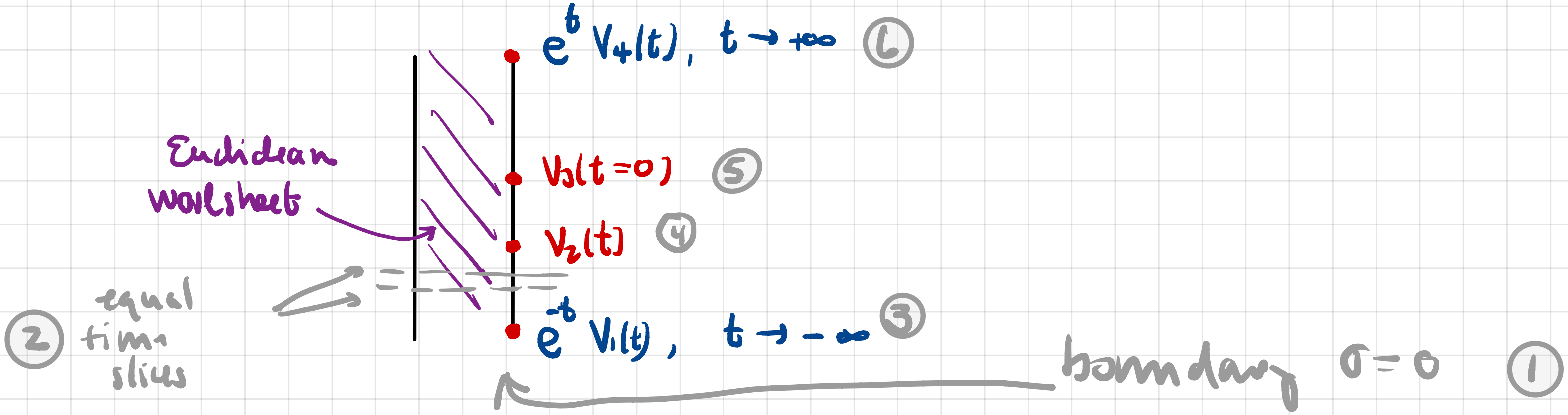
metric: $dz\,d\bar{z} = e^{2t}(dt^2 + d\sigma^2)$ (Weyl equivalent to $dt^2+d\sigma^2$)

so $\dfrac{dz\,d\bar{z}}{|z|^2} = dt^2 + d\sigma^2$ metric on the upper-half plane (UHP)

Indeed: $\mathrm{Im}\,z = \frac{1}{2i}(z-\bar{z}) = \frac{1}{2i}e^t(e^{i\sigma}-e^{-i\sigma}) = e^t \sin\sigma$

but $0 \leq \sigma < \pi \implies \sin\sigma \geq 0 \implies \mathrm{Im}\,z \geq 0$

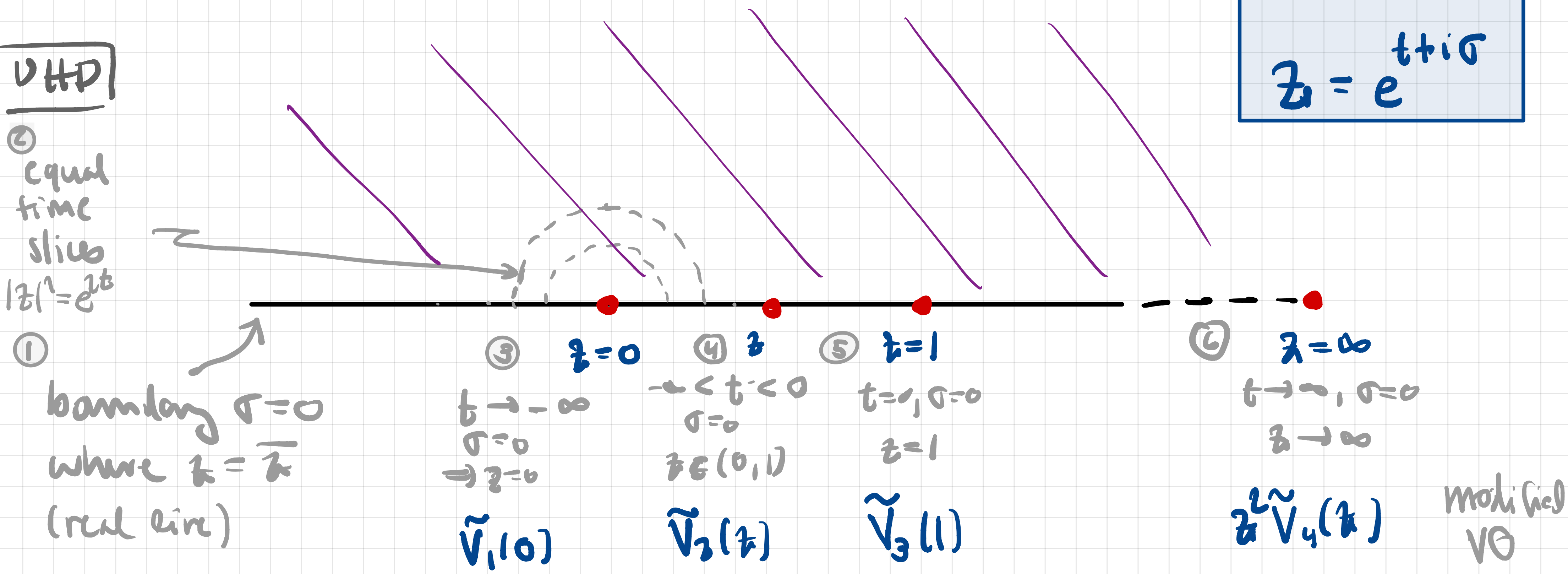
The strip



is mapped to the UHP with four marked points

UHP

$$z = e^{t+i\sigma}$$



Modification of the vertex operators due to the conformal transformation:

Recall that a vertex operator $V(\tau)$ transforms as

$$V(\tau) \longrightarrow V(\tilde{\tau}) = \left(\frac{d\tau}{d\tilde{\tau}}\right)^{1} V(\tau)$$

primary with $h=1$

Then

$$\tilde{V}(\underbrace{z=\bar{z}}_{\sigma=0}) = \frac{dt}{dz} V(t) = z^{-1} V(t)$$

$$z = e^{t+i\sigma} \Longrightarrow \frac{dz}{dt} = z$$

So, the conformal transformations modify the VO according to $\tilde{V}(z = \bar{z}) = \frac{1}{z} V(t)$
$= e^{-t} V(t)$

- incoming state $(t \to -\infty)$ $\quad |0; k_1\rangle = \lim_{t \to -\infty} e^{-t} V_1(t) |0; 0\rangle$

$$\Rightarrow \quad \lim_{\substack{t \to -\infty \\ z \to 0}} \underbrace{e^{-t}}_{z^{-1}} \underbrace{V_1(t)}_{z \tilde{V}_1(z = \bar{z})} = \lim_{z \to 0} \tilde{V}_1(z)$$

$z = e^{t}$

- $V_2(t)$: $\quad -\infty < t < 0$

$$\underbrace{V_2(t)\, dt}_{\text{we have this in the amplitude}} = \underbrace{z \tilde{V}_2(z)}_{V_2(t)} \frac{1}{z} dz = \tilde{V}_2(z)\, dz$$

$dt = \frac{dt}{dz} dz = \frac{1}{z} dz$

- $V_3(t=0)$: $\quad V_3(t=0) = 1 \cdot \tilde{V}_3(1)$

as $t = 0 \to z = 1$

- outgoing state $\quad \langle 0; -k_4| = \lim_{t \to \infty} e^{t} V_4(t)$

$$\Rightarrow \quad \lim_{t \to \infty} \underbrace{e^{t}}_{z} \underbrace{V_4(t)}_{z \tilde{V}_4} = \lim_{z \to \infty} z^2 \tilde{V}_4(z)$$

We can now write the amplitude on the strip as an <u>amplitude on the UHP</u>...

Let's discuss first conformal transformations and the gauge fixing procedure in more generality (which is usefull when computing more general amplitudes)

The group of global conformal transformations of the UHP is $PSL(2,\mathbb{R})$
Mobious group

$$z \longmapsto \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbb{R}, \quad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$$

This is a three dimensional group of residual gauge symmetries.

One can check that this group maps 1) the UHP into itself

2) 1-1 mappings of the UHP into itself are of the form $z \to \tilde{z}(z)$ ie analytic functions of z.

Hence, $PSL(2,\mathbb{R})$ is a subgroup of the full conformal group

3) Generators of $PSL(2,\mathbb{R})$

L_{-1} : $z \to z+b$

L_0 : $z \to az$

L_1 : $z \to \dfrac{z}{1+cz}$ special conf transformation $\left(w=-\frac{1}{z} \longmapsto w-c\right)$

Remark $\quad L_n = -z^{n+1}\partial_z \qquad (\bar{L}_n = -\bar{z}^{n+1}\partial_{\bar{z}})$

$z=0 \qquad L_n = -z^{n+1}\partial_z$ non singular at $z=0$ only for $n \geq -1$

$z=\infty \qquad$ let $w = -\frac{1}{z}$

$$L_n = -z^{n+1}\left(\frac{1}{z^2}\right)\partial_w = -z^{n-1}\partial_w = -(-w)^{1-n}\partial_w$$

$$= (-1)^n\left(\frac{1}{w}\right)^{n-1}\partial_w$$ non singular at $w=0$ only for $n \leq +1$

Globally defined conf transf on $S^2 = \mathbb{C}\cup\infty$ are generated by (L_{-1}, L_0, L_1)

$L_{-1} = -\partial_z \qquad$ translation $\qquad z \longmapsto z+b$

$L_0 = -z\partial_z \qquad z \longmapsto az$

$L_{+1} = -z^2\partial_z \qquad$ special conf transformations $\qquad z \longmapsto \frac{z}{cz+1}$

(or $\quad w = -\frac{1}{z} \longmapsto w-c$)

Important properties of PSL$(2;\mathbb{R})$

▷ One can find a transformation which maps any distinct three points $\{z_1, z_3, z_4\}$ the points $\{0, 1, \infty\}$

Indeed $$z \longmapsto \frac{z_{43}}{z_{31}} \, \frac{z - z_1}{z_4 - z}$$ where $z_{ij} = z_i - z_j$

(one can easily prove this by asking which $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2,\mathbb{R})$ maps $z_1 \to 0$, $z_2 \to 1$, $z_4 \to \infty$)

One can use this to gauge fix the three point amplitude

- Of particular interest for us is the fact that $PSL(2,\mathbb{R})$ preserves the cyclic ordering of any four points on the boundary $(\sigma=0)$

Consider the four points $\{z_1, z_2, z_3, z_4\}$ on the boundary st

$$z_1 < z_2 < z_3 < z_4$$

Then the map above maps

$$z_2 \longmapsto \frac{z_{21}\, z_{43}}{z_{31}\, z_{42}} \in (0,1)$$

conformal cross ratio

so it maps $(z_1, z_2, z_3, z_4) \longrightarrow \left(0, \frac{z_{12}\, z_{34}}{z_{13}\, z_{24}}, 1, \infty\right)$

ie fixing three points $\{z_1, z_3, z_4\}$ at $\{0, 1, \infty\}$ the fourth point $0 < z_2 < 1$.

The preservation of the cyclic ordering of four points on the boundary must imply some cyclic symmetry of the four point amplitude.

↳ Next 4-point amplitude (continued)

covariant amplitudes & the Faddeev-Popov trick