String Theory 1

Lecture # 11

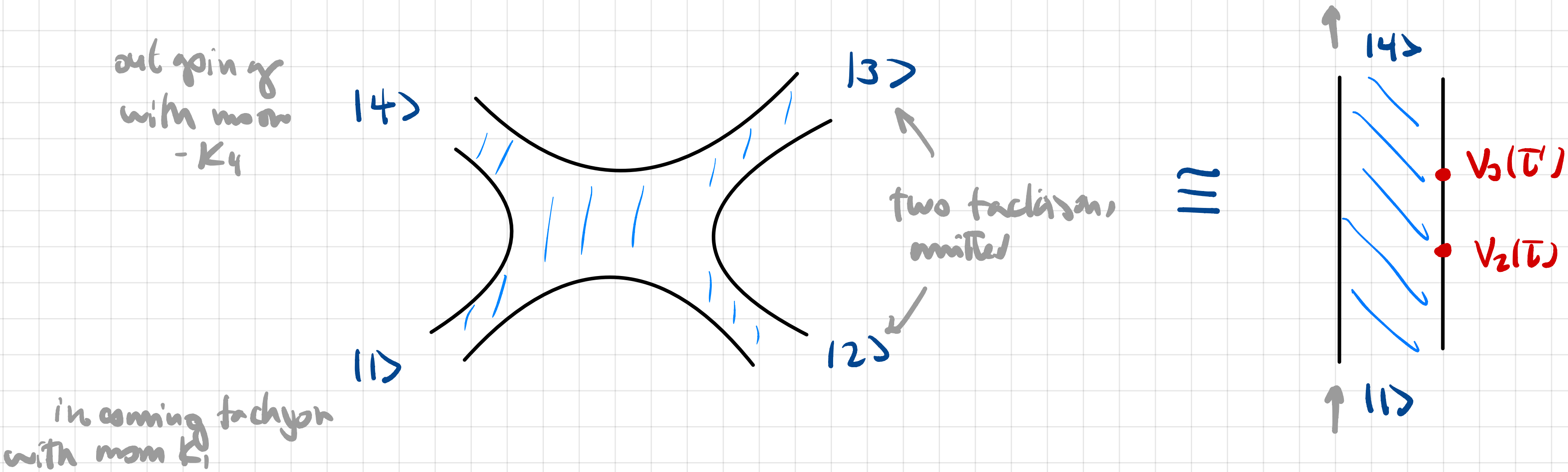
3 Interactions

3.1 Generalities
3.2 Vertex operators: introduction
3.3 Vertex operators: open string
3.4 The state vertex correspondence open strings
3.5 Vertex operator: closed string
3.6 3-point interactions
3.7 4-point tachyon amplitude
3.8 Comments on the general picture

3.7 The Veneziano amplitude continued

or 4-tachyon amplitude

[This section: features of amplitudes than can be generalized to n-point amplitudes in open & closed string amplitudes]



Last lecture

$$\mathcal{A}_4(k_1,k_2,k_3,k_4) = g_o^2 \int_{\tau'>\tau} d\tau\, d\tau' \langle 0;-k_4|\, V_3(\tau')V_2(\tau)\,|0;k_1\rangle \Big/ \mathrm{Vol(conf)}$$

Lorentzian WS

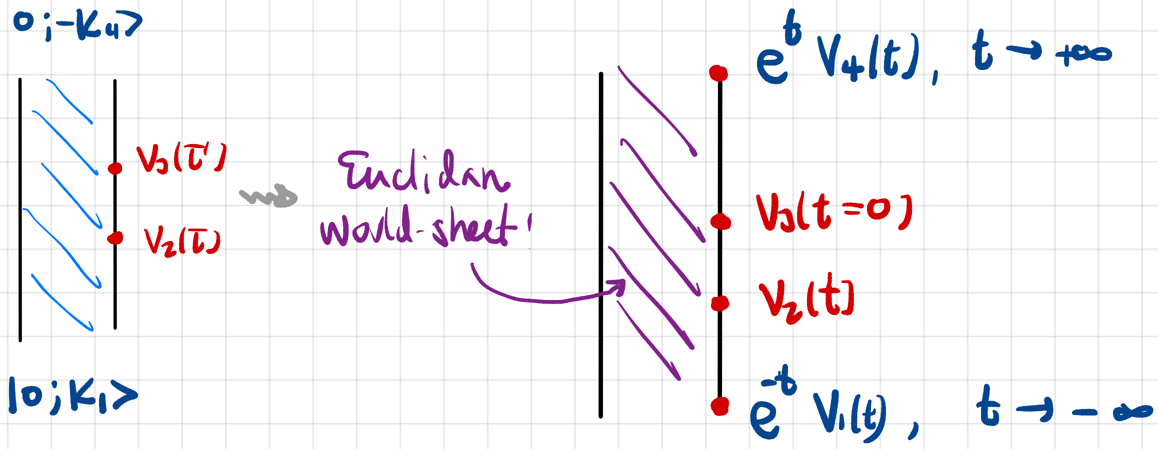
fix $\tau'=0$

$$= g_o^2 \int_{-\infty}^{0} d\tau \,\langle 0;-k_4|\, V_3(0)\, V_2(\tau)\, |0;k_1\rangle$$

$\tau = -it$

$$= g_o^2 \int_{-\infty}^{0} dt \,\langle 0;-k_4|\, V_3(0)\, V_2(-it)\, |0;k_1\rangle$$

Euclidean WS



use the state-vertex correspondence

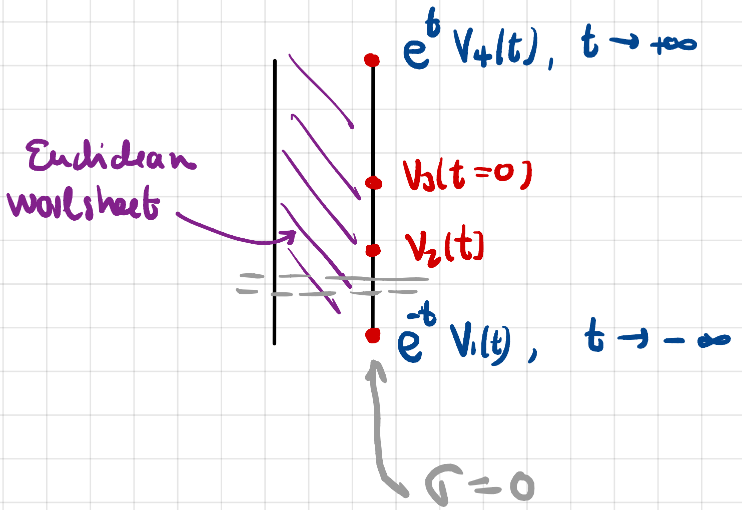
- Similar expressions for n-point amplitudes!
- Amplitudes now have an interpretation in Euclidean worldsheet

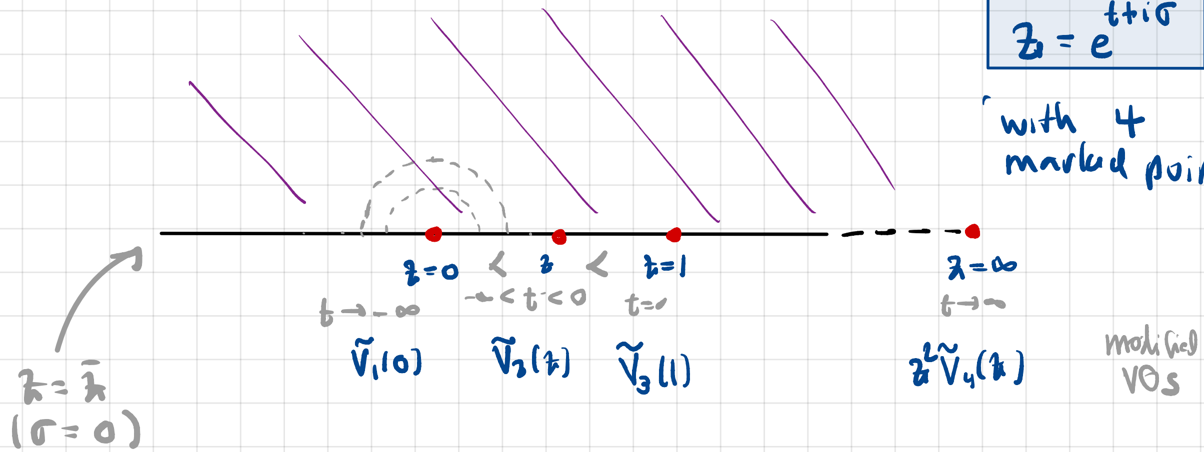
so: * work in Euclidean WS
* learn about general properties of amplitudes
(using conformal symmetries of the WS)

Consider a Euclidean conformal map

$$z = e^{t+i\sigma}\ ,\quad \bar{z} = e^{t-i\sigma}$$

new coordinates on EWS





UHP
$z = e^{t+i\sigma}$
with 4 marked points

↳ Want to write the amplitude on the UHP
but now write it covariantly and fix
the gauge using the Faddeev–Popov "trick"

need to learn more about conformal transformations

Recall from last lecture

The group of global conformal transformations of the UHP is $PSL(2,\mathbb{R})$
Möbius group

$$z \longmapsto \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbb{R}, \quad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$$

This is a three dimensional group of residual gauge symmetries.

(1-1 mappings of UHP $\to$ UHP generated by $\{L_{-1}, L_0, L_1\}$)

L_{-1}: $z \mapsto z+b$

L_0: $z \mapsto az$

L_1: $z \mapsto \frac{z}{1-cz}$

$\left(w = -\frac{1}{z} \mapsto w - c\right)$

$$z = -\frac{1}{w} \to -\frac{1}{w-c}$$

$$= \frac{z}{1-cz}$$

Important properties of $PSL(2;\mathbb{R})$

- One can find a transformation which maps any distinct three points $\{z_1, z_3, z_4\}$ to the points $\{0, 1, \infty\}$

Indeed
$$z \longmapsto \frac{z_{43}}{z_{31}} \frac{z - z_1}{z_4 - z} \qquad \text{where } z_{ij} = z_i - z_j \ (i \neq j)$$

One can easily prove this by asking which $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{R})$ maps $z_1 \to 0$, $z_3 \to 1$, $z_4 \to \infty$

1) $0 = \dfrac{a z_1 + b}{c z_1 + d} \Rightarrow b = -a z_1$

2) $1 = \dfrac{a z_3 - (a z_1)}{c z_3 + d} \Rightarrow c z_3 + d = a z_{31} \Rightarrow d = a z_{31} - c z_3$

3) $0 = \dfrac{c z_4 + d}{a z_4 + b} \Rightarrow c z_4 + a z_{31} - c z_3 = 0 \Rightarrow a z_{31} = -c z_{43}$

$$z \mapsto \frac{a z - a z_1}{-\frac{a z_{31}}{z_{43}} z + a z_{31} + \frac{a z_{31}}{z_{43}} z_3}$$
$$= \frac{z - z_1}{\frac{z_{31}}{z_{43}}(-z + z_{43} + z_3)} = \frac{z_{43}}{z_{31}} \frac{z - z_1}{z_4 - z}$$

(One can use this to gauge fix the three point amplitude)

- Of particular interest for us is the fact that $PSL(2,\mathbb{R})$ preserves the ordering of $\boxed{\text{any}}$ four points on the boundary $(\sigma=0)$

Consider the four points $\{z_1, z_2, z_3, z_4\}$ on the boundary st

$$z_1 < z_2 < z_3 < z_4$$

Then the map above maps

$$z_2 \longmapsto \frac{z_{21}\, z_{43}}{z_{31}\, z_{42}} \in (0,1) \qquad \text{conformal cross ratio}$$

so it maps $(z_1, z_2, z_3, z_4) \longrightarrow \left(0, \frac{z_{12}\, z_{34}}{z_{13}\, z_{24}}, 1, \infty\right)$

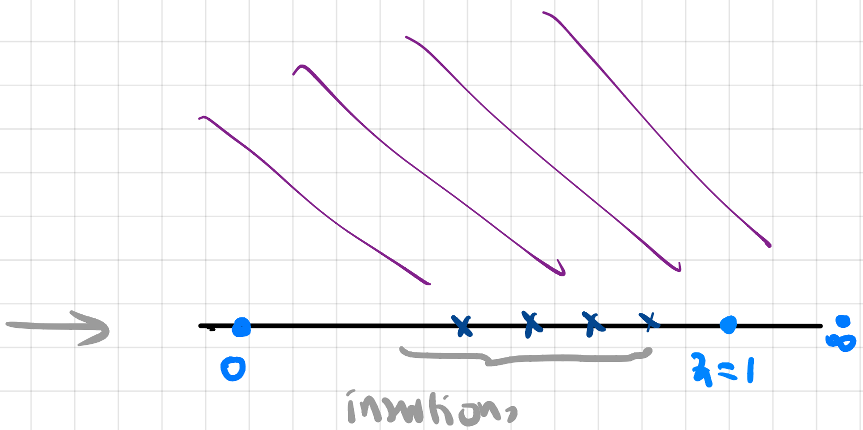
ie fixing three points $\{z_1, z_3, z_4\}$ at $\{0, 1, \infty\}$ the fourth point $0 < z_2 < 1$.

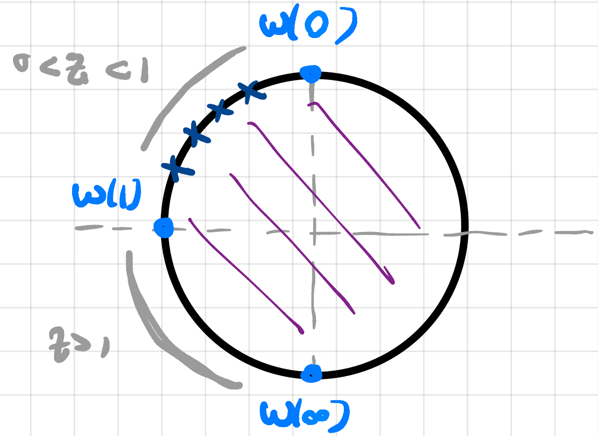
The preservation of the cyclic ordering of four points on the boundary implies a cyclic symmetry of the four point amplitude (see eg GSW)

Side remark (see GSW chapter 7)

cyclic symmetry: an elegant way to make the cyclic symmetry of amplitudes manifest is by introducing the map (new) $z \longmapsto w = \frac{1-z}{1-iz}$

which maps UHP $\longrightarrow$ unit disk





- boundary $\longrightarrow$ boundary of the unit disk

(z real $|w|^2 = \frac{z\bar{z}+1}{z\bar{z}+1} = 1$)

Return to 4-point amplitude : $z = e^{t+i\sigma}$

$$A_4(k_1,k_2,k_3,k_4) = g_0^2 \int_{-\infty}^{0} dt \underbrace{\langle 0; -k_4 | V_3(0)\, V_2(it) | 0; k_1 \rangle}_{\lim_{z' \to \infty} z'^{2} \langle \tilde{V}_4(\frac{1}{z'}) \tilde{V}_3(1) \tilde{V}_2(t) \tilde{V}_1(0) \rangle}$$

we have the interval on the boundary $\int_{-\infty}^{0} dt \longrightarrow \int_0^1 dz$

We say that the interval on the boundary $(0,1)$ is the moduli space of conformal structures on the UHP with four marked points.

We will return to this notion of moduli space later

Covariance and the Fadeev-Popov gauge fixing:

We could have written the four-point amplitude covariantly (in EWS)

$$g_o^2 \underbrace{\int d\tau_1 d\tau_2 d\tau_3 d\tau_4}_{\text{overcounts equivalent configurations}} \langle \tilde{V}_4(\tau_4) \tilde{V}_3(\tau_3) \tilde{V}_2(\tau_2) \tilde{V}_1(\tau_1) \rangle \Big/ \underbrace{\mathrm{Vol}(SL(2;\mathbb{R}))}_{\text{need this here due to the overcounting of configurations}}$$

and then use the Faddeev-Popov trick to fix the gauge, (that is to deal with any residual gauge symmetries).

see Peskin-Schroeder chapter 9 for details (also in AQFT)

Faddeev-Popov "Trick": want to fix the gauge by setting

$$z_1 = z_1^0 \qquad z_3 = z_3^0 \qquad z_4 = z_4^0$$

recall we only have the freedom to fix 3 points

Then by the FP trick we have:

$$g_o^2 \int dz_1\, dz_2\, dz_3\, dz_4\, \delta(z_1 - z_1^0)\, \delta(z_3 - z_3^0)\, \delta(z_4 - z_4^0)$$

to account for the gauge choice

$$\times \left| \mathrm{Det} \frac{\partial(z_1, z_3, z_4)}{\partial(\lambda_{-1}, \lambda_0, \lambda_1)} \right|$$

Faddeev-Popov determinant

$\{\lambda_{-1}, \lambda_0, \lambda_1\}$ parameters of $PSL(2,\mathbb{Z})$

new measure for the amplitude

$$\times \langle \tilde{V}_4(z_4)\, \tilde{V}_3(z_3)\, \tilde{V}_2(z_2)\, \tilde{V}_1(z_1) \rangle$$

and we drop $\mathrm{Vol}(SL(2,\mathbb{R}))$

$$\left| \mathrm{Det} \frac{\partial(z_1, z_3, z_4)}{\partial(\lambda_{-1}, \lambda_0, \lambda_1)} \right| = \text{Jacobian of the transformation from } z_1, z_3, z_4 \text{ to } \underbrace{\lambda_1, \lambda_0, \lambda_{-1}}_{\text{parameter of the gauge group}}$$

$$= z_{43}\, z_{31}\, z_{41}$$

where $\delta z = \lambda_{-1} + \lambda_0 z + \lambda_1 z^2$ infinitesimal Möbius Transf

expand $\tilde{z} = \frac{az+b}{cz+d}$ around identity matrix $a = d = 1$, $b = c = 0$

$\{L_{-1}, L_0, L_1\}$

$z \mapsto z+b$, $z \mapsto az$, $z \mapsto \frac{z}{1-cz}$

$\left(w = -\frac{1}{z} \mapsto w - c\right)$

$z = -\frac{1}{w} \to -\frac{1}{w-c}$

$= \frac{z}{1-cz}$

$\lambda_{-1} = \delta b$, $\lambda_0 = \delta a - \delta d$, $\lambda_1 = -\delta c$

L_{-1} translations, L_0, L_1

$$\mathrm{Det} \frac{\partial(z_1, z_3, z_4)}{\partial(\lambda_{-1}, \lambda_0, \lambda_1)} = \begin{array}{cl} \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_3 & z_4 \\ z_1^2 & z_3^2 & z_4^2 \end{vmatrix} & \begin{array}{l} \leftarrow \partial z_i / \partial \lambda_{-1} \\ \leftarrow \partial z_i / \partial \lambda_0 \\ \leftarrow \partial z_i / \partial \lambda_1 \end{array} \end{array} \qquad i = 1, 3, 4$$

$$= z_3 z_4^2 + z_4 z_1^2 + z_1 z_3^2 - (z_1^2 z_3 + z_3^2 z_4 + z_1 z_4^2)$$

$$= (z_4 - z_3)(z_3 - z_1)(z_4 - z_1)$$

$$g_o^2 \int \overbrace{dz_1\, dz_2\, dz_3\, dz_4 \, |(z_4-z_3)(z_3-z_1)(z_4-z_1)|}^{d\mu \,=\, \text{measure on the UHP}} \, \delta(z_1 - z_1^0)\, \delta(z_3 - z_3^0)\, \delta(z_4 - z_4^0)$$

$$\times \left\langle \tilde{V}_4(z_4)\, \tilde{V}_3(z_3)\, \tilde{V}_2(z_2)\, \tilde{V}_1(z_1) \right\rangle_{UHP}$$

choose $z_1^0 = 0$, $z_3^0 = 1$, $z_4^0 = \Lambda \to \infty$

$$= \lim_{\Lambda \to \infty} g_o^2 \int dz_2 \, \underbrace{\overset{z_{43}}{(\Lambda-1)}\,\overset{z_{41}}{\Lambda} \left\langle \tilde{V}_4(\Lambda)\right.}_{\Lambda^2 \langle \tilde{V}_4(\Lambda)|} \tilde{V}_3(1)\, \tilde{V}_2(z_2)\, \underbrace{\left.\tilde{V}_1(0) \right\rangle_{UHP}}_{|1\rangle}$$

$$= g_o^2 \int_0^1 dz \, \langle 4 |\, \tilde{V}_3(1)\, \tilde{V}_2(z)\, | 1 \rangle$$

PS 3 This is the Veneziano amplitude.

$$\mathcal{A}(k_1, k_2, k_3, k_4) = g_o^2\, \delta(k_1 + k_2 + k_3 + k_4)\ A(s,t)$$

Regge slope

where • $A(s,t) = B(-\alpha(s), -\alpha(t))$, $\alpha(x) = 1 + \alpha' x$

Regge trajectory

• $B(a,b) = \dfrac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$ Beta function

• $\Gamma(a) = \int_0^\infty dt\ t^{x-1} e^{-t}$ Euler's Gamma function

• Mandelstam variables
$$\begin{cases} s = -(k_1 + k_2)^2 \\ u = -(k_1 + k_4)^2 \\ t = -(k_1 + k_3)^2 \end{cases}$$

$$s + u + t = \Sigma m_i^2$$

Consider $A(s,t) = B(-\alpha(s), -\alpha(t)) = \dfrac{\Gamma(-\alpha(s))\,\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}$

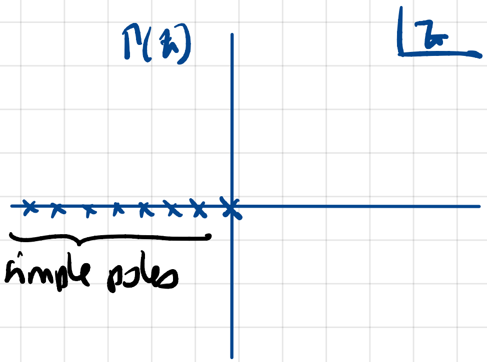
- poles at $-\alpha(s) = 0, -1, -2, \dots$; $-\alpha(t) = 0, -1, -2, \dots$

(from numerator of the Gamma-function)

that is at $s = \frac{1}{\alpha'}(n-1)$, $n = 0, 1, 2, \dots$

precisely the masses of level n open string states (infinitely many)

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \quad \mathrm{Re}(z) > 0$$



$\Gamma(z)$ has no zeros

$$\Gamma(z+1) = z\,\Gamma(z)$$

(use to analytically continue Γ to the whole z-plane)

Behaviour near singularities: near $z = -n$, n non-negative integer

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n-1)(z+n)} \sim \frac{(-1)^n}{n!}\frac{1}{z+n}$$

using $\Gamma(z+1) = z\,\Gamma(z)$ repeatedly

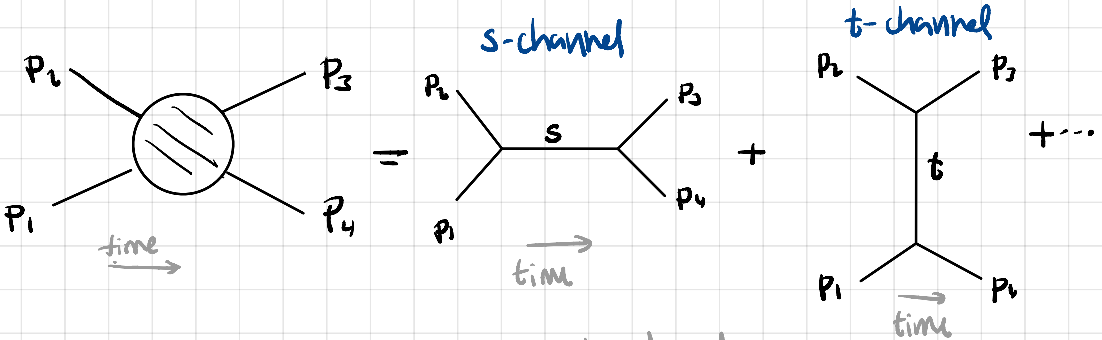
► Dolen-Horn-Schmid duality (see GSW chapter 1)

$$A(s,t) = A(t,s)$$

obvious from expression above in terms of B-function

Compare with QFT

leading (non trivial) contributions: come from tree diagr



resonances in s-channel ~ intermediate particles in scattering

virtual exchanged particles in t-channel

$A(s,t)$ invariant under $s \leftrightarrow t$

$$A(s,t) = A(t,s) = -\sum_{n=0}^{\infty} \frac{1}{n!}\,(\alpha(s)+1)(\alpha(s)+2)\cdots(\alpha(s)+n)\,\frac{1}{\alpha(t)-n}$$

obvious

OTOH, can expand as a series to exhibit the poles in the t-channel **!**

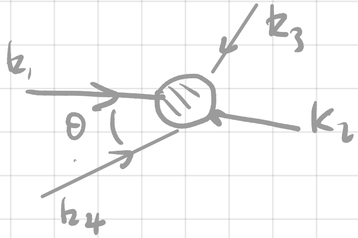
" " " " " " " s-channel

UV behaviour (see GSW chapter 1)

$A(s,t) \sim F(\theta_s)^{-\alpha(s)}$ exponentially soft behaviour ("falls off exp with s)

↳ the function of an angle θ_s

$\frac{t}{s} \approx -\sin^2\frac{\theta}{2}$ $\qquad s \to \infty$



Compare QFT: fixed angle scattering of point particles

$$A(s,t) \sim -\sum_{J=0}^{J_{max}} \frac{(-s)^J}{t-M_J^2} \underset{\substack{s\to\infty \\ t \text{ fixed}}}{\sim} s^{J_{max}}$$

t-channel

- exchange of spin J & M_J particles
- bad UV behaviour (also at higher loops)
- no s-channel poles!

string interpretation: infinite tower of states & UV divergences "cancelled"

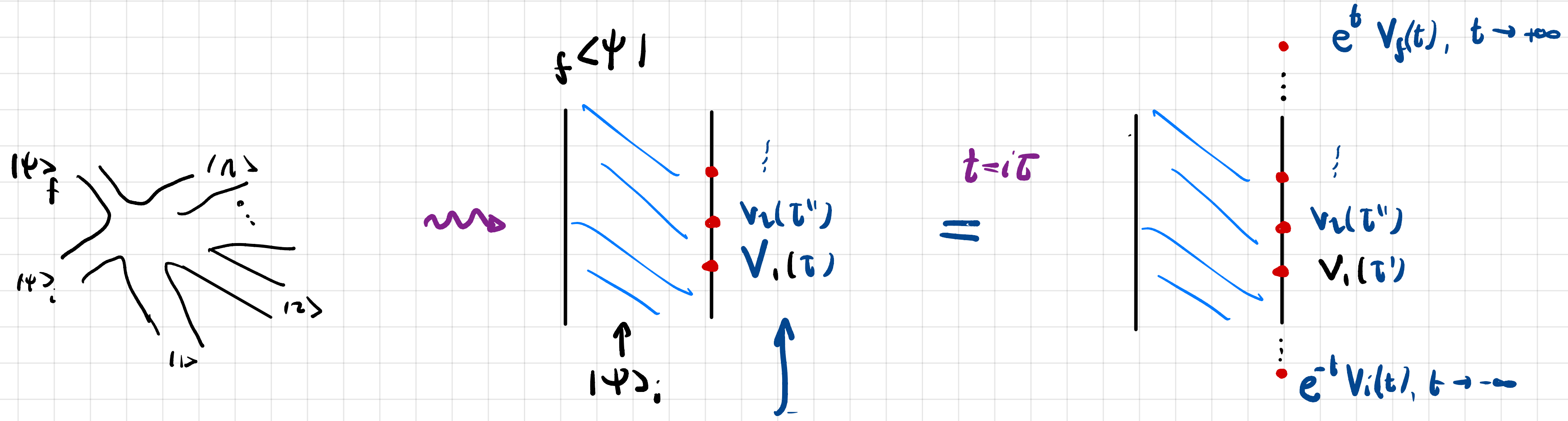
3.8 Comments on the general picture

In string perturbation theory we are interested in the amplitude for the scattering of asymptotic in and out states (the S-matrix)

We have discussed a number of ideas and tools for computing amplitudes. (& so far tree level OS)

Wrap up this chapter on interactions with a number of comments on the lessons learned and on the general picture for scattering amplitudes

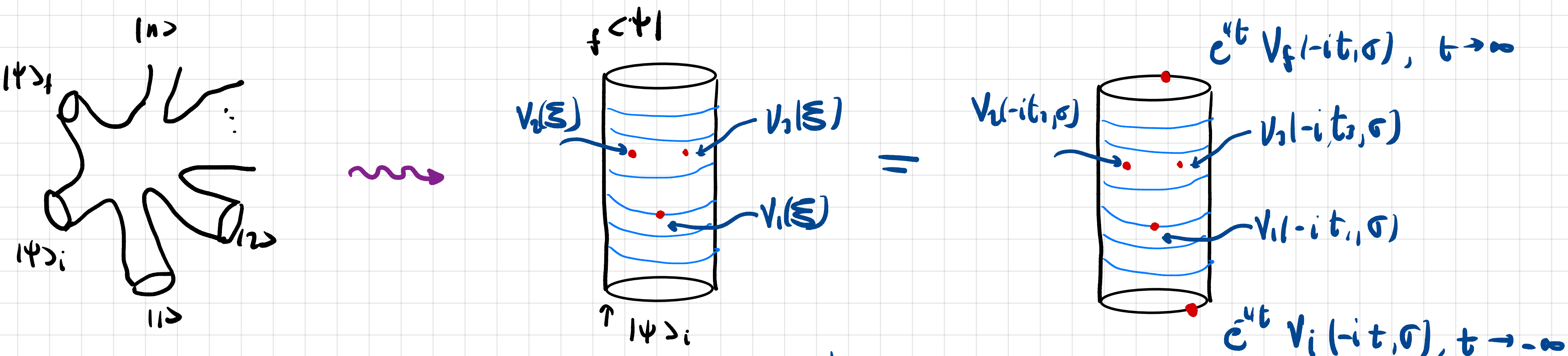
open strings (tree level)



vertex operators inserted on the boundary of Σ

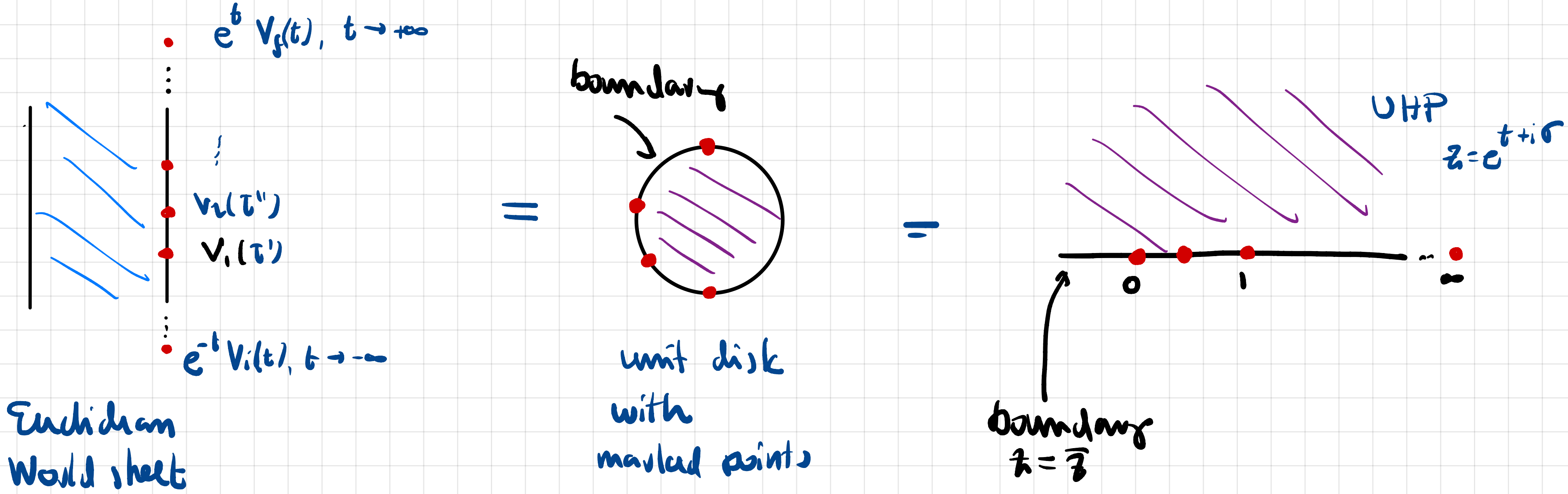
Euclidean World sheet

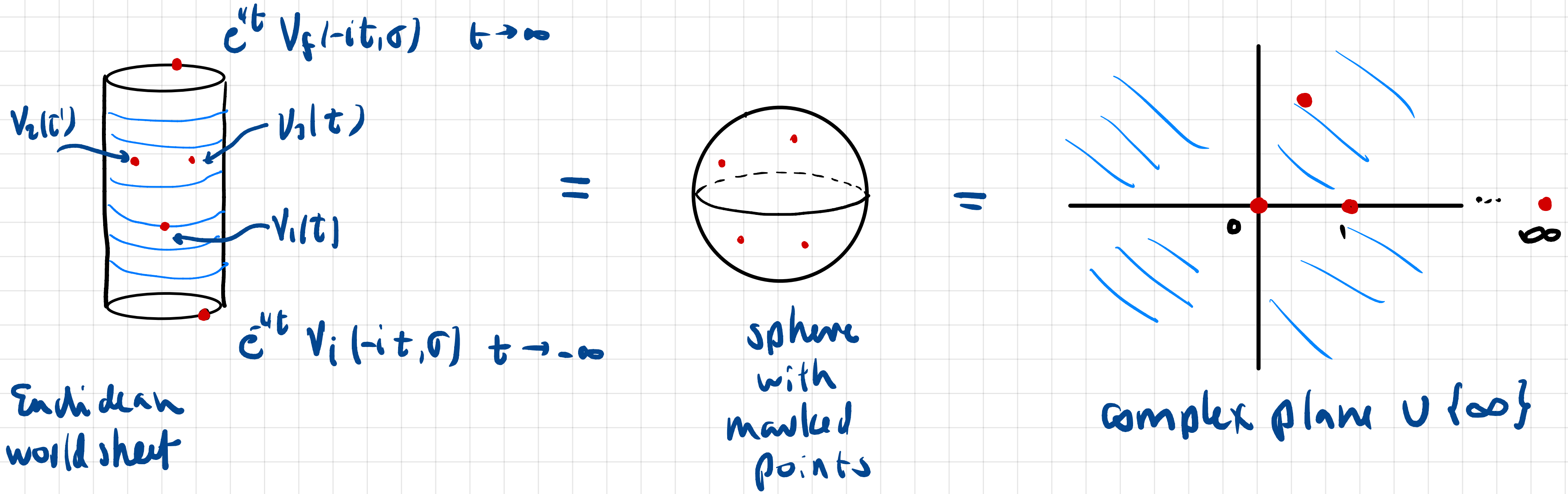
closed strings (tree level)



vertex operators inserted on the interior of Σ

Euclidean world sheet

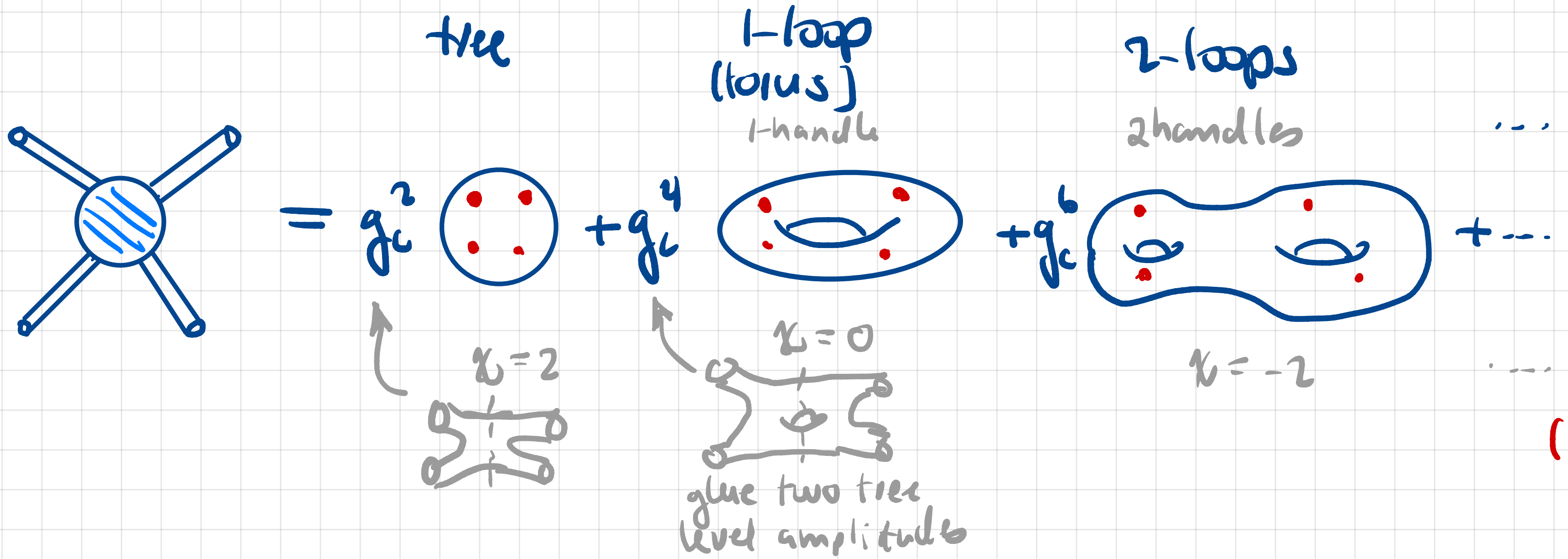

$e^{t} V_f(t)$, $t \to +\infty$
$V_2(\tau'')$
$V_1(\tau')$
$e^{-t} V_i(t)$, $t \to -\infty$
Euclidean World sheet
=
boundary
unit disk with marked points
=
UHP
$z = e^{t+i\sigma}$
0
1
∞
boundary $z = \bar{z}$


$e^{4t} V_f(-it,\sigma)$ $t \to \infty$
$V_2(t')$
$V_3(t)$
$V_1(t)$
$e^{4t} V_i(-it,\sigma)$ $t \to -\infty$
Euclidean world sheet
=
sphere with marked points
=
0
1
∞
complex plane $\cup \{\infty\}$

Beyond tree level in string perturbation theory:

The string perturbation series is a genus expansion that is, a sum of Euclidean world sheets with different topology.

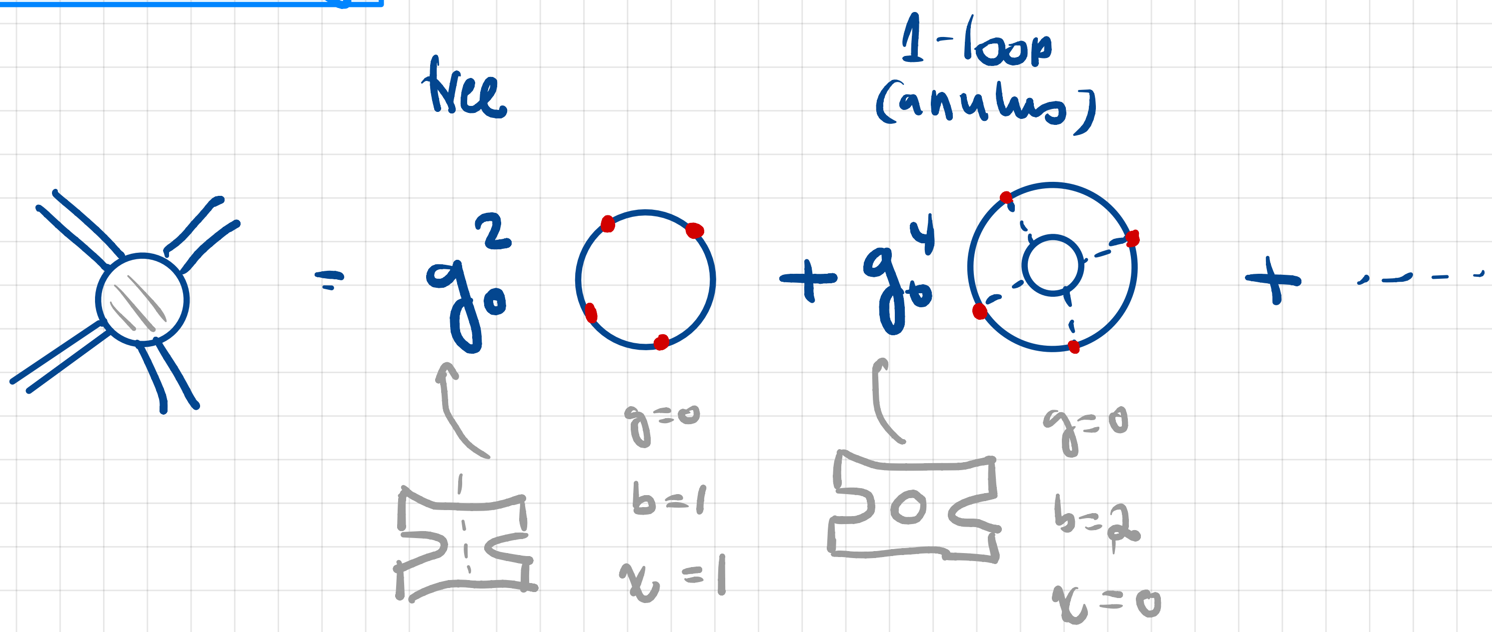
Closed string



$\chi = 2(1-g)$
(Euler number)

- sum over all topologies (Riemann surfaces) $\boxed{\text{without}}$ boundaries
- these surfaces are classified by the number of handles g
- one diagram at each loop

Open string



$\chi = 2(1-g) - b$
Euler number
b = # of boundary components

- sum over all topologies (Riemann surfaces) $\boxed{\text{with}}$ boundaries
- these surfaces are classified by the number of handles g and the number of boundaries b
- one diagram at each order in perturbation theory

The relation between couplings

Recall: we cannot add any interaction terms to S_p without breaking conformal and Weyl invariance except for

$$\frac{1}{4\pi}\int_\Sigma d^2\xi \sqrt{-\det\gamma}\, R[\gamma] + \frac{1}{2\pi}\int_{\partial\Sigma} ds\, K[\gamma] = \chi = 2-2g-b$$

PS1

topological invariant

Consider then the action $S = S_p + \lambda\chi$, $\lambda \in \mathbb{R}$

S has the same dynamics as S_p

However, in the path integral formalism

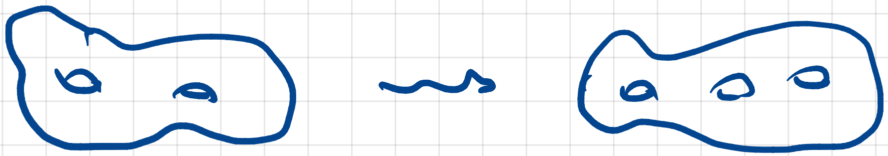
integrated vertex insertion for $|i\rangle$

$$\mathcal{A}(|1\rangle, \dots, |n\rangle) = \sum_{\text{topologies}} \int \frac{\mathcal{D}[X,\gamma]}{\mathrm{Vol}(\mathrm{conf}_{\mathrm{rc}})} e^{-S[X,\gamma]} \prod_{i=1}^{n} V_{|i\rangle}$$

$$= \sum_{\text{topologies}} (e^{\lambda})^{-\chi} \int \frac{\mathcal{D}[X,\gamma]}{\mathrm{Vol}(\mathrm{conf}_{\mathrm{rc}})} e^{-S_p[X,\gamma]} \prod_{i=1}^{n} V_{|i\rangle}$$

Same series expansion as above with expansion parameter $\boxed{g_s = e^{\lambda}}$.

Add a handle so new diagram has an extra closed string loop



$$\chi \longrightarrow \chi - 2$$

$$(g_s)^{-\chi} \longrightarrow (g_s)^{-\chi} \cdot g_s^2$$

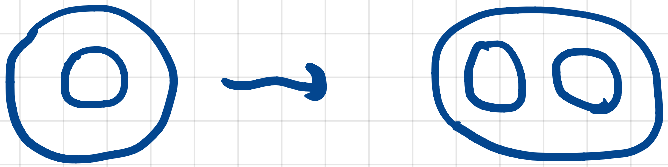
$\chi = 2(1-g)$

$\chi(g+1) - \chi(g)$

$= -2$

Identify $g_s = g_c = e^{\lambda}$

Add an interior boundary so new diagram has an extra open string loop



$$\chi \longrightarrow \chi - 1$$

$$g_s^{-\chi} \longrightarrow (g_s)^{-\chi} g_s$$

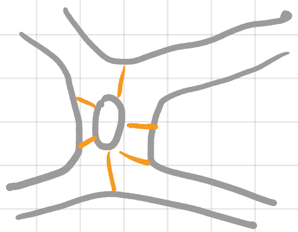
$\chi = 2(1-g) - b$

$\chi(b+1) - \chi(b) = -1$

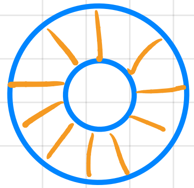
Identify $g_o^2 = g_s$

Then $\boxed{g_s = g_o^2 = g_c}$

Open-closed duality: a single geometry can have two interpretations



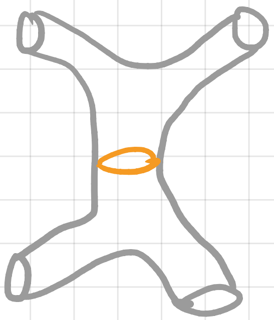
open string

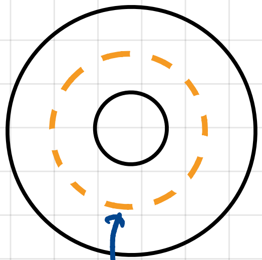


g_o^4 **one loop** open string amplitude

↳ topology of a cylinder!

Reinterpret: **tree level** amplitude of a closed string!





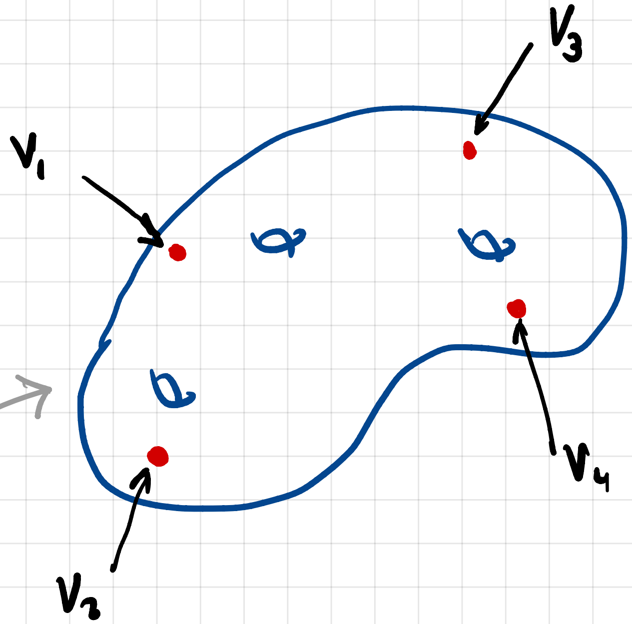
closed string

g_c^2 tree level closed string amplitude

Consistency: $g_c = g_o^2$ ✓

(See GSW for the computation)

General scattering process: say for the closed string



genus g Riemann surface (complex surfaces)

$$\mathcal{A}_g(\psi_1, \ldots, \psi_n) = \int_{\mathcal{M}_{g,n}} \underbrace{[d\mu]}_{\substack{\text{measure}\\ \text{on moduli}\\ \text{space}}} \langle V_1 V_2 V_3 \cdots V_n \rangle_{\Sigma_{g,n}(\mu)}$$

Moduli space of Riemann surfaces with marked points, (parameters which characterise the world sheet as a complex 2d manifold without boundary)

parametrizes Weyl equiv class of metrics $\cong$ $\mathbb{C}$-structures

parametrises Weyl equivalent class of metrics (complex structure)

- M_Σ & $[d\mu]$ $\rightarrow$ complicated

however low genus isn't so bad (we did tree level examples)

(1-loop amplitude calculations are rather interesting)

Next: strings in background fields

strings propagating in non trivial backgrounds