String Theory 1

Lecture # 12

4. Strings in background fields

4.1 Introduction

For strings propagating in $\mathbb{M}^{1,25}$, we have identified various massless fields in the bosonic string spectrum, including a graviton.

We expect then that a theory of space-time gravity should emerge, so spacetime should be allowed a nontrivial metric (or indeed a a nontrivial topology)

We expect a D=26 dim theory of gravity emerging with a Hilbert-Einstein action.

Moreover, we should be able to describe the dynamics of string excitations propagating in non-trivial backgrounds.

The action for a string propagating in a spacetime with metric $G_{\mu\nu}(X)$ is

$$S_\sigma[\gamma, X] = -\frac{1}{4\pi\alpha'}\int d^2\xi \sqrt{\gamma}\,\gamma^{ab}\,\partial_a X^\mu\, \partial_b X^\nu \underbrace{G_{\mu\nu}(X)}_{\text{target space metric}}$$

So far we have only considered a flat target spacetime $G_{\mu\nu} = \eta_{\mu\nu}$

Classically this is Weyl invariant so taking $\gamma_{ab} = e^{2\phi(\sigma)}\eta_{ab}$

$\boxed{\text{NON-LINEAR } \sigma\text{-MODEL}}$

NLσM

$$S_\sigma[\gamma, X] = -\frac{1}{4\pi\alpha'}\int d^2\xi\, \partial_a X^\mu\, \partial^a X^\nu\, G_{\mu\nu}(X)$$

describes an interacting 2dim QFT with couplings encoded in the target space metric $G_{\mu\nu}(X)$

complicated! compare $G_{\mu\nu} = \eta_{\mu\nu} \Rightarrow$ free field theory

(We will generalise S_σ later to include the other massless states)

In this chapter we discuss how a $D=26$ dimensional gravitational theory emerges: we will do this from the effective field theory point of view.

KEY: we require that the quantum theory is Weyl invariant.

First however, we use this action to try to make sense of the graviton states in the spectrum of the free string

Recall massless spectrum of the closed string propagating in $M^{1,25}$

$$\gamma_{\mu\nu}\, \alpha_{-1}^{\mu}\, \tilde{\alpha}_{-1}^{\nu} |0;k\rangle \qquad \gamma_{\mu\nu} \text{ traceless symmetric}$$

↳ with $k^{\mu}\gamma_{\mu\nu}=0$

$$\gamma_{\mu\nu} \sim \gamma_{\mu\nu} + k_{\mu}\xi_{\nu} + k_{\nu}\xi_{\mu}\ , \quad k\cdot\xi=0$$

<u>Spacetime interpretation</u>: right degrees of freedom expected of a graviton in the traceless harmonic gauge GR

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \gamma_{\mu\nu}(x)$$

$$\gamma_{\mu\nu}(x) \sim \gamma_{\mu\nu} + \boxed{\partial_{\mu}\xi_{\nu}(x) + \partial_{\nu}\xi_{\mu}(x)}$$

→ infinitesimal diffeomorphisms parametrized by ξ which in momentum space become $\gamma_{\mu\nu}(k) \to \gamma_{\mu\nu}(k) + \xi_{\mu}k_{\nu} + \xi_{\nu}k_{\mu}$ with $\xi\cdot k=0$

In the σ-model consider: $G_{\mu\nu}(X) = \eta_{\mu\nu} + h_{\mu\nu}(X)$ → small perturbation of flat space

$$\Rightarrow \quad e^{-S_\sigma} = e^{-S_0}\left(1 + \frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi \, h_{\mu\nu}(X)\, \partial_a X^\mu \partial^a X^\nu + \dots\right)$$

S_0: $G = \eta$

In the path integral $\rightsquigarrow \int \mathcal{D}X \mathcal{D}\gamma \, e^{-S_0} \, \mathcal{V}$

ie insertion of an operator $\mathcal{V} \sim \int_\Sigma d^2\xi \, h_{\mu\nu}(X)\, \partial_a X^\mu \partial^a X^\nu$ in the path integral

This must be a vertex operator corresponding to a physical state, the graviton, if $h_{\mu\nu}$ satisfies the appropriate conditions, ie if

$$h_{\mu\nu} = \gamma_{\mu\nu} : e^{ik\cdot X} : , \quad \gamma_{\mu\nu} \text{ traceless symmetric}$$

$\mathcal{V}$ generates a gravitational plane wave with polarization $\gamma_{\mu\nu}$

4.2 Background field expansion and the Weyl anomaly

To analyze the quantum NLσM we use the covariant background field expansion, which is a perturbation theory in which one separates the 2 dim fields as

$$X^\mu(\xi) = X_0^\mu(\xi) + \sqrt{\alpha'}\, Y^\mu(\xi) \qquad [\sqrt{\alpha'}] = L, \; Y \text{ dimensionless}$$

background part or "expectation value" satisfying EOM. For our purposes we take this to be a constant.

dynamical quantum fluctuation

One then expands the NLσM action around X_0^μ and get an expansion in powers of the quantum field Y about X_0

$$G_{\mu\nu}(X)\, \partial_a X^\mu \partial^a X^\mu = \alpha' \Big(G_{\mu\nu}(X_0^\mu) + \sqrt{\alpha'}\, \partial_\rho G_{\mu\nu}(X_0) Y^\rho(\xi) + \frac{\alpha'}{2} \partial_\rho \partial_\sigma G_{\mu\nu}(X_0) Y^\rho(\xi) Y^\sigma(\xi) + \cdots \Big) \partial_a Y^\mu \partial^a Y^\nu$$

Each term represents an interaction for the fluctuations Y.

What is the expansion parameter?

The quantum perturbation theory is an expansion in is powers of $\sqrt{\alpha'}$ ($\sqrt{\alpha'}$ is an $\hbar$-like parameter)

We need to expand in terms of an effective dimensionless parameter: noting that $\partial_\rho G \sim 1/r_c$

r_c = characteristic radius of the curvature of target space, our effective dimensionless coupling constant is of order $\sqrt{\alpha'}/r_c$.

Then we obtain a perturbative expansion if

$$\sqrt{\alpha'} \sim l_s \ll r_c$$

string length << typical space-time length scales

Remark: this means that perturbative string theory has a double expansion in g_s & α'

For $l_s \ll r_c$ we then work with a weakly coupled σ-model perturbation theory (in the usual sense of a perturbative QFT framework; from this one can read-off Feynman rules for diagrams ...).

In other words, we have a large radius expansion corresponding in spacetime to an EFT-like expansion with cutoff $M_s \sim (\alpha')^{-1/2}$.

(When $l_s \approx r_c$ this interpretation breaks down and instead we have a strongly coupled theory.)

Returning to $S_\sigma[G]$: this is classically conformally invariant however this is not necessarily true after quantisation because the NLσM is an interacting theory.

The interactions typically lead to (unphysical) divergences of the WS correlation functions. To deal with this we resort to regularisation & renormalisation techniques. Fortunately the theory with action S_σ is renormalisable.

However these techniques inevitably introduce an explicit scale dependence of the correlation functions (see AQFT) hence the theory is no longer conformally invariant.

(YM theory is classically conformally invariant but on quantisation the theory develops a scale dependence)

The lack of scale invariance in a QFT is described in terms of the β-function (which arises when computing the UV divergences in Feynman diagrams)

Recall $T_{ab} = -\frac{2}{T}\frac{1}{\sqrt{\gamma}}\frac{\delta S}{\delta \gamma^{ab}} = 0$ & in particular $T_{+-} = 0$

classically $T_{+-} = 0 \longleftrightarrow$ Weyl invariance

At the quantum level however

$$T_{+-} = -\frac{1}{2\alpha'}\beta_{\mu\nu}\,\partial_+ X^\mu \cdot \partial_- X^\nu$$

gets corrected at 1-loop

Beta function

$$\beta \sim M \frac{\partial G}{\partial M}$$

In fact, even for $G_{\mu\nu} = \eta_{\mu\nu}$: $T_{+-} = -\frac{1}{12}(D-26)\,R^{(2)}$

The theory is conformal invariant if $\boxed{\beta = 0}$

We would like to insist that the 2 dim QFT on the world sheet (ie NLσM) to be Weyl invariant at the quantum level. This implies, in particular, that the theory is conformally invariant.

(Conformal symmetry is a gauge theory we want to preserve in the quantum theory; recall this was essential for the counting of the theory: construction of states, vertex operators and amplitudes were all based on having a CFT on the WS)

We need then to compute the β-function. The requirement $\beta = 0$ necessary to preserve Weyl invariance places restrictions on the target space fields.

However the NLσ-M is $\boxed{\text{not}}$ so easy to analyze.

So how do we proceed? Return to the NLσM action & as discussed earlier, we analyze the quantum NLσM from the perturbation theory obtained by the covariant background field expansion with

$$X^\mu(\xi) = X_0^\mu + \sqrt{\alpha'}\, Y^\mu(\xi)$$

and expand around X_0^μ.

So how do we proceed? Return to the NLσM action & as discussed earlier, we analyze the quantum NLσM from the perturbation theory obtained by the covariant background field expansion with

$$X^\mu(\xi) = X_0^\mu + \sqrt{\alpha'}\, Y^\mu(\xi)$$

and expand around X_0^μ.

As the NLσM action is invariant under field redefinitions

$$X^\mu \longmapsto \tilde{X}^\mu(X)$$ (target space coordinate change)

accompanied by the corresponding transformation of $G_{\mu\nu}$

we can choose Riemann normal coordinates

$$\Gamma^\mu{}_{\nu\rho}\big|_x = 0 \quad ; \qquad R^\mu{}_{\nu\rho\sigma}\big|_x = (\partial_\rho \Gamma^\mu_{\sigma\nu} - \partial_\sigma \Gamma^\mu_{\rho\nu})\big|_x$$

simplifies computations!

Then $\left(G_{\mu\nu}(X) = G_{\mu\nu}(X_0) + \sqrt{\alpha'}\,\partial_\rho G_{\mu\nu}(X_0) Y^\rho(\xi) + \frac{\alpha'}{2} \partial_\rho \partial_\sigma G_{\mu\nu}(X_0) Y^\rho(\xi) Y^\sigma(\xi) + \cdots \right)$

$$G_{\mu\nu}(X_0 + \sqrt{\alpha'} Y) = \underset{\eta_{\mu\nu}}{G_{\mu\nu}(X_0)} - \frac{1}{3}\sqrt{\alpha'} \underbrace{R_{\mu\rho\nu\sigma}(X_0)}_{\text{Riemann Tensor of } M \text{ at } X_0} Y^\rho Y^\sigma + \mathcal{O}(Y^3)$$

$$\Rightarrow \quad S_\sigma[X] = -\frac{1}{4\pi} \int d^2\xi \Big\{ \eta_{\mu\nu}\, \partial_a Y^\mu \cdot \partial^a Y^\nu \quad \leftarrow \text{kinetic term}$$

$$- \frac{1}{3}\sqrt{\alpha'} R_{\mu\rho\nu\sigma}(X_0) (\partial_a Y^\mu)(\partial^a Y^\nu) Y^\rho Y^\sigma + \cdots \Big\}$$

leading quartic interaction term

interacting QFT with an infinite set of coupling constants

- We are ready to read off the Feynmann rules for diagrams for the two dimensional world sheet theory.
- Moreover we can compute the (one loop) divergences that contribute to the renormalisation of the couplings.

$\hookrightarrow$ introduces a scale M breaking conformal invariance

The Weyl anomaly:

$S_0[X,\gamma]$ is classically conformally invariant but not necessarily quantum mechanically

$$T_{+-} \sim \beta_{\mu\nu}\, \partial X^\mu \cdot \partial X^\nu = 0 \qquad \text{ie} \qquad \beta = 0$$

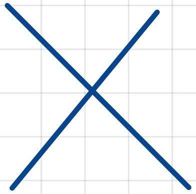
$$\beta_{\mu\nu}(G) \sim M \frac{\partial G_{\mu\nu}(X,M)}{\partial M}$$

describes how couplings (metric) depend on the energy scale M

Compute $\beta_{\mu\nu}$ to one loop

expect $\beta \neq 0$ from UV divergences in Feynman diagrams

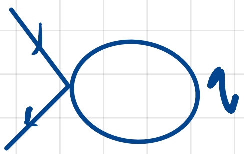
Quartic interaction (in momentum space)



$$\sim R_{\mu\rho\nu\sigma}\, p^{\mu}_{a}\, p^{a\nu}$$

$p^{\mu a}$ 2dim momentum of the scalar field Y^μ

Consider the one-loop correction to the kinetic term:

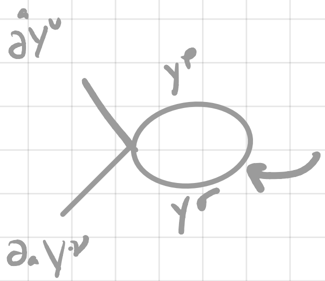


$$= \int \frac{d^2 q}{(2\pi)^2}\, \overbrace{\underline{R_{\mu\rho\nu\sigma}}\, p^{\mu}_{a}\, p^{a\nu}\, \underline{\eta^{\rho\sigma}}}^{R_{\mu\nu}}\, \frac{1}{q^2}$$

propagator on the loop

one-loop logarithmic divergence

In position space



propagator diverges as $\xi \to \xi'$

$$Y^\rho(\xi)\, Y^\sigma(\xi') \sim \frac{1}{2}\, \eta^{\rho\sigma} \log|\xi - \xi'|$$

This divergence can be determined using "dimensional regularisation"

$$\int \frac{d^d q}{(2\pi)^d} R_{\mu\rho\nu\sigma}\, p^\mu \cdot p^\nu \frac{\eta^{\rho\sigma}}{q^2} \underset{d=2+\epsilon}{=} \frac{1}{2\pi\epsilon} R_{\mu\nu}\, p^\mu \cdot p^\nu + \cdots$$

leading order divergence

$R_{\mu\nu} = R_\mu{}^\sigma{}_{\nu\sigma}$ target-space Ricci tensor

We now cancel this divergence by adding a counterterm (terms that need substracted to S_σ to get a finite theory)

$$R_{\mu\rho\nu\sigma}\, \partial Y^\mu \cdot \partial Y^\nu Y^\rho Y^\sigma \longrightarrow R_{\mu\rho\nu\sigma} Y^\rho Y^\sigma \partial Y^\mu \cdot \partial Y^\nu \underbrace{- \frac{1}{\epsilon} R_{\mu\nu}\, \partial Y^\mu \cdot \partial Y^\nu}_{\Delta S}$$

These divergences (and those from higher loops!) can be absorbed by

a wave function renormalisation of the fields Y^μ

$$Y^\mu \longrightarrow Y^\mu + \frac{\alpha'}{6\epsilon} R^\mu{}_\nu Y^\nu + \mathcal{O}(Y^2)$$

together with a functional renormalisation of $G_{\mu\nu}$

$$G_{\mu\nu} \longrightarrow G_{\mu\nu} + \frac{1}{2\epsilon} R_{\mu\nu}$$

not an entirely easy computation!

This gives the one-loop β-functional (obtained from the $\frac{1}{\epsilon}$ poles)

$$\beta_{\mu\nu} \propto \alpha' R_{\mu\nu}$$

D. Tong Lecture notes ; also GSW chapter 3

The condition for conformal invariance (to leading order in α') is

$$\beta_{\mu\nu} = 0 \quad \Rightarrow \quad R_{\mu\nu} = 0 \qquad !$$

target space must be Ricci-flat

that is, the string moves in a background spacetime which satisfies vacuum Einsteins eqs in 26 dim ($R_{\mu\nu} - \frac{1}{26} G_{\mu\nu} R = 0$).

consistency condition on the world sheet. $\longleftrightarrow$ spacetime equation of motion (spacetime dynamics !)

Higher orders in α': one gets stringy corrections to Einstein's eqs

$$\beta_{\mu\nu}(G_{\mu\nu}) = \alpha' R_{\mu\nu} + \underbrace{\frac{(\alpha')^2}{2} R_{\mu\kappa\rho\sigma} R_{\nu}{}^{\kappa\rho\sigma}}_{} = 0 \quad \text{to } O(\alpha'^2)$$

string theory predicts specific small corrections to Einstein's in $D=26$ @ large radius.

4.3 Including other massless fields

Apart from the graviton, we identified other massless fields in the closed string bosonic spectrum:

previously: identified the spacetime metric perturbations as insertions of the graviton vertex operator.

We can extend the Polyakov action further such that the effect in the path integral is to generate insertions of operators for the Ramond-Kalb $B_{\mu\nu}$ and the dilaton Φ fields

• The Ramond-Kalb antisymmetric field: $B_{\mu\nu}\, dx^\mu \wedge dx^\nu$

One can add to the Polyakov action the term

$$S^{(B)}[X] = -\frac{1}{4\pi\alpha'} \int d^2\xi \; \epsilon^{\alpha\beta} B_{\mu\nu}(X)\, \partial_\alpha X^\mu \partial_\beta X^\nu$$

which is reparametrization and Weyl invariant (also power counting renormalizable).

Moreover under spacetime gauge transformations

$$B \longrightarrow B + d\Lambda \quad , \quad \Lambda \text{ a 1-form}$$

the action $S^{(B)}$ changes by a surface term (exercise).

$(d\Lambda)_{\mu\nu} \sim \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$

• The dilaton Φ : We can add

$$S^{[\Phi]}[X;\gamma] = \frac{1}{4\pi\alpha'}\int d^2\xi \sqrt{\gamma}\, \Phi(X)\, R^{(2)}(\gamma)\, \alpha'$$

renormalizable

[For Φ = constant the integrand is a total derivative.]

This term however is $\boxed{\text{not}}$ Weyl invariant classically

$$\gamma \to e^{2\omega(\sigma)}\gamma \implies R^{(2)} \to e^{-2\omega}\left(R^{(2)} - 2\nabla^2\omega\right)$$

PS 1

$$S^{[\Phi]}[X,\gamma] \longrightarrow S^{\Phi}[X,\gamma] + \frac{1}{4\pi\alpha'}\int d^2\sigma \underbrace{\sqrt{\gamma}\,\Phi(X)\left(-2\nabla^2\omega\right)\alpha'}_{\text{not a total derivative if } \Phi \neq \text{const}}$$

One can show however that a classical Weyl variation of S^{Φ} can be cancelled by an $\mathcal{O}(\alpha')$ variation of $S^{(G)} + S^{(B)}$!

We have now a more general NLσM

$$S_\sigma = S^{(G)} + S^{(B)} + S^{(\Phi)}$$

$$S^{(G)}[\gamma, X] = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi \; G_{\mu\nu}(X)\,\partial_a X^\mu \partial^a X^\nu$$

$$S^{(B)}[X] = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi \; \epsilon^{\alpha\beta} B_{\mu\nu}(X)\,\partial_\alpha X^\mu \partial_\beta X^\nu$$

$$S^{[\Phi]}[X;\gamma] = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi \sqrt{\gamma}\; \Phi(X)\, R^{(2)}(\gamma)\,\alpha'$$

$S_\sigma \leftarrow$ field theory on Σ

with target space M which carries a geometrical structure $(G_{\mu\nu}, B_{\mu\nu}, \Phi)$.

Comment on the dilaton term:

Recall $$S^{(\Phi)} = \frac{1}{4\pi}\int d^2\xi \sqrt{\gamma}\, R^{(2)}(\gamma)\,\Phi(X)$$

We recognize this term as a generalisation of the topological term $\lambda\chi \propto \int d^2\xi\, \lambda R^{(2)}$ which is related to the string coupling constant.

NLσM: string coupling not a constant (it is a field $\Phi(X)$)

$S^{(\Phi)}$ is a term in an interacting theory in a background with $\Phi = \Phi_0 =$ constant

$\Phi(X) = \Phi_0 + \sqrt{\alpha'}\,\partial_\mu \Phi\, Y^\mu + \cdots$

in the weak coupling limit where $\boxed{g_s = e^{\Phi_0}}$

Generally: the string couplings are not parameters of the theory, they are dynamical.
(In the α'-expansion of the σ-model, we obtain an infinite set of couplings]

These considerations illustrate the fact that in string theory there are no continuous parameters.

Parameters are determined by eg expectation values of dynamical spacetime fields.

An $\boxed{\text{involved}}$ computation of the β-functional extending the one-loop computation for $S^{[G]}$ gives for the full σ-model action $S^{[G]} + S^{[B]} + S^{[\Phi]}$:

$$\beta^{G}_{\mu\nu} = \alpha' \Big(\underbrace{R_{\mu\nu} - \frac{1}{4} H_{\mu\lambda\sigma} H_{\nu}{}^{\lambda\sigma}}_{\text{1-loop } G+B} + \underbrace{2 \nabla_\mu \nabla_\nu \Phi}_{\text{classical } \Phi} \Big)$$

$H = dB$

$H_{\mu\nu\rho} = 3\,\partial_{[\mu} B_{\nu\rho]}$

$$\beta^{B}_{\mu\nu} = \alpha' \Big(\underbrace{-\frac{1}{2} \nabla^\lambda H_{\lambda\mu\nu} + (\nabla^\lambda \Phi) H_{\lambda\mu\nu}}_{\text{1-loop } G+B} \Big)$$

$$\beta^{\Phi} = \underbrace{\frac{1}{6}(D-26)}_{\text{1-loop } G+B} + \alpha' \Big(\underbrace{(\nabla_\mu \Phi)(\nabla^\mu \Phi)}_{\text{1-loop } \Phi} \underbrace{- \frac{1}{2} \nabla^2 \Phi - \frac{1}{24} H_{\mu\nu\rho} H^{\mu\nu\rho}}_{\text{two loop } G+B} \Big)$$

references: Friedan's thesis; Callan & Thorlacius "sigma models & string theory"; Tseytlin "Conformal anomaly in a 2dim σ-model"

Next

$\hookrightarrow$ Space-time effective action

We want to interpret the vanishing of the β-function as spacetime equations of motion.

4.4 Space-time effective action

We want to interpret the vanishing of the β-function as spacetime equations of motion.

Indeed, one can show that they arise as the Euler-Lagrange equations for the effective action

$$S_D^S = \frac{1}{2\kappa_0^2} \int d^D x \sqrt{-G}\, e^{-2\Phi} \left(\frac{2(26-D)}{3\alpha'} + R(G) - \frac{1}{12}|H|^2 + 4|\nabla \Phi|^2 \right)$$

"string frame" action (G, B, Φ in S_{LC}^S are the fields that appear in the σ-model action)

κ_0 related to Newton's constant: see next