String Theory 1

Lecture # 13

4. Strings in background fields

4.1 Introduction

4.2 Background field expansion and the Weyl anomaly

4.3 Including other massless fields

4.4 Space-time effective action

Last lecture: more general NLσM by adding new terms to the 2dim Polyakov action

$$S_\sigma = S^{(G)} + S^{(B)} + S^{(\Phi)}$$

$$S^{(G)}[\gamma, X] = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi \; G_{\mu\nu}(X)\,\partial_a X^\mu \partial^a X^\nu \qquad \text{(Polyakov)}$$

$$S^{(B)}[X] = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi \; \epsilon^{\alpha\beta} B_{\mu\nu}(X)\,\partial_\alpha X^\mu \partial_\beta X^\nu \longleftarrow \left\{ \begin{array}{l} \text{under spacetime transf} \\ B \longrightarrow B + d\Lambda\,, \quad \Lambda \text{ a 1-form} \\ S^{(B)} \longrightarrow S^{(B)} + \text{surface term} \end{array} \right.$$

$$S^{[\Phi]}[X;\gamma] = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi \sqrt{\gamma}\; \Phi(X)\, R^{(2)}(\gamma)\,\alpha'$$

S_σ : interacting 2dim QFT on Σ with target space M which carries a geometrical structure $(G_{\mu\nu}, B_{\mu\nu}, \Phi)$.

- the effect in the path integral is to generate insertions of vertex operators for the closed string massless fields ie the target space metric $G_{\mu\nu}$, the Ramond-Kalb $B_{\mu\nu}$ and the dilaton Φ

eg: identified the spacetime metric perturbations as insertions of the graviton vertex operator etc

- S_σ is reparametrisation invariant however

$$\begin{cases} S^{(G)} + S^{(B)} \text{ is classically Weyl invariant} \\ \text{but } S^{(\Phi)} \text{ is not (unless } \Phi = \text{constant)} \end{cases}$$

- complicated! as couplings depend on X^μ

To analyze the quantum NLσM

1. study the perturbation theory obtained by the covariant background field expansion with

$X^\mu(\xi) = X_0^\mu + \sqrt{\alpha'}\, Y^\mu(\xi)$ and expand around X_0^μ

$$S_\sigma[X] = -\frac{1}{4\pi}\int d^2\xi \Big\{ \underbrace{G_{\mu\nu}(X_0)}_{\eta_{\mu\nu}}\, \partial Y^\mu \cdot \partial Y^\nu$$ kinetic term

$$-\frac{1}{3}\sqrt{\alpha'}\, R_{\mu\rho\nu\sigma}(X_0)\, Y^\rho Y^\sigma (\partial Y^\mu)\cdot(\partial Y^\nu) + \cdots \Big\}$$

leading quartic interaction term

interacting QFT with an infinite set of coupling constants

⇝ perturbative α' expansion of the NLσM in powers of the fluctuation valid for

$$\sqrt{\alpha'} \sim \ell_s \ll r_c \quad \text{(large radius expansion)}$$

↪ we work with a weakly coupled σ-model

⇝ can read off Feynmann rules for the 2dim WS theory & compute the divergences that contribute to the renormalisation of the couplings.

② insist that the 2 dim QFT on the world sheet (i.e. NLσM) to be Weyl invariant at the quantum level. This implies, in particular, that the theory is conformally invariant.

Not only $S^{(\Phi)}$ is classically not Weyl invariant but the renormalisation process introduces a scale thus breaking conformal invariance

↳ computation of the β-function

$$T_{+-} \sim \beta_{\mu\nu}\, \partial X^\mu \partial X^\nu$$

Preservation of Weyl invariance at the quantum level iff $\beta^{(G)} = 0$, $\beta^{(B)} = 0$, $\beta^{(\Phi)} = 0$

Considering only $S^{[0]}$:

the condition for conformal invariance (to leading order in α') is

$$\beta^{[G]}_{\mu\nu} = 0 \quad \Rightarrow \quad R_{\mu\nu} = 0 \quad !$$

target space must be Ricci-flat

that is, the string moves in a background spacetime which satisfies vacuum Einsteins eqs in 26 dim ($R_{\mu\nu} - \frac{1}{26} G_{\mu\nu} R = 0$).

consistency condition on the world sheet. $\longleftrightarrow$ spacetime equation of motion (spacetime dynamics !)

An $\boxed{\text{involved}}$ computation of the β-functional to first order in α' gives for the full σ-model action

$$\beta^G_{\mu\nu} = \alpha'\Big(\underbrace{R_{\mu\nu} - \tfrac{1}{4} H_{\mu\lambda\sigma} H_\nu{}^{\lambda\sigma}}_{\text{1-loop } G+B} + \underbrace{2\nabla_\mu \nabla_\nu \Phi}_{\text{classical } \Phi}\Big)$$

$H = dB$

$H_{\mu\nu\rho} = 3\,\partial_{[\mu} B_{\nu\rho]}$

$$\beta^B_{\mu\nu} = \alpha'\Big(\underbrace{-\tfrac{1}{2}\nabla^\lambda H_{\lambda\mu\nu} + (\nabla^\lambda \Phi) H_{\lambda\mu\nu}}_{\text{1-loop } G+B}\Big)$$

$$\beta^\Phi = \underbrace{\tfrac{1}{6}(D-26)}_{\text{1-loop } G+B} + \alpha'\Big(\underbrace{(\nabla_\mu \Phi)(\nabla^\mu \Phi)}_{\text{1-loop } \Phi} \underbrace{- \tfrac{1}{2}\nabla^2 \Phi - \tfrac{1}{24} H_{\mu\nu\rho} H^{\mu\nu\rho}}_{\text{two loop } G+B}\Big)$$

references: Friedan's thesis; Callan & Thorlacius "Sigma models & string theory"; Tseytlin "Conformal anomaly in a 2dim σ-model"

4.4 Space-time effective action

We want to interpret the vanishing of the β-function as spacetime equations of motion.

Indeed, one can show that they arise as the Euler-Lagrange equations for the effective action

$$S_D^S = \frac{1}{2\kappa_0^2} \int d^D x \sqrt{-G}\, e^{-2\Phi} \left(\frac{2(26-D)}{3\alpha'} + R(G) - \frac{1}{12}|H|^2 + 4|\nabla\Phi|^2 \right)$$

"string frame" action (G, B, Φ in S_{ec}^S are the fields that appear in the σ-model action)

κ_0 related to Newton's constant: see next

For space-time computations one often uses the "Einstein frame":

let $\tilde{\Phi} = \Phi - \Phi_0$, $\tilde{G} = e^{\frac{1}{6}\tilde{\Phi}} G$

$\Phi(X) = \Phi_0 + \sqrt{\alpha'}\,\partial_\mu \Phi\, Y^\mu + \ldots$

$$S_D^{(E)} = \frac{1}{2\kappa^2} \int d^D x \sqrt{-\tilde{G}} \left(\frac{2(26-D)}{3\alpha'} + \tilde{R}(G) - \frac{1}{12} e^{-\frac{1}{3}\tilde{\Phi}} |H|^2 - \frac{1}{6} |\nabla \tilde{\Phi}|^2 \right)$$

$\kappa = \kappa_0 e^{\Phi_0} = (8\pi G_N)^{1/2}$

Gravitational coupling controls spacetime q-effects

indices raised and lowered with $\tilde{G}$

Einstein-Hilbert term takes the canonical form with gravitational coupling
$\kappa = (8\pi G_N)^{1/2}$

space time theory is GR coupled to additional fields

$\sqrt{\alpha'} \sim l_s \ll r_c$ (EFT with cutoff $M_s \sim 1/\sqrt{\alpha'}$)

The spacetime action should capture the classical limit when $E \ll M_s$.
The stringy corrections to this can be seen from the corrected β function

$$\beta = \beta^{(0)} + \alpha' \beta^{(1)} + (\alpha')^2 \beta^{(2)} + \cdots$$

harder ...

(For example $\beta^G_{\mu\nu} \sim \alpha' R_{\mu\nu} + \frac{(\alpha')^2}{2} R_{\mu\nu\lambda\sigma} R_\nu{}^{\lambda\rho\sigma} + \cdots$)

The corrected β-functional is interpreted as Euler-Lagrange equation for an α' corrected action:

$$S_{26} = S_{26}^{(0)} + \alpha' S_{26}^{(1)} + (\alpha')^2 S_{26}^{(2)} + \cdots$$

EFT (expansion with cutoff scale M_s)

$\frac{1}{M_s^2}$

4-derivative terms

$\frac{1}{M_s^4}$

6-derivative terms

↳ effective action obtained after integrating out massive modes

Remarks on the energy scales

Observations about the energy scales involved in the space-time effective action obtained by requiring that its EOM are the same as the vanishing of the Beta functions.

▶ The gravitational coupling

The Einstein frame is constructed such that the Einstein-Hilbert term takes the canonical form with gravitational coupling

$$\kappa = \kappa_0 e^{\Phi_0} = (8\pi G_N)^{1/2} \sim (M_{\text{Planck}})^{-\frac{1}{2}(D-2)}$$

related to the Planck mass

⇝ scale above which quantum gravitational effects become important

▸ We also have the string scale

$$\alpha' \sim M_s^{-2} \sim \ell_s^{+2}$$

scale at which size of the string becomes important

(Recall that we obtained an effective theory from the NLσM large radius expansion with cutoff M_s)

⇝ The string scale: controls stringy corrections (world sheet quantum corrections).

(eg deviations from GR in terms of higher derivative terms)

- The gravitational coupling κ and the string scale α' are related by the string coupling e^{Φ_0}

We have a dimensionless ratio

$$\frac{M_s}{M_{Pl}} \sim e^{+\frac{2}{D-2}\Phi_0}$$

This controls higher contributions in the genus expansion (higher loop orders)

Effective action is an action for the dynamics at energy scales $E \ll M_s$ in the limit $e^{\Phi_0} \to 0$

[suppress spacetime quantum effects]

Next: compactifications

↳ illustrate
- $\mathbb{R}^{1,24} \times S^1_R$
- T-duality

5 Compactifications

Target space: $\mathcal{M}_{D=26} = \mathbb{R}^{1,25-d} \times X_d$

Idea Kaluza-Klein: 1920

5 dim GR with $\mathcal{M}_5 = \mathbb{R}^{1,3} \times S^1_R$

EFT $\hookrightarrow$ 4 dim GR + field eqs of EM

Consider S^1-compactifications of the bosonic string ie

$$\mathcal{M}_{26} = \mathbb{R}^{1,24} \times S^1_R$$

circle of radius R

fields X^μ — $X^i \quad i=0,\dots,24$; $X^{25} \sim X^{25} + 2\pi R$

parametrising circle of radius R

We will discuss this from our two perspectives

1. From the spacetime EFT
   → Kaluza-Klein mechanism to obtain an effective theory on $\mathbb{R}^{1,24}$

(more generally $\mathcal{M} = \mathbb{R}^{1,25-d} \times X_d$ where the geometry & topology of X_d determine the parameters of the LEET on $\mathbb{R}^{1,25-d}$)

2. From the world sheet CFT perspective with target space $\mathbb{R}^{1,24} \times S^1_R$

[see BLT, DTong lectures]

5.1 Spacetime EFT approach

for the closed bosonic string theory

Kaluza-Klein ansatz for the fields to obtain an effective action in (1,24]-dimensions.

Fields $X^\mu \begin{cases} X^i & i=0,\ldots,24 \\ X^{25} & \text{coordinate on } S^1_R \end{cases}$

$G_{ij}(X^i, X^{25})$

metric $\quad G_{\mu\nu}\, dX^\mu dX^\nu = G_{ij}\, dX^i dX^j + e^{2\sigma}\left(dX^{25} + A_i\, dX^i\right)^2$

$G_{25,25} = e^{2\sigma}, \quad G_{25,i} = e^{2\sigma} A_i$

KR field $\quad B_{\mu\nu}\, dX^\mu \wedge dX^\nu = B_{ij}\, dX^i \wedge dX^j + \tilde{A}_i\, dX^i \wedge dX^{25}$

$B_{i,25} = \tilde{A}_i$

dilaton $\quad \Phi = \Phi_{(25)} + \frac{1}{2}\sigma$

see Polchinski 8.1

▸ One then rewrites the effective action $S_{(26)}$ in terms of

(X^i, X^{25}) $\qquad$ $G_{ij}, A_i, e^{2\sigma}$ $\qquad$ $i,j = 0,1,\dots,24$

$B_{ij}, \tilde{A}_i$

$\Phi_{(25)}$

This is a long computation, but that is ok.

Recall
$$S_{26} = \frac{1}{2K_0^2}\int d^{26}x \sqrt{-G}\, e^{-2\Phi}\left(R - \frac{1}{12}|H|^2 + 4|\nabla\Phi|^2\right)$$

For example:

$$R(G_{26}) = R(G_{25}) - \frac{1}{2}e^{2\sigma} F(A)^{ij} F(A)_{ij} - 2e^{-\sigma}\nabla^i\nabla_i e^{\sigma}$$

etc

- All these fields depend on X^i but also on X^{25}.

Due to the identification $X^{25}(\tau,\sigma) \sim X^{25}(\tau,\sigma) + 2\pi R$

we can expand these fields in Fourier modes with respect to X^{25}:

$$\mathcal{F}(X^i, X^{25}) = \sum_{n\in\mathbb{Z}} \mathcal{F}_n(X^i)\, e^{in\frac{1}{R}X^{25}}$$

independent of X^{25}

- Finally we integrate $S_{(26)}$ over X^{25} to obtain a theory in 25-dimensions

We will not be able to do all this explicitly (but see below for the dilaton)

↳ long computation indeed!

$n=0$ in Fourier series for the fields

Note however that, as we will see, the two modes (typically) give the massless sector of the theory.

The zero modes $(n=0)$ are

25 dim
- metric $G_{ij}(x^i)$
- KR field $B_{ij}(x^i)$
- 2x 1-form gauge fields

what is the gauge symmetry?

A (graviphoton) & $\hat{A}$ (KR-photon)

correspond to $U(1)\times U(1)$ gauge fields

- 2 scalars: σ, $\Phi_{(25)}$

The gauge field A : under a spacetime diffeomorphism $\delta X^\mu = \epsilon^\mu(X)$

the metric changes as $\delta G_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$.

Thus under $\delta X^{25} = \epsilon(X^i)$ reparametrisation of X^{25} direction

we find $\delta A_i = \partial_i \epsilon$ $\left(A_i = G_{25,i} \Rightarrow \delta A_i = \delta G_{25,i} = \partial_i \epsilon\right)$

So indeed we interpret A_i as a $U(1)$ gauge field and the gauge symmetry $\boxed{\text{descends}}$ from the 26-dimensional diffeomorphism invariance.

A_i is called the graviphoton.

The gauge field $\tilde{A}$: the KR field changes as $\delta B_{\mu\nu} = \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu$

Then under $\delta X^{25} = \lambda(X^i)$

we find $\delta \tilde{A}_i = \partial_i \lambda$ $\left(\tilde{A}_i = B_{i,25} \Rightarrow \delta\tilde{A}_i = \delta B_{i,25} = +\partial_i \lambda\right)$

$\tilde{A}_i$ is called the KR-photon

Let's look at the dilaton more carefully.

$$\Phi(X^\mu) = \Phi(X^i, X^{25})$$ recall $X^{25} \sim X^{25} + 2\pi R$

We expand this field (and any other fields) in Fourier modes with respect to X^{25}:

$$\Phi(X^\mu) = \sum_{n\in\mathbb{Z}} e^{in\frac{1}{R}X^{25}} \underbrace{\phi_n(X^i)}_{\text{independent of } X^{25}} \qquad \underbrace{\phi_n = \phi^*_{-n}}_{\text{because } \Phi \text{ is real-valued}}$$

Dilaton terms in the action $S_{(26)}$:

$$|\nabla_{26}\Phi|^2 = \partial_i\Phi\partial^i\Phi + (\partial_{25}\Phi)^2 = \sum_{n,m} e^{i(n+m)\frac{1}{R}X^{25}} \underbrace{\left\{\partial_i\phi_n\partial^i\phi_m - \frac{nm}{R^2}\phi_n\phi_m\right\}}_{\text{Independent of } X^{25}}$$

Then

in principle there is a factor $\sqrt{G}e^{2\phi}$

$$\int d^{26}x\, |\nabla_{26}\Phi|^2 = \int d^{25}x\, 2\pi R \sum_{n=-\infty}^{\infty}\left\{\partial_i\phi_n\partial^i\phi_{-n} + \frac{n^2}{R^2}\underbrace{\phi_n\phi_{-n}}_{|\phi_n|^2}\right\}$$

$\Rightarrow$ the massless dilaton $\Phi(X^\mu)$ of the 26-dimensional EFT $\rightarrow \mathbb{R}^{1,24} \times S^1_R$
gives rise to a discrete infinite tower of scalar fields ϕ_n,
the Kaluza-Klein modes, with mass $M_n^2 = \frac{1}{R^2} n^2$

For small R all are heavy modes except the massless mode ($n=0$)

can ignore for distance scale R with $E \ll \frac{1}{R} \sim M_{KK}$

$\rightsquigarrow$ the effective theory in $\mathbb{R}^{1,24}$ involving only the massless KK-modes.

The massive KK modes Φ_n $(n \neq 0)$ are charged under the graviphoton A_i

Recall that a diffeomorphism $\delta X^{25} = \epsilon(X^i)$ results in a gauge transformation $A_i \longrightarrow A_i + \partial_i \epsilon$. This has an effect on the KK-modes:

consider: $$\Phi(X^M) \xrightarrow[X^{25} \to X^{25} + \epsilon(X^i)]{} \sum_{n \in \mathbb{Z}} e^{in\frac{1}{R}(X^{25}+\epsilon)} \phi_n(X^i)$$

reparametrisation of X^{25} direction
(rotation on the S^1_R)

hence $$\phi_n \longrightarrow e^{in\epsilon/R} \phi_n$$

$\frac{n}{R} = p^{25}$

electric charge under A_i

That is the graviphoton charge is the KK-momentum

One can show that there are $\boxed{\text{no}}$ excitations charged under the $U(1)$ symmetry associated to the Ramond-Kalb-photon

Massless sector of the effective 25-dimensional theory:

$G_{\mu\nu}(x) \longrightarrow G_{\mu\nu}(x^i) : \{ G_{ij}(x^i), G_{i,25}(x^i), G_{25,25}(x^i) \}$

25 dim Graviton — graviphoton $e^{2\sigma} A$ — radion $e^{2\sigma}$

$B_{\mu\nu}(x) \longrightarrow B_{\mu\nu}(x^i) : \{ B_{ij}(x^i), B_{i,25}(x^i) \}$

25 dim KR field — KR-photon $\tilde{A}$

$\Phi(x) \longrightarrow \Phi(x^i)$ 25 dim dilaton

Remark: we have introduced a new scale $M_{KK} \sim \frac{1}{R}$

At energy scales $E \ll \frac{1}{R} \sim M_{KK}$ modes with $n \neq 0$ "decouple"

We should not trust the EFT analysis for $M_{KK} \sim M_s$

However, one can perform an exact analysis of the world sheet CFT!

Next

$\hookrightarrow$ World-sheet perspective (closed string)