String Theory 1

Lecture # 14

5 Compactifications continued

Consider S^1-compactifications of the bosonic string ie

$\mathcal{M}_{26} = \mathbb{R}^{1,24} \times S^1_R$ fields X^μ $\begin{cases} X^i & i = 0, \dots, 24 \\ X^{25} \sim X^{25} + 2\pi R \end{cases}$

circle of radius R

parametrises circle of radius R

We will discuss this from our two perspectives

① From the spacetime EFT

→ Kaluza-Klein mechanism to obtain an effective theory on $\mathbb{R}^{1,24}$

(more generally $\mathcal{M} = \mathbb{R}^{1,25-d} \times X_d$ where the geometry & topology of X_d determine the parameters of the LEET on $\mathbb{R}^{1,25-d}$)

② From the world sheet CFT perspective with target space $\mathbb{R}^{1,24} \times S^1_R$

[see BLT, DTong lectures]

5.1 Spacetime EFT approach (continued)

for the closed bosonic string theory

Kaluza-Klein ansatz for the fields to obtain an effective action in (1,24)-dimensions.

$$(G_{\mu\nu}) = \begin{pmatrix} G_{ij} & \frac{1}{2}e^{2\sigma}A_i \\ \frac{1}{2}e^{2\sigma}A_i & e^{2\sigma} \end{pmatrix}, \quad (B_{\mu\nu}) = \begin{pmatrix} B_{ij} & \frac{1}{2}\tilde{A}_i \\ -\frac{1}{2}\tilde{A}_i & 0 \end{pmatrix}, \quad \Phi = \Phi_{(25)} + \frac{1}{2}\sigma$$

↳ into

$$S_{26} = \frac{1}{2\kappa_0^2}\int d^{26}x \sqrt{-G}\, e^{-2\Phi}\left(R - \frac{1}{12}|H|^2 + 4|\nabla\Phi|^2\right)$$

so for example: $R(G_{26}) = R(G_{25}) - \frac{1}{2}e^{2\sigma} F(A)^{ij}F(A)_{ij} - 2e^{-\sigma}\nabla^i\nabla_i e^{\sigma}$
etc

- Due to the identification $X^{25}(\tau,\sigma) \sim X^{25}(\tau,\sigma) + 2\pi R$ we can expand these fields in Fourier modes with respect to X^{25}

$$\mathcal{F}(X^i, X^{25}) = \sum_{n\in\mathbb{Z}} \mathcal{F}_n(X^i)\, e^{in\frac{1}{R}X^{25}} :$$

independent of X^{25}

- Finally we integrate $S_{(26)}$ over X^{25} to obtain a theory in 25-dimensions

We will not be able to do all this explicitly $\rightsquigarrow$ long computation indeed!

- zero modes: fields in 25-dims $\rightsquigarrow$ massless-sector

metric G_{ij}
KR-field B_{ij}
2 $u(1)$ gauge fields: graviphoton A_i, KR-photon $\tilde{A}_i$
2 scalars $\Phi_{(25)}$, σ

Let's look at the $\underline{\text{dilaton}}$ more carefully.

$$\Phi(X^\mu) = \Phi(X^i, X^{25})$$ recall $X^{25} \sim X^{25} + 2\pi R$

We expand this field (and any other fields) in Fourier modes with respect to X^{25}:

$$\Phi(X^\mu) = \sum_{n\in\mathbb{Z}} e^{in\frac{1}{R}X^{25}} \underbrace{\phi_n(X^i)}_{\text{independent of } X^{25}} \qquad \underbrace{\phi_n = \phi_{-n}^*}_{\text{because } \Phi \text{ is real-valued}}$$

Dilaton terms in the action $S_{(26)}$:

$$|\nabla_{26}\Phi|^2 = \partial_i\Phi\partial^i\Phi + (\partial_{25}\Phi)^2 = \sum_{n,m} e^{i(n+m)\frac{1}{R}X^{25}} \underbrace{\left\{\partial_i\phi_n\,\partial^i\phi_m - \frac{nm}{R^2}\phi_n\phi_m\right\}}_{\text{Independent of } X^{25}}$$

Then

$$\int d^{26}x\, |\nabla_{26}\Phi|^2 \underset{\text{integrate wrt } X^{25}}{=} \int d^{25}x\, 2\pi R \sum_{n=-\infty}^{\infty}\left\{\partial_i\phi_n\,\partial^i\phi_{-n} + \frac{n^2}{R^2}\underbrace{\phi_n\phi_{-n}}_{|\phi_n|^2}\right\}$$

$\Rightarrow$ the massless dilaton $\Phi(X^\mu)$ of the 26-dimensional EFT $\mathbb{R}^{1,24} \times S^1_R$ gives rise to a discrete infinite tower of scalar fields ϕ_n, the Kaluza-Klein modes, with mass $M_n^2 = \frac{1}{R^2} n^2$

For small R all are heavy modes except the massless mode $(n=0)$

can ignore for distance scale R with $E << \frac{1}{R} \sim M_{KK}$

$\rightsquigarrow$ the effective theory in $\mathbb{R}^{1,24}$ involving only the massless KK-modes.

The massive KK modes Φ_n ($n \neq 0$) are charged under the graviphoton A_i

Recall that a diffeomorphism $\delta X^{25} = \epsilon(X^i)$ results in a gauge transformation $A_i \longrightarrow A_i + \partial_i \epsilon$. This has an effect on the KK-modes:

Consider:
$$\Phi(X^\mu) \xrightarrow[X^{25} \to X^{25} + \epsilon(X^i)]{} \sum_{n \in \mathbb{Z}} e^{in\frac{1}{R}(X^{25}+\epsilon)} \phi_n(X^i)$$

reparametrisation of X^{25} direction

Hence under a gauge transformation

$$\phi_n \longrightarrow e^{in\epsilon/R} \phi_n$$

$\frac{n}{R} = p^{25}$

electric charge under A_i

That is, the graviphoton charge is the KK-momentum

One can show that there are $\boxed{\text{no}}$ excitations charged under the $U(1)$ symmetry associated to the Ramond-Kalb-photon

EFT for the massless sector of the 25-dimensional theory:

$G_{\mu\nu}(X) \longrightarrow G_{\mu\nu}(X^i) : \{ \underbrace{G_{ij}(X^i)}_{\text{25 dim Graviton}}, \underbrace{G_{i,25}(X^i)}_{\substack{\text{graviphoton} \\ e^{2\sigma}A}}, \underbrace{G_{25,25}(X^i)}_{\substack{\text{radion} \\ e^{2\sigma}}} \}$

in the NLσM pert theory $\langle G_{25,25} \rangle = R^2$

$B_{\mu\nu}(X) \longrightarrow B_{\mu\nu}(X^i) : \{ \underbrace{B_{ij}(X^i)}_{\text{25 dim KR field}}, \underbrace{B_{i,25}(X^i)}_{\substack{\text{KR-photon} \\ \tilde{A}}} \}$

$\Phi(X) \longrightarrow \Phi(X^i)$ 25 dim dilaton

<u>Remark:</u> we have introduced a new scale $M_{KK} \sim \frac{1}{R}$

At energy scales $E \ll \frac{1}{R} \sim M_{KK}$ modes with $n \neq 0$ "decouple"

We should not trust the EFT analysis for $M_{KK} \sim M_s$

However, one can perform an $\boxed{\text{exact}}$ analysis of the world sheet CFT!

5.2 World-sheet perspective (closed string)

The target space for the two dimensional NLσM is $\mathbb{R}^{1,24} \times S^1_R$.

X^{25} is a field on S^1_R which is periodic ie $X^{25} \sim X^{25} + 2\pi R$

This non trivial topology has very interesting consequences.

- space-time translation by $2\pi R$: $e^{2\pi i R \hat{P}_{25}}$ should act as identity

generates translations along X^{25}

$$e^{2\pi i R \hat{P}_{25}} |\cdots, k_{25}\rangle = e^{2\pi i R k_{25}} |\cdots, k_{25}\rangle = |\cdots, k_{25}\rangle$$

iff $\boxed{k_{25} = \frac{m}{R}}$ $m \in \mathbb{Z}$ just as in the EFT analysis!

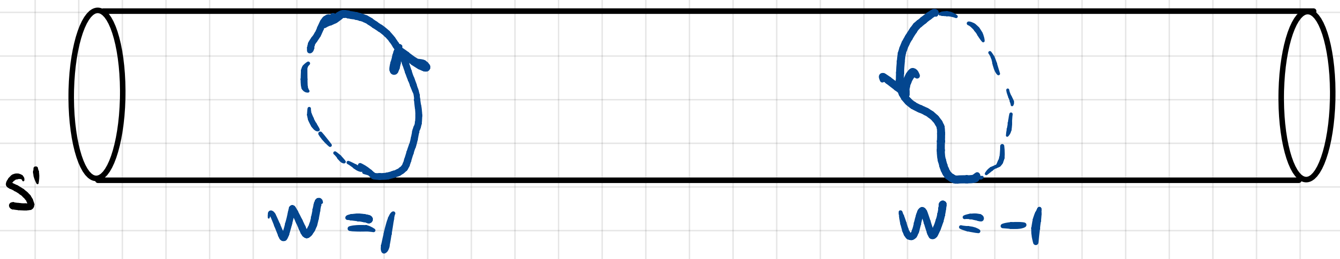
quantized

$$X^{25}(\tau, \sigma+2\pi) = X^{25}(\tau,\sigma) + 2\pi R w \qquad w \in \mathbb{Z}$$

(that is, X^{25} only needs to be periodic $\sigma \to \sigma + 2\pi$ up to $2\pi R$ shifts)

w is called the winding number

Term $2\pi R w$ $\Rightarrow$ closed strings wrapped on S^1_R and counts how many times the string wraps around S^1_R



winding is a stringy effect: there is nothing like this in the EFT we discussed

In the 26 dim EFT: these are solitons!

Spectrum of the string with target space $\mathbb{R}^{1,24} \times S^1_R$

Mode expansion of X^i $i = 0, \ldots, 24$ remains unchanged

just as in lecture 4 $(\mu \to i)$

separately periodic up to a zero mode

$$X^i_L(\xi^+) = \frac{1}{2} x^i + \frac{1}{2} \alpha' p^i \xi^+ + i \sqrt{\frac{\alpha'}{2}} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{1}{n} \tilde{\alpha}^i_n e^{-in\xi_+}$$

$$X^i_R(\xi^-) = \frac{1}{2} x^i + \frac{1}{2} \alpha' p^i \xi^- + i \sqrt{\frac{\alpha'}{2}} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{1}{n} \alpha^i_n e^{-in\xi^-}$$

where x^i, p^i, $\tilde{\alpha}^i_n$ and α^i_n are the Fourier coeffts (operators!)

$$\alpha^i_0 = \tilde{\alpha}^i_0 = \sqrt{\frac{\alpha'}{2}} p^i \quad \text{from periodicity} \quad \sigma \to \sigma + 2\pi$$

Mode expansion of X^{25} (which respects $X^{25}(\tau,\sigma+2\pi) = X^{25}(\tau,\sigma) + 2\pi R w$)

$$X^{25}(\tau,\sigma) = x^{25} + \alpha'\tau p^{25} + wR\sigma + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^{25} e^{-in\xi^-} + \tilde{\alpha}_n^{25} e^{-in\xi^+}\right) \qquad p^{25} = \frac{n}{R}$$

$$= X_R^{25}(\xi^-) + X_L^{25}(\xi^+)$$

where

$$X_R^{25}(\xi^-) = \frac{1}{2}x_R^{25} + \frac{1}{2}\alpha' p_R \xi^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{25} e^{-in\xi^-}, \qquad p_R = \frac{n}{R} - \frac{Rw}{\alpha'}$$

$$X_L^{25}(\xi^+) = \frac{1}{2}x_L^{25} + \frac{1}{2}\alpha' p_L \xi^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^{25} e^{-in\xi^+}, \qquad p_L = \frac{n}{R} + \frac{Rw}{\alpha'}$$

This is just as in $\mathbb{R}^{1,25}$ except that $\alpha_0^{25} = \sqrt{\frac{\alpha'}{2}}\, p_R \neq \tilde{\alpha}_0^{25} = \sqrt{\frac{\alpha'}{2}}\, p_L$

$$\alpha_0^{25} + \tilde{\alpha}_0^{25} = \sqrt{2\alpha'}\, p^{25} = \sqrt{2\alpha'}\,\frac{n}{R}; \qquad \alpha_0^{25} - \tilde{\alpha}_0^{25} = -\sqrt{\frac{2}{\alpha'}}\, Rw$$

The string states

Because $\tilde{\alpha}_0^{25}$ & α_0^{25} only differ by a c-number, the commutators of oscillators remain unchanged.

Then the Fock space is identical to that derived for $\mathbb{R}^{1,25}$ and states are organised into levels labelled by the e-values of N & $\tilde{N}$.

The oscillator vacuum state however is now labelled by

$$\underbrace{|0,\tilde{0};k}_{\substack{\text{25 dim momentum} \\ k \in \mathbb{R}^{1,24}}},\underbrace{n,w\rangle}_{} \qquad n, w \in \mathbb{Z}$$

$p^{25} = \frac{n}{R}$ from $X^{25}(\tau, \sigma + 2\pi) = X^{25}(\tau,\sigma) + 2\pi R w$

with

$$p^i |0,\tilde{0};k,n,w\rangle = k^i |0,\tilde{0};k,n,w\rangle$$

$$p^{25}_{L,R} |0,\tilde{0};k,n,w\rangle = \left(\frac{n}{R} \pm \frac{wR}{\alpha'}\right) |0,\tilde{0};k,n,w\rangle$$

General state in Fock space: $\prod \alpha^i_{-m} \prod \tilde{\alpha}^i_{-k} |0;\tilde{0};k;n;w\rangle$

Physical spectrum: Virasoro operators & constraints

↳ as before with some reorganisation

$$L_0 = \frac{1}{2}\left[\alpha_0 \cdot \alpha_0 + (\alpha_0^{25})^2\right] + \boxed{\sum_{n>0} \alpha_{-n}\cdot\alpha_n + \sum_{n>0} \alpha_{-n}^{25}\alpha_n^{25}} \leftarrow N$$

(1+24) dim inner product

$$L_m = \frac{1}{2}\sum_n \alpha_{m-n}\cdot\alpha_n + \frac{1}{2}\sum \alpha_{m-n}^{25}\alpha_n^{25} \qquad m \neq 0$$

Similar expressions for $\tilde{L}_m$

Physical states: $\begin{cases} L_m|\varphi\rangle = 0 & \tilde{L}_m|\varphi\rangle = 0 \quad \forall m>0 \\ (L_0-1)|\varphi\rangle = 0 & (\tilde{L}_0-1)|\varphi\rangle = 0 \end{cases}$

Because the algebraic structures are identical, the discussions about normal ordering go through as before.

The <u>main novelty</u> is the mass shell & level matching conditions

Mass-shell and level matching conditions

$$(L_0-1)|\phi\rangle=0 \quad \& \quad (\tilde{L}_0-1)|\phi\rangle=0$$

$$L_0-1=\frac{\alpha'}{4}(\underbrace{p\cdot p}_{-M_{25}^2}+p_R^2)+N-1 \longrightarrow M_{25}^2=p_R^2+\frac{4}{\alpha'}(N-1)$$

$$p_R=\frac{n}{R}-\frac{Rw}{\alpha'}$$

$$p_L=\frac{n}{R}+\frac{Rw}{\alpha'}$$

$$\tilde{L}_0-1=\frac{\alpha'}{4}(\underbrace{p\cdot p}_{-M_{25}^2}+p_L^2)+\tilde{N}-1 \longrightarrow M_{25}^2=p_L^2+\frac{4}{\alpha'}(\tilde{N}-1)$$

Then:

$$M_{(25)}^2=\frac{n^2}{R^2}+\frac{R^2w^2}{\alpha'^2}+\frac{2}{\alpha'}(N+\tilde{N}-2)$$

contribution to the mass from momentum along the compact direction

contribution to the mass of the string winding around the circle |w| times (stringy contributions)

mass shell condition

$$N-\tilde{N}=nw$$

level-mismatching condition

For $w=0$ this matches results from EFT

lowest energy state: tachyon $(N=\tilde{N}=0,\ m=0,\ w=0:\ M_T^2=-4/\alpha')$

Massless spectrum: for $N = \tilde{N} = 1$ ($\Rightarrow mw = 0$) and $n = w = 0$

26-dim
- graviton: $\gamma_{ij}\, \alpha^i_{-1} \tilde{\alpha}^j_{-1} |0; \tilde{0}; K, 0, 0\rangle$
- KR B-field: $B_{ij}\, \alpha^i_{-1} \tilde{\alpha}^j_{-1} |0; \tilde{0}; K, 0, 0\rangle$
- dilaton: scalar from $\alpha^i_{-1} \tilde{\alpha}^j_{-1} |0, \tilde{0}; K, 0, 0\rangle$
- graviphoton and KR-photon: $(\xi \cdot \alpha_{-1} \tilde{\alpha}^{25}_{-1} \pm \xi \cdot \tilde{\alpha}_{-1} \alpha^{25}_{-1}) |0, \tilde{0}; K, 0, 0\rangle$

(graviphoton from the 26 dim metric + another photon from the 26 dim KR field)

radion $\alpha^{25}_{-1} \tilde{\alpha}^{25}_{-1} |0, \tilde{0}; K\; 0, 0\rangle$

(identified with the scalar σ)

massless string spectrum $\longleftrightarrow$ massless spectrum from KK reduction of EFT

(For certain values of R, there are more massless states) *

$$M^2_{(25)} = \frac{n^2}{R^2} + \frac{R^2 w^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2)$$ depends on R

$$N - \tilde{N} = n w \qquad \tilde{N} = N - n w$$

$$M^2 = \frac{n^2}{R^2} + \frac{R^2 w^2}{\alpha'^2} + \frac{2}{\alpha'}(2N - nw - 2)$$

$$= \Big(\frac{n}{R} - \frac{Rw}{\alpha'}\Big)^2 + \frac{4}{\alpha'}(N-1)$$

+ve or zero

How many massless modes?

$$\Big(\frac{n}{R} - \frac{Rw}{\alpha'}\Big)^2 = \frac{4}{\alpha'}(1-N) \Rightarrow N = 0, 1$$

≥ 0

$$\tilde{N} = N - nw = -nw, \ 1 - nw$$

$$\tilde{N} \geq 0 \ \Big\langle \begin{array}{ll} nw \leq 0 & N=0 \\ nw \leq 1 & N=1 \end{array}$$

etc see later

States with nontrivial circle momentum (n/R) and winding (w)

In general these are not massless. (See problem sheet)

The simplest is the "winding" tachyon $|0;0;k;n=0,w\rangle$

($N=\tilde{N}=0$ so $nw=0$)

$$M^2_{(25)} = \frac{R^2}{\alpha'^2}w^2 - \frac{4}{\alpha'} = \frac{1}{\alpha'^2}\left(R^2w^2 - 4\alpha'\right) \qquad R^2w^2 < 4\alpha'$$

Remark: (Problem sheet)

For certain values of R there are more massless states & gauge symmetry can be enhanced

In particular, at $R=\sqrt{\alpha'}$ gauge symmetry is $SU(2)\times SU(2)$!

Vertex operators

As before:

one constructs the state $\leftrightarrow$ vertex operator correspondence.

Vertex operator for the tachyon:

$$V_{n,\omega}(p) \sim \int d^2\xi \; : e^{ip\cdot X} \underbrace{e^{ip_L X^{25} + i p_R X^{25}}}_{\text{assigns } (n,\omega) \text{ to any state}} :$$

It turns out that the vertex operator for the graviphoton & KR-photon is given by

$$V_{\pm}(\zeta, k) = \frac{1}{\sqrt{2\alpha'}} \int d^2\xi \; \zeta \cdot (\partial_+ X \, \partial_- X^{25} \pm \partial_- X \, \partial_+ X^{25}) :e^{ik\cdot X}:$$

$U(1) \times U(1)$ symmetry

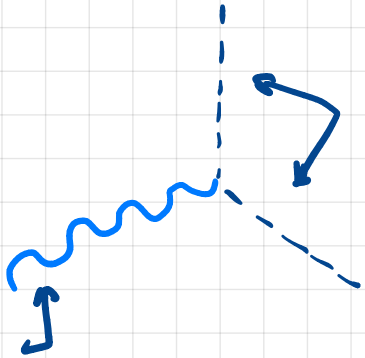
$$\zeta \cdot k = 0 \;, \quad k^2 = 0$$

Checking that KK-modes with $n \neq 0$ & winding modes are charged under the $U(1) \times U(1)$ gauge symmetries:

$$\underbrace{U(1)}_{A_i \text{ graviphoton}} \times \underbrace{U(1)}_{\tilde{A}_i}$$

Consider the 3-amplitude

In the 25-dim theory this is the tree level contribution to an emission/absorption of a KR-photon from a w-tachyon to a w-tachyon with winding number w



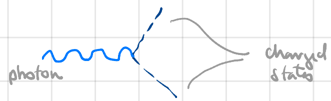
winding tachyon $(n=0)$

$|0, \tilde{0}; k, 0, w\rangle$

$M^2_{(25)} = \frac{1}{\alpha'^2}(R^2 w^2 - 4\alpha')$

Kalb-Ramond photon

$(\xi \cdot \alpha_{-1} \tilde{\alpha}^{25}_{-1} - \xi \cdot \tilde{\alpha}_{-1} \alpha^{25}_{-1}) |0, k; 0, 0\rangle$



read off charge of the state: coefficient in front of the coupling of the fields

Vertex operator for the KR photon

$$V_{KR}(\xi, k) = \frac{1}{\sqrt{2\alpha'}} \int d\tau d\sigma \; \xi \cdot (\partial_+ X \partial_- X^{25} - \partial_- X \partial_+ X^{25}) e^{ik \cdot X}$$

Computing the amplitude:

$$A \sim \langle 0, -k_3; 0, w |\, V_{KR}(\xi, k_2)\, | 0, k_1; 0\, w \rangle$$

$$= \frac{1}{\sqrt{2\alpha'}} \underline{\langle 0, -k_3; 0, w} \,|\, (\xi \cdot \partial_+ X\, \partial_- X^{25} - \xi \cdot \partial_- X\, \partial_+ X^{25})\, e^{ik_2 \cdot X}\, \underline{|0, k_1; 0, w\rangle}$$

$$= \frac{1}{\sqrt{2\alpha'}} \langle 0, -k_3; 0, w |\, (\xi \cdot \tilde{\alpha}_0\, \alpha_0^{25} - \xi \cdot \alpha_0\, \tilde{\alpha}_0^{25})\, |0, k_1 + k_2; 0, w\rangle$$

$$= \frac{1}{\sqrt{2\alpha'}}\, \xi \cdot (k_1 + k_2)\, \langle 0, -k_3; 0, w |\, \underbrace{(\alpha_0^{25} - \tilde{\alpha}_0^{25})}\, |0, k_1 + k_1; 0, w\rangle \longrightarrow \alpha_0^{25} - \tilde{\alpha}_0^{25} = -\sqrt{\frac{2}{\alpha'}}\, R w$$

$$= \left(\frac{1}{\alpha'} R w\right)\, \xi \cdot k_3\, \delta^{(25)}(k_1 + k_2 + k_3)$$

winding tachyon charged under the KR U(1) gauge field $\tilde{A}$ with charge q $w \frac{R}{\alpha'}$

Similar computation for the graviphoton:

momentum $\frac{n}{R}$ is the charge under A_i

This agrees with the KK reduction

More generally, states with circle momentum & winding numbers (n, w) have graviphoton $U(1)$ charge $\frac{1}{2}(p_L + p_R) = \frac{n}{R}$

and KR-photon $U(1)$ charge $\frac{1}{2}(p_L - p_R) = \frac{Rw}{\alpha'}$

Remark: we have introduced a new scale R

R is called a modulus

(more general compactifications give rise to moduli spaces)

In fact, we have a one parameter family of compactifications with $R \in (0, \infty)$

Or do we?

$(0, \infty)$ containts values of R which give rise to indistinguishable physical theories.

!

↳ next : T duality