String Theory 1

Lecture # 15

5 Compactifications

$\mathbb{R}^{1,24} \times S^1_R \qquad X^{25} \sim X^{25} + 2\pi r$

Two perspectives continued ...

5.1 Spacetime EFT approach

Kaluza-Klein mechanism on the 26 dim EFT

- We obtained a $\mathbb{R}^{1,24}$ EFT for the massless sector

$\{ G_{ij}, e^{2\sigma} A, e^{2\sigma} \}$ — 25 dim graviton, graviphoton, radion

$\{ B_{ij}, \tilde{A} \}$ — 25 dim KR field, KR-photon

ϕ — dilaton

2 U(1) gauge symmetries

Action for Gravity + B + U(1) x U(1) + scalar in 25 dims

- Plus a discrete infinite tower of massive states (KK-modes) $M^2_{KK} \sim \frac{1}{R^2}$

These are charged under the graviphoton: charge $\frac{n}{R}$ (KK-momentum)

There are no modes charged under the U(1) corresponding to the KR photon.

- EFT good for $E \ll M^2_{KK}$ $\quad M_{KK} \sim \frac{1}{R}$ — can ignore massive modes

5.2 The world sheet perspective   exact description

2dim World sheet NLσM with target space with a nontrivial topology

$$X^i \longrightarrow X^i \qquad i = 0, \dots, 24$$

$$X^{25} \sim X^{25} + 2\pi R \qquad (X^{25} \text{ parametrises a circle } S^1_R)$$

States in the string Hilbert space are similar to those of $\mathbb{R}^{1,25}$, however:

we have states with quantized momentum ($p^{25} = \frac{n}{R}$)   (no KK-modes)

and quantized winding modes.

The winding modes come from the periodicy condition

$$X^{25}(\tau, \sigma + 2\pi) = X^{25}(\tau, \sigma) + 2\pi R w \qquad w \in \mathbb{Z} \qquad (X^{25}(\tau,\sigma) \text{ periodic up to } 2\pi R w)$$

The mode expansion of $X^i(\tau,\sigma)$, $i=0,\dots,24$, is as for $\mathbb{R}^{1,25}$, but the expansion for X^{25} changes

$$X^{25}(\tau,\sigma) = x^{25} + 2\alpha' p^{25}\tau + \underline{2Rw\sigma} + \text{oscillator modes}, \qquad p^{25} = \frac{n}{R} \qquad n \in \mathbb{Z}$$

The states are of the form: $\prod \alpha^{\mu}_{-n} \prod \tilde{\alpha}^{\nu}_{-m} \, |K; n; w\rangle$

K: 25-dim momentum evalue

n: $p^{25} = \frac{n}{R}$

w: $X^{25}(\tau,\sigma+2\pi) = X^{25}(\tau,\sigma) + 2\pi R w$

and physical states must satisfy the conditions:

$(L_0 - 1)|\phi\rangle = 0$
$(\tilde{L}_0 - 1)|\phi\rangle = 0$

$$\begin{cases} M^2_{(25)} = \dfrac{n^2}{R^2} + \underbrace{\dfrac{R^2 w^2}{\alpha'^2}} + \dfrac{2}{\alpha'}(N + \tilde{N} - 2) \\[2ex] N - \tilde{N} = n w \end{cases}$$

contribution to the mass from momentum along the compact direction

contribution to the mass of the string winding around the circle $|w|$ times $\left(\frac{Rw}{\alpha'}\right)^2 = (2\pi R T w)^2$

mass shell condition

$\boxed{M_{(25)} \text{ depends on } R!}$

level-(mis)matching condition

Also $L_m|\phi\rangle = 0 \quad \forall m > 0$ (and similarly for $\tilde{L}_m$)

Massless spectrum: for any R (states with $m=w=0$, $N=\tilde{N}=1$)

- 25 dim graviton $\quad \gamma_{ij}\, \alpha_{-1}^{i}\, \tilde{\alpha}_{-1}^{j} |0,K;0,0\rangle$

- 25 dim B-field $\quad B_{ij}\, \alpha_{-1}^{i}\, \tilde{\alpha}_{-1}^{j} |0,K;0,0\rangle$

- dilaton: scalar from the trace part of γ: $\Phi_{(25)}$

- 2x 25 dim $U(1)\times U(1)$ gauge fields $\quad \xi\cdot(\alpha_{-1}\tilde{\alpha}_{-1}^{25} \pm \tilde{\alpha}_{-1}\alpha_{-1}^{25})|0,K;0,0\rangle$

(graviphoton from the 26 dim metric + another photon from the 26 dim KR field)

- scalar ("radion") $\quad \alpha_{-1}^{25}\, \tilde{\alpha}_{-1}^{25} |0,K\rangle \otimes |0,0\rangle$

identified with a scalar σ

massless string spectrum $\longleftrightarrow$ massless spectrum from KK reduction of EFT

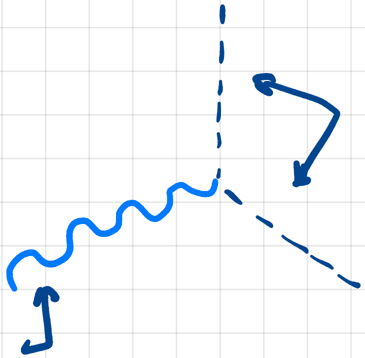
For certain values of R (eg $\alpha' = \sqrt{R}$!) there are more!

Checking that KK-modes with $n \neq 0$ & winding modes are charged under the $\underbrace{U(1)}_{A_i}\times\underbrace{U(1)}_{\tilde{A}_i}$ gauge symmetries:

A_i graviphoton

Consider the 3-amplitude

In the 25-dim theory this is the tree level contribution to an emission/absorption of a KR-photon from a w-tachyon to a w-tachyon with winding number w



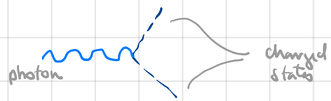
winding tachyon ($n=0$)

$|0,\tilde{0};k,0,w\rangle$

$M^2_{(25)} = \frac{1}{\alpha'^2}(R^2 w^2 - 4\alpha')$

Kalb-Ramond photon

$(\xi\cdot\alpha_{-1}\tilde{\alpha}^{25}_{-1} - \xi\cdot\tilde{\alpha}_{-1}\alpha^{25}_{-1})\,|0,k;0,0\rangle$



read off charge of the state: coefficient in front of the coupling of the fields

Vertex operator for the KR photon

$$V_{KR}(\xi,k) = \frac{1}{\sqrt{2\alpha'}}\int d\tau\, d\sigma\; \xi\cdot(\partial_+X\,\partial_-X^{25} - \partial_-X\,\partial_+X^{25})\,e^{ik\cdot X}$$

Computing the amplitude:

$$A \sim \langle 0; -k_3, 0, w | \, V_{KR}(\varsigma, k_2) \, | 0; k_1, 0\, w \rangle$$

$$= \frac{1}{\sqrt{2\alpha'}} \underline{\langle 0; -k_3}, 0, w | \left(\varsigma \cdot \partial_+ X \partial_- X^{25} - \varsigma \cdot \partial_- X \partial_+ X^{25} \right) e^{ik_2 \cdot X} \underline{|0; k_1, 0, w\rangle}$$

$$= \frac{1}{\sqrt{2\alpha'}} \langle 0; -k_3, 0, w | \left(\varsigma \cdot \tilde{\alpha}_0 \alpha_0^{25} - \varsigma \cdot \alpha_0 \tilde{\alpha}_0^{25} \right) |0; k_1 + k_2, 0, w\rangle$$

$$= \frac{1}{\sqrt{2\alpha'}} \varsigma \cdot (k_1 + k_2) \langle 0; -k_3, 0, w | \underbrace{\left(\alpha_0^{25} - \tilde{\alpha}_0^{25} \right)} | 0; k_1 + k_2, 0, w \rangle \longrightarrow \alpha_0^{25} - \tilde{\alpha}_0^{25} = -\sqrt{\frac{2}{\alpha'}} R w$$

$$= \left(\frac{1}{\alpha'} R w \right) \varsigma \cdot k_3 \, \delta^{(25)}(k_1 + k_2 + k_3)$$

winding tachyon charged under the KR $U(1)$ gauge field $\tilde{A}$ with charge q $w \frac{R}{\alpha'}$

Similar computation for the graviphoton:

momentum $\frac{n}{R}$ is the charge under A_i

This agrees with the KK reduction

More generally, states with circle momentum & winding numbers (n, w) have graviphoton $U(1)$ charge $\frac{1}{2}(p_L^{25} + p_R^{25}) = \frac{n}{R}$

and KK-photon $U(1)$ charge $\frac{1}{2}(p_L^{25} - p_R^{25}) = \frac{Rw}{\alpha'}$

$$p^{25}_{L,R} = \left(\frac{n}{R} \pm \frac{wR}{\alpha'} \right)$$

Remark: we have introduced a new scale R

In fact, we have a one parameter family of compactifications with $R \in (0, \infty)$ R is called a modulus

(More general compactifications give rise to a moduli space)

However $(0, \infty]$ contains values of R which give rise to indistinguishable physical theories.

5.3 T-duality (closed strings)

Symmetry of the spectrum: observe that the formulas

$$M^2_{(2\pi)} = \frac{n^2}{R^2} + \frac{1}{(\alpha')^2} w^2 R^2 + \frac{2}{\alpha'}(N+\tilde{N}-2) \quad , \quad N-\tilde{N} = nw$$

are invariant under: $n \leftrightarrow w$ & $R \leftrightarrow \frac{\alpha'}{R} = \hat{R}$

$\Rightarrow$ compactifications on S_R & $S_{\alpha'/R}$ have the $\boxed{\text{same}}$ spectrum.

[Note that $R = \sqrt{\alpha'}$ is a fixed point of this transformation: something special happens at this point.]

We will see that, in fact, this is an exact symmetry of the CFT

$\boxed{\text{T-duality}}$

as compactifications on S_R & $S_{\hat{R}}$ with $\hat{R} = \frac{\alpha'}{R}$ are indistinguishable as physical theories.

The interchange $n \leftrightarrow w$ means that

momentum excitations $\longleftrightarrow$ winding mode excitations

T-duality as an exact symmetry of the CFT

BUT, we have $\boxed{\text{only}}$ shown that the spectrum is the same for two theories where

$$R \longleftrightarrow \hat{R} = \frac{\alpha'}{R}$$

and simultaneously

$$(n, w) \longleftrightarrow (w, n)$$

We need to consider the $\boxed{\text{full CFT}}$ to prove this is an $\boxed{\text{exact}}$ symmetry of the CFT

any transformation which reorganizes the CFT data which does not modify the physical (null state conditions & correlation functions)

Recall $\alpha_0^{25} = \sqrt{\frac{\alpha'}{2}}\left(n\frac{1}{R} - \frac{R}{\alpha'}w\right) = \sqrt{\frac{\alpha'}{2}}\, p_R^{25}$

$p_{L,R}^{25} = \left(\frac{n}{R} \pm \frac{wR}{\alpha'}\right)$

$\tilde{\alpha}_0^{25} = \sqrt{\frac{\alpha'}{2}}\left(n\frac{1}{R} + \frac{Rw}{\alpha'}\right) = \sqrt{\frac{\alpha'}{2}}\, p_L^{25}$

Then under $R \leftrightarrow \frac{\alpha'}{R}$ and $n \leftrightarrow w$

$$\alpha_0^{25} \longrightarrow \sqrt{\frac{\alpha'}{2}}\left(w\frac{R}{\alpha'} - \frac{1}{R}n\right) = -\alpha_0^{25}$$

$$\tilde{\alpha}_0^{25} \longrightarrow \sqrt{\frac{\alpha'}{2}}\left(w\frac{R}{\alpha'} + \frac{1}{R}n\right) = \tilde{\alpha}_0^{25}$$

interchanges the graviphoton with the KK-photon!

$2\times$ 25 dim
$u(1)\times u(1)$ gauge fields

$\zeta\cdot(\alpha_{-1}\tilde{\alpha}_{-1}^{25} \pm \tilde{\alpha}_{-1}\alpha_{-1}^{25})|0,k;0,0\rangle$

We extend the action of the transformation to the oscillator modes

define:

$$X_R^{25}(\xi^-) \longleftrightarrow \hat{X}_R^{25}(\xi^-) = -X_R^{25}(\xi^-)$$

$$X_L^{25}(\xi^+) \longleftrightarrow \hat{X}_L^{25}(\xi^+) = X_L^{25}(\xi^+)$$

$$X^i(\xi^\pm) \longleftrightarrow \hat{X}^i(\xi^\pm) = X^i(\xi^\pm) \qquad i=0,\dots,24$$

new coordinate fields

Mode expansions:

$$X^{25}(\tau,\sigma) = X_L^{25}(\xi^+) + X_R^{25}(\xi^-) = x^{25} + 2\alpha' \frac{n}{R}\tau + 2Rw\sigma + \dots$$

$$\longleftrightarrow$$

$$\hat{X}^{25}(\tau,\sigma) = X_L^{25}(\xi^+) - X_R^{25}(\xi^-) = \hat{x}^{25} + 2\alpha' \frac{n}{R}\sigma + 2Rw\tau + \dots$$

- circle radius R

$\sigma \to \sigma + 2\pi$

$X^{25}(\tau,\sigma+2\pi) = X^{25}(\tau,\sigma) + 2\pi R w$

- momentum $p^{25} = \frac{n}{R}$

- circle radius $\frac{\alpha'}{R}$

$\sigma \to \sigma + 2\pi$

$\hat{X}^{25}(\tau,\sigma+2\pi) = X^{25}(\tau,\sigma) + 2\pi \frac{\alpha'}{R} n$

$\hat{R}$

$2\pi \hat{R}$ periodic!

- momentum $\hat{p}^{25} = \frac{wR}{\alpha'}$

X & $\hat{X}$ have the same energy momentum tensor

$$T_{\pm\pm} = \partial_\pm X \cdot \partial_\pm X = \partial_\pm \hat{X} \, \partial_\pm \hat{X}$$

$\partial_+ \hat{X}^{\tau\sigma} = \partial_+ X_L^{\tau\sigma} = \partial_+ X^{\tau\sigma}$
$\partial_- \hat{X}^{\tau\sigma} = -\partial_- X_R^{\tau\sigma} = -\partial_- X^{\tau\sigma}$

So one can recover L_m & $\tilde{L}_m$ as Fourier modes

$\Rightarrow$ CFTs of X & $\hat{X}$ are the $\boxed{\text{same}}$ with $\hat{R} = \dfrac{\alpha'}{R}$

As a consequence of this duality the moduli space of circle compactifications of the bosonic string is not $(0, \infty)$ but instead

$\boxed{R \in (0, \sqrt{\alpha'}]}$ or equivalently $R \in [\sqrt{\alpha'}, \infty)$

$R = \sqrt{\alpha'}$ fixed point

Fixed point of the duality transformation:

$$R \longleftrightarrow \hat{R} = \frac{\alpha'}{R} \quad \text{when} \quad \boxed{R = \sqrt{\alpha'}}$$

$R = \sqrt{\alpha'}$ is special $\rightsquigarrow$ more massless states and enhanced gauge symmetry

$R = \sqrt{\alpha'}$

$$M^2_{(25)} = \frac{n^2}{R^2} + \frac{1}{(\alpha')^2} w^2 R^2 + \frac{2}{\alpha'}(N + \tilde{N} - 2) \overset{\downarrow}{=} \frac{1}{R^2}\left(n^2 + w^2 + 2(N + \tilde{N} - 2)\right)$$

so $M^2_{25} = 0$ when $n^2 + w^2 + 2(N + \tilde{N}) = 4$

& $nw = N - \tilde{N}$

$$n^2 + w^2 + 2(N+\tilde N) = 4 \qquad R = \sqrt{-\alpha'}$$

$$\& \quad nw = N - \tilde N$$

$$(n-w)^2 = 4 - 2(N+\tilde N) - 2nw = 2(2 - (\cancel{N}+\tilde N) - (N - \cancel{\tilde N}))$$

$$= 4(1-N) \quad \text{so} \quad \underline{N = 0,1}$$

$$(n+w)^2 = 4 - 2(N+\tilde N) + 2nw = 2(2 - (\cancel{N}+\tilde N) + (\cancel{N}-\tilde N))$$

$$= 4(1-\tilde N) \quad \text{so} \quad \underline{\tilde N = 0,1}$$

possible cases: $(N,\tilde N) = (0,0), (0,1), (1,0), (1,1)$

$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$nw=0$	$nw=-1$	$nw=1$	$nw=0$
$n^2+w^2=4$	$n=-w$	$n=w$	
	$w=\pm 1$	$w=\pm 1$	

$N = \tilde{N} = 0 \implies nw = 0 \quad n^2 + w^2 = 4$

$n = 0 \quad w = \pm 2 \quad |0, \tilde{0}; 0, \pm 2; K\rangle$

$w = 0 \quad n = \pm 2 \quad |0, \tilde{0}; \pm 2, 0; K\rangle$

4 new scalars

$N = 1 \quad \tilde{N} = 0 \implies nw = 1 \quad \left\langle \begin{array}{l} n = w = 1 \\ n = w = -1 \end{array} \right.$

$\lambda \cdot \alpha_{-1} |0, \tilde{0}; \pm 1, \pm 1, K\rangle$ 2x (1 gauge field + 1 scalar)

$N = 0 \quad \tilde{N} = 1 \implies nw = -1 \quad \left\langle \begin{array}{l} n = -w = 1 \\ n = -w = -1 \end{array} \right.$

$\tilde{\lambda} \cdot \tilde{\alpha}_{-1} |0, \tilde{0}; \mp 1, \pm 1, K\rangle$ 2x (1 gauge field + 1 scalar)

✓ $N = \tilde{N} = 1 \implies \begin{array}{l} nw = 0 \\ n^2 + w^2 = 0 \end{array} \implies n = w = 0$

$\gamma_{\mu\nu} \alpha_{-1}^{\mu} \tilde{\alpha}_{-1}^{\nu} |0, \tilde{0}; 0, 0; K\rangle$ graviton, KR, 2 gauge fields, 2 scalars !

There are in fact

4 extra massless vectors which enhance the $U(1) \times U(1)$ symmetry to $SU(2) \times SU(2)$

$$\xi \cdot \alpha_{-1}^{25} \tilde{\alpha}_{-1} |0, \tilde{0}; 0, 0; K\rangle, \quad \tilde{\xi} \cdot \alpha_{-1} \tilde{\alpha}_{-1}^{25} |0, \tilde{0}; 0, 0; K\rangle$$

$$\lambda \cdot \alpha_{-1} |0, \tilde{0}; \pm 1, \pm 1, K\rangle \qquad \tilde{\lambda} \cdot \tilde{\alpha}_{-1} |0, \tilde{0}; \mp 1, \pm 1, K\rangle$$

and

8 additional scalar fields $\longrightarrow$ $(\underline{3}, \underline{3})$ rep of $SU(2) \times SU(2)$

φ, σ

$$|0, \tilde{0}; 0, \pm 2; K\rangle \qquad |0, \tilde{0}; \pm 2, 0; K\rangle$$

$$\alpha_{-1}^{25} |0, \tilde{0}; \pm 1, \pm 1, K\rangle \qquad \alpha_{-1}^{25} |0, \tilde{0}; \mp 1, \pm 1, K\rangle$$

(BLT 10.2 for details)

5.4 Open strings and T-duality

What happens to T-duality?

closed strings it was crucial that strings can wind around S^1

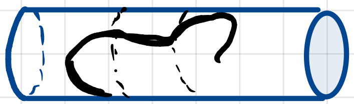
Recall: open string boundary conditions compatible with Poincaré invariance in 26 dimensions

$$\frac{\partial}{\partial\sigma} X^\mu(\tau,\sigma) = 0 \quad \text{at} \quad \sigma = 0, \pi$$

Neumann boundary conditions

(ends of the string are free to move in spacetime)

Consider now compactifying on a circle



$\rightarrow$ no winding modes !

while KK-momentum momentum modes still make sense

Recall, for open strings:

Soln of $\partial_+ \partial_- X^\mu = 0$

$$X_L^\mu(\xi^+) = \frac{1}{2} x^\mu + \alpha' p^\mu \xi^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{1}{m} \alpha_m^\mu e^{-im\xi^+}$$

$$X_R^\mu(\xi^-) = \frac{1}{2} x^\mu + \alpha' p^\mu \xi^- + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{1}{m} \alpha_m^\mu e^{-im\xi^-}$$

Neumann boundary condition $\quad \partial_\sigma X^\mu \Big|_{\sigma=0,\pi} = 0 \quad \Rightarrow \alpha_m^\mu = \tilde{\alpha}_m^\mu$

Dirichlet boundary condition $\quad X^\nu \Big|_{\sigma=0,\pi} = c^\nu$

$$\Rightarrow \quad x^\nu = c^\nu \;, \quad p^\nu = 0 \;, \quad \alpha_m^\nu = -\tilde{\alpha}_m^\nu$$

compactify on a circle with X^{25} parametrising the circle of radius R & consider an open string with NN boundary conditions in the coordinate X^{25} ($2\pi R$ periodic)

so both ends of the string move freely on the circle S^1_R

Follow the same procedure as for the bosonic string.

What happens when interchanging

$$X_L^{25} \longleftrightarrow X_L^{25} \qquad X_R^{25} \longleftrightarrow -X_R^{25} \qquad ?$$

What should we expect?

Should we expect a dual string for which there is a winding quantum number but no KK-momentum?

The proposed dual coordinate is

$$\hat{X}^{25}(\tau,\sigma) = X_L^{25}(\xi^+) - X_R^{25}(\xi^-)$$

$$= 2\alpha' p^{25}\sigma + \sqrt{2\alpha'}\sum_{m\neq 0}\frac{1}{m}\alpha_m^{25} e^{-in\tau}\sin(m\sigma)$$

$$= \frac{2\alpha'}{R} n\,\sigma + osc = 2\hat{R} n\,\sigma + osc$$

$\hat{R} = \frac{\alpha'}{R}$ dual circle $S^1_{\hat{R}}$

- periodicity $\hat{X}^{25}(\tau,\sigma+\pi) = \hat{X}^{25}(\tau,\sigma) + \underline{2\pi\hat{R}n}$

- no term linear in τ ie the dual string has no momentum in the circle direction : translation invariance along S^1 is broken

- Moreover dual string wraps around the dual circle n times

Boundary conditions of the dual string: at $\sigma = 0, \pi$

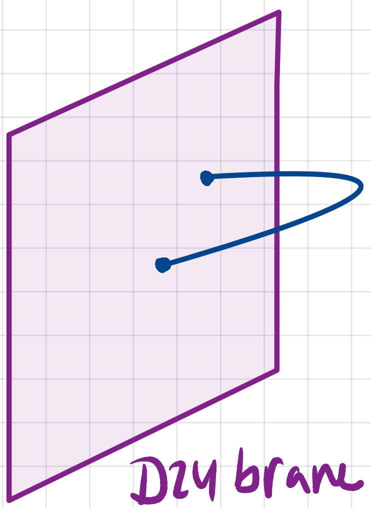
$$\hat{X}^{25}(\tau,\sigma)\Big|_{\sigma=0} = 0$$

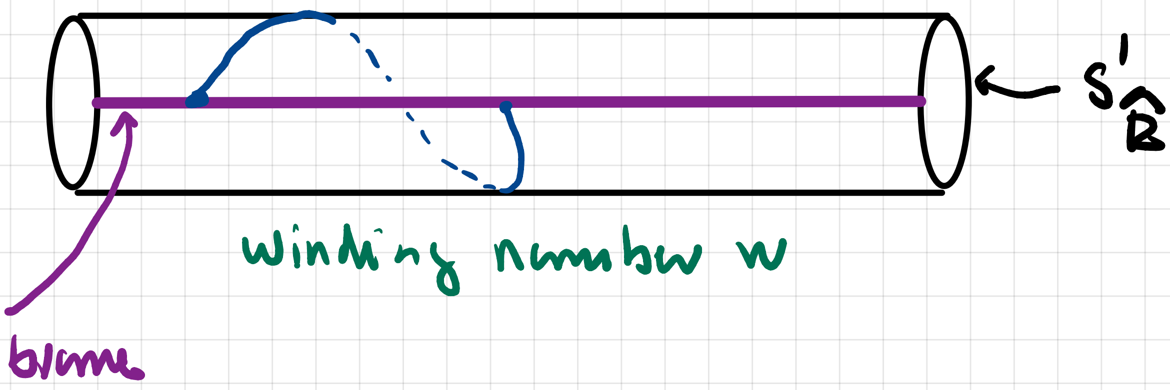
$$\hat{X}^{25}(\tau,\sigma)\Big|_{\sigma=\pi} = 2\alpha' \frac{n}{R}\pi = 2\pi n \hat{R}$$

position of the end points of the dual string are fixed.

→ This is a Dirichlet boundary condition!

The dual open string is attached to a (1+24) dimensional hyperplane, a $\boxed{\text{D24-brane}}$





Under a T-duality transformation :

open string with Neuman boundary condition on S^1_R $\longleftrightarrow$ open string with Dirichlet boundary condition on $S^1_{\hat{R}}$

[momentum $\frac{n}{R}$ along S^1_R $\longleftrightarrow$ no momentum along $S^1_{\hat{R}}$

no winding around S^1_R $\longleftrightarrow$ winding around $S^1_{\hat{R}}$]

The subspace where the string ends are attached to is called a D-brane

convention : a Dp-brane is a D-brane with p spatial dimensions

(so it is p+1 dimensional)

T-duality

$\longleftrightarrow$

open string with Neumann boundary conditions compactified on S^1_R

dual open string with Dirichlet boundary conditions compactified on $S^1_{\hat{R}}$, $\hat{R} = \alpha'/R$

D25 space-filling brane
↳ open string ends are free to move on spacetime

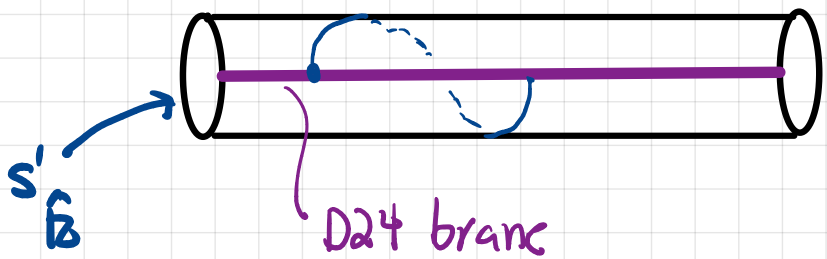
endpoints of the string live on a D24 brane

$p^{25} = \frac{n}{R}$ quantized

no translational symmetry along $S^1_{\hat{R}}$

no winding

string can wind around $S^1_{\hat{R}}$



massless sector: (both sides)
25 dimensional U(1) gauge fields

Final remark:

$\mathbb{R}^{1,d+1} \times M_d$ is not the most general possibility

eg: for the superstring

- BH solns
- $AdS_3 \times M_7$
- ⋮

↳ next: Epilogue on D-branes.