

ST 2

TT 2026

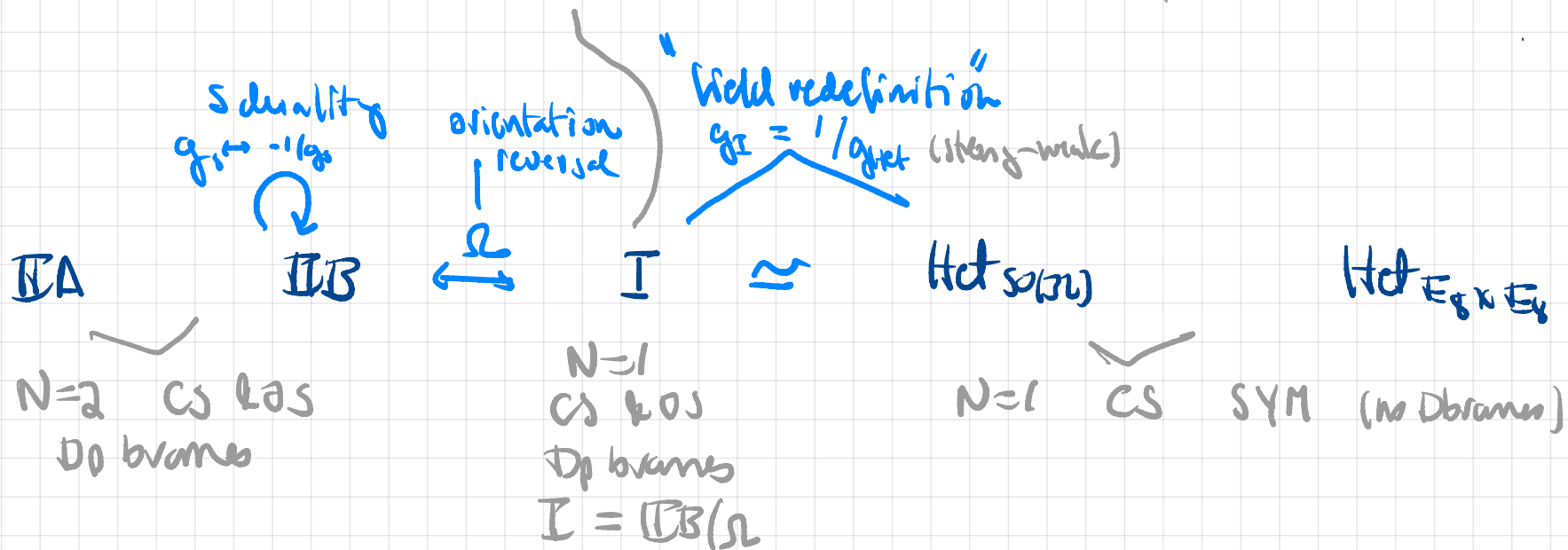
#8



Relations between string theories

L-1

- must have D9 branes (spacetime filling, couple to C_{10} vol form)
- cancel anomalies (Green-Schwarz): need a stack of 16 x D9
 \rightarrow $U(1)^{16}$ enhanced to $SO(32)$ on the D9 branes



S^1 ↓

9 dims:

↓

⋮

IIA [$S^1_{R_1}$] \simeq IIB [$\hat{S}^1_{\hat{R}_1}$]

$\hat{R}_1 = \frac{\alpha'}{R_1}$

(T-dual)

part of a large moduli space in 9 dims

11 dims

M-theory

↪ strong coupling limit of IIA
(i.e. $g_s \rightarrow \infty \Rightarrow$ M theory)

↑ strong/weak
IIA = M [S¹]

- M-theory is not a string theory
- write 11dim eff action

$$S_M = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-G} \left(R - \frac{1}{2} (G_4)^2 \right) - \underbrace{\int G_3 \wedge G_4 \wedge G_4}_{\text{topological}} \quad G_4 = dG_3$$

- Not known if a 9-theory exists
G₃ couples to a 3dim membrane M₂ (int a D-brane)
- M [Interval] = H^{et} E₉ × E₉

M_6 has to be a spin manifold

KK ansatz: $(G_{MN}) = \left(\begin{array}{c|c} \eta_{\mu\nu} & * \\ \hline * & G_{mn} \end{array} \right)$ G_{mn} metric on M_6

10 dim Lorentz group $SO(1,9) \rightarrow SO(1,3) \times SO(6)$
 $\cong SU(2) \times SU(2)$ \leftarrow rotation group in 6 dim
 4 dim Lorentz group $SO(6) \cong SU(4) \rightarrow Spin(6)$

spinors decompose $E \rightarrow \sum E^{rd} \otimes E^{cd}$ (tensors also decompose)

$\frac{16}{MW} \rightsquigarrow ((2,1)_W, \bar{4}) \oplus ((1,2)_W, 4)$
 E : 10 dim MW spinor
 \uparrow susy transform parameters
 \leftarrow spinors of $Spin(6) \sim SU(4)$

T^6 : $4 \times ((2,1)_W \oplus (1,2)_W)$ $N=4$ SUSY
 \leftarrow All over all of them (T^6 is Flat)

Supersymmetric backgrounds

ϵ supersymmetry transf parameter in 10dim

Q corresponding supercharges

Supersymmetric vacuum: $\epsilon \cdot Q | vacuum \rangle = 0$

So if there are background fields $\Phi = \{G_{MN}, B_{MN}, C\text{-forms}, \dots\}$

then $\langle \delta_\epsilon \Phi \rangle_{vacuum} = 0$

$\delta_\epsilon \Phi_{bosonic} \sim \Phi_{fermionic} \equiv 0 \Rightarrow$ no fermion condensates in the background

so $\langle \Phi_{fermionic} \rangle_{vacuum} = 0$

\Rightarrow Then: $\delta_\epsilon \Phi_{fermionic} \equiv 0$

$\Phi_{fermionic} = \{ \text{Gravitinos, dilatino, gauginos (Hd)} \}$

Supersymmetric variation of the gravitino ψ_M 4

$$\delta_\epsilon \psi_M = \nabla_M \epsilon + \cancel{H_M} \epsilon + e^{\frac{1}{2}} \sum_P \cancel{F_P} \Gamma_M \epsilon = 0$$

\rightarrow = (bosonic fields) $\cdot \epsilon$

$$\nabla_M = \partial_M + \frac{i}{2} \omega_M^{AB} \Gamma_{AB} \quad \Gamma_A \text{ 10 dim } \Gamma\text{-matrices, } \Gamma_{AB} = \Gamma_{[AB]}$$

$$\cancel{H_M} \equiv H_{MNP} \Gamma^{NP}, \quad \cancel{F_P} \equiv F_{P M_1 \dots M_{10}} \Gamma^{M_1 \dots M_{10}}$$

spin connection

$$\omega_M^{ab} = \frac{1}{2} (\Omega_{MNP} - \Omega_{NPM} + \Omega_{PNM}) e^{Na} \Omega^{2b} \quad e \rightarrow 10\text{-bein}$$

$$\Omega_{MNP} = (\partial_M \hat{e}_N^a - \partial_N \hat{e}_M^a) \Gamma_{ab}$$

Need to solve $\delta_\epsilon \psi_M = 0$ for susy vacuum configurations of the fields: e_M^a (metric), H , F , ϕ , ϵ

Many interesting solutions!

(Het: also $\delta_\epsilon (\text{gauginos}) = 0$)

► det $\Phi = 0$, $H = 0$, $F = 0$ $\omega = 0$ (simplest)

→ $\partial_M \epsilon = 0$ constant spinors

flat space

M_6 is flat ex $M_6 = T^6$

preserves all susy (SUSY = 0 satisfied \forall cons of G)

► $H = F = \phi = 0$
(no background fluxes)

$\omega \neq 0$ \Rightarrow non zero curvature

$\nabla_M \epsilon = 0$

covariantly constant spinors (Killing spinors ϵ)

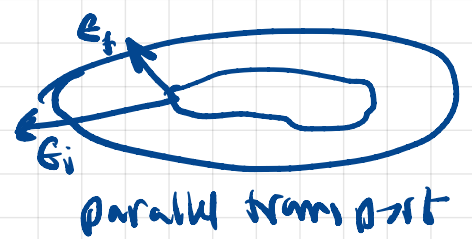
$[\nabla_M, \nabla_N] \epsilon = \frac{i}{2} R_{MN}{}^{AB} \Gamma_{AB} \epsilon$ so $0 = R_{mn}{}^{ab} \Gamma_{ab} \epsilon = 0$

$\Rightarrow R_{mn} = 0$

M_6 is Ricci flat

$\nabla_m \epsilon = 0$ (6 dim)

Γ_{ab} generate a subgroup \mathcal{H}_6 of $Cliff(6)$ leaving at least one component of ϵ invariant i.e. $\nabla_m \epsilon = 0$



$SO(6)$ transf
 $G_i \rightarrow G_f = \Gamma G_i$
measured by $\nabla \epsilon$

\mathcal{H}_6 (holonomy group)

$\mathcal{H}_6 \subseteq SO(6)$
reduced holonomy group

Spinors

$$so(6) \longrightarrow su(3)$$

$$\underline{4} \longrightarrow \underline{1} \oplus \underline{3}$$

component of $\epsilon^{6 \text{ dim}}$ which is inv. under parallel transport

$\Rightarrow N=1$ ONLY in 4 dims

$$\epsilon \rightarrow \frac{16}{MW} \longrightarrow ((2,1), \bar{1}) \oplus ((1,2), 1)$$

or
inst

Type I, Het on $M_6 \rightarrow 4 \text{ dim } N=1$

IIA, IIB $\rightarrow 4 \text{ dim } N=2$

\leftarrow from two ten dimensional spinors

M_6 Ricci-flat \rightsquigarrow Calabi-Yau

Full classification: compactly to other dimensions

T²
K3
3-fold
⋮
n-fold

M_{2N} 4hd = SU(N) < SO(2N) CY

Other dimensions

M₇ 4hd = G₂ < SO(7)
M₆ 4hd = Spin(7) < SO(8) } manifolds with exceptional holonomy

Can have CY x torus

Non-zero fluxes:

IIB AdS₅ x S⁵ F₅ ≠ 0 along S₅ & AdS₅
(λ = -1)

Het AdS₄ x G₂ H₃ ≠ 0

⋮

Next IIA/IIB on a CY & mirror symmetry

CALABI-YAU MANIFOLDS

Mathematical objects of interest: 6 dim

algebraic varieties with certain special properties

set of solutions of

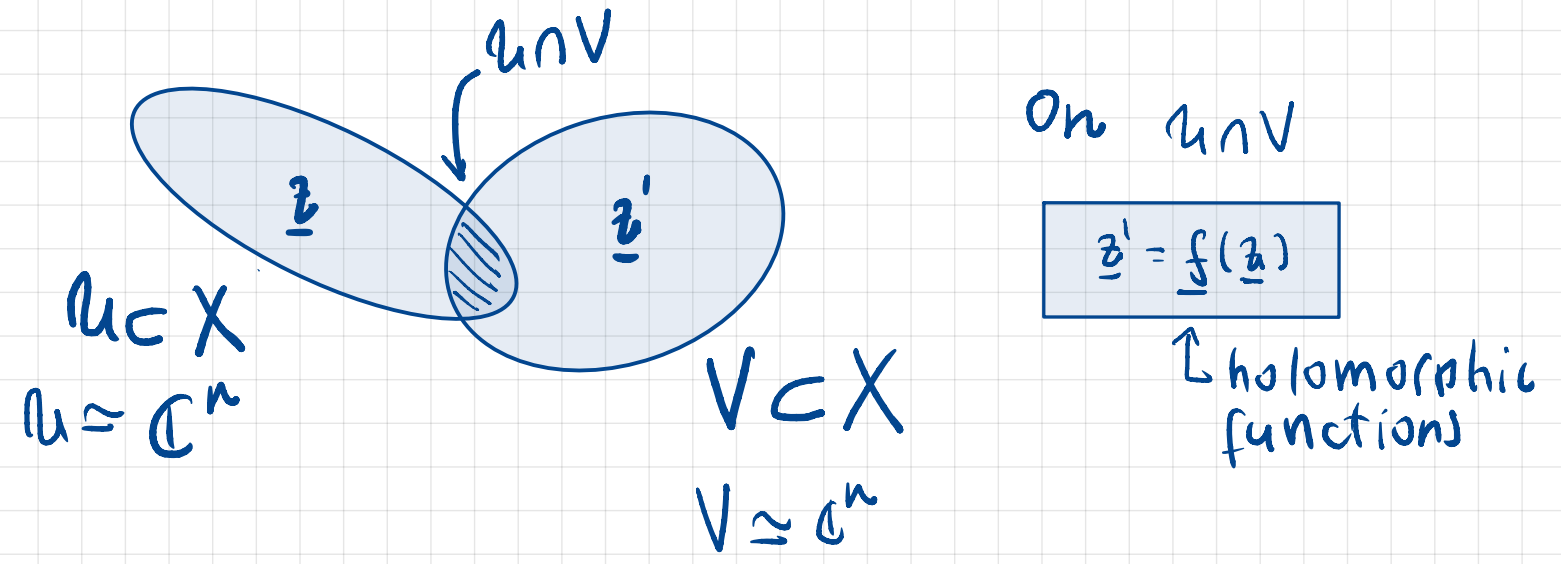
$$\underline{|P(\underline{y}, \underline{x}) = 0, \underline{x} \in \mathbb{A}|}$$

↑ polynomials with complex coefficients \underline{y}

Calabi-Yau manifolds

Definition: a Calabi-Yau manifold is a complex manifold which is Kähler and admits a Ricci-flat metric (that is, $c_1 = 0$)

complex manifold: an n -dim complex manifold X is a $2n$ -dimensional real differentiable manifold on which one can choose \mathbb{C} -local coordinates $\{z^1, \dots, z^n\}$ st the transition functions between two coordinate patches are holomorphic maps $\mathbb{C}^n \rightarrow \mathbb{C}^n$



The complex coordinate system with holomorphic transition functions is called a **complex structure on X**

An important example for us today is

\mathbb{P}^n n -dim complex projective spaces

$$(z^1, \dots, z^{n+1}) \in \mathbb{C}^{n+1} \setminus \{0\}$$

subject to the identifications

$$(z^1, \dots, z^{n+1}) \simeq \lambda (z^1, \dots, z^{n+1}) \quad \forall \lambda \in \mathbb{C}, \lambda \neq 0$$

Exercise: $\mathbb{P}^1 = S^2$

Forms decompose in (p, q) -type

$\Omega^{(p, q)}$

12

$$C_{r\text{-form}} = \sum_{\substack{p, q \\ p+q=r}} \frac{1}{p!q!} C_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{\bar{j}_1} \wedge \dots \wedge d\bar{z}^{\bar{j}_q}$$

$z = x + iy$

For example: w a two form \rightarrow $\underbrace{(2,0)}_{w_{ij}} + \underbrace{(1,1)}_{w_{i\bar{j}}} + \underbrace{(0,2)}_{w_{\bar{i}\bar{j}}}$

exterior derivative $d = \partial + \bar{\partial}$

$$\partial : \Omega^{p, q} \rightarrow \Omega^{p+1, q}$$

$$\bar{\partial} : \Omega^{p, q} \rightarrow \Omega^{p, q+1}$$

$$\partial = \sum dz^i \frac{\partial}{\partial z^i}$$

$$\bar{\partial} = \sum d\bar{z}^{\bar{j}} \frac{\partial}{\partial \bar{z}^{\bar{j}}}$$

Cohomology

$$H_d^{(r)} = \frac{\ker(d) \leftarrow d\text{-closed forms}}{\operatorname{Im}(d) \leftarrow d\text{-exact form}} \quad [d] \in H^r$$

(dα = 0 α ~ α + dβ)

dim H^(r) = b_r Betti numbers ← topological invariants

For complex manifolds

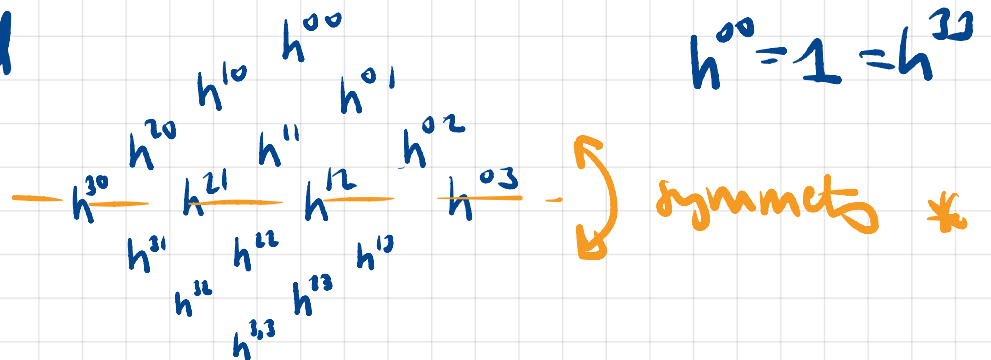
$$H_0^{(p,q)} = \frac{\ker(\partial)}{\operatorname{Im}(\partial)}$$

$$H_0^{(p,q)} = \frac{\ker(\bar{\partial})}{\operatorname{Im}(\bar{\partial})}$$

$$h^{p,q} = \dim H_0^{(p,q)}$$

Hodge numbers of M

arrange in a Hodge diamond



similar for $\bar{h}^{p,q}$

Definition: a Calabi-Yau manifold is a complex manifold which is Kähler and admits a Ricci-flat metric ($c_1 = 0$) (14)

A Kähler manifold is a complex Riemannian manifold with a (hermitian) metric g which can be written as

($g_{i\bar{j}} = 0, g_{j\bar{i}} = 0$
hermitian
metric)

$$g_{i\bar{j}} = \frac{\partial}{\partial z^i} \frac{\partial}{\partial \bar{z}^j} K(z, \bar{z})$$

↑ real valued "function"
(Kähler potential)

That is, the (1,1)-form

$$\omega \equiv i g_{i\bar{j}} dz^i \wedge d\bar{z}^j \quad (\text{hermitian form})$$

is closed:

$$d\omega = 0$$

In fact, ω determines a cohomology class 15

$$[\omega] \in H^2(X) \quad (\omega \sim \omega + d\alpha)$$

called the Kähler class.

The equation $d\omega = 0$

is a linear differential equation which can have many solutions $\{e_1, \dots, e_N\}$ ($N = b_2$)

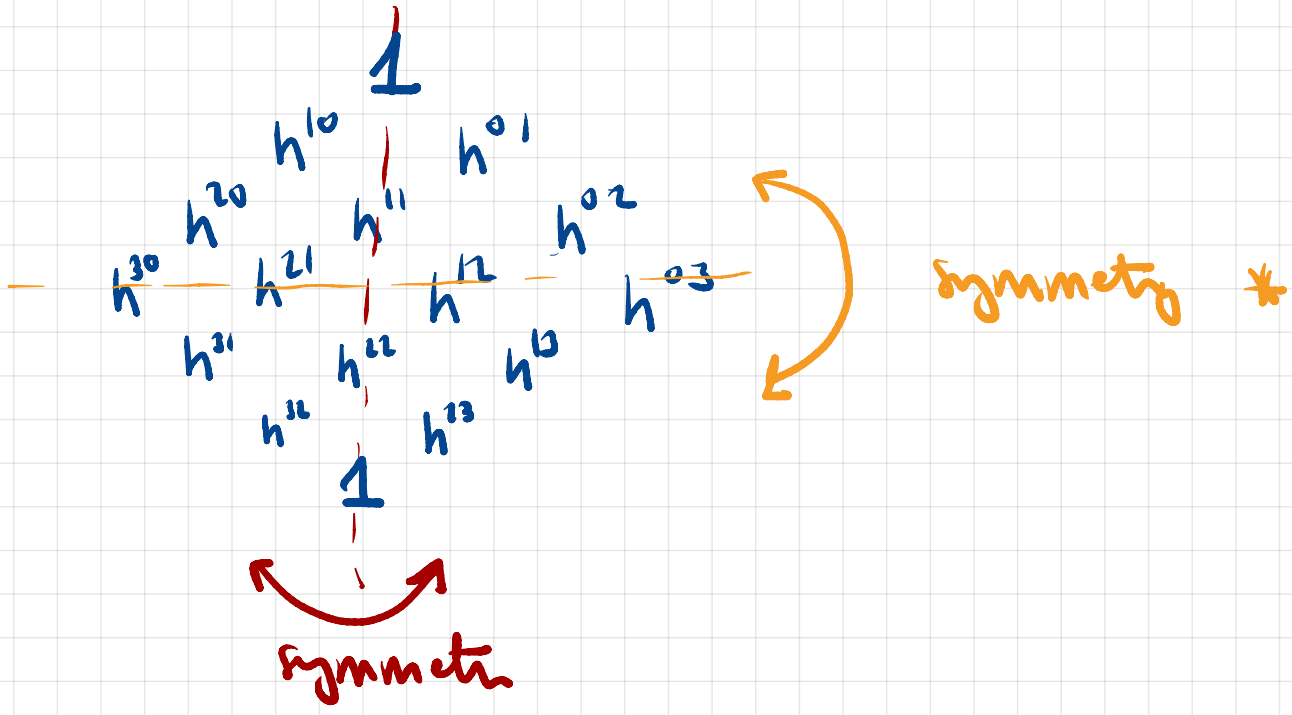
so the most general solution is

$$\omega = \sum_{i=1}^N t^i e_i \quad \begin{array}{l} \uparrow \\ \{t^1, \dots, t^N\} \end{array} \quad \begin{array}{l} \text{Kähler class} \\ \text{parameters} \end{array}$$

(\mathbb{P}^n is Kähler.)

Theorem: If M is Kähler then $h^{p,q} = \bar{h}^{q,p}$

using complex conjugation $\bar{h}^{p,q} = h^{q,p} \Rightarrow \underline{h^{p,q} = h^{q,p}}$



$h^{0,0} = b_0 = 1$ constant function

$h^{3,3} = 1$ volume form

Definition: a Calabi-Yau manifold is a complex manifold which is Kähler and admits a Ricci-flat metric (that is, $c_1 = 0$)

$c_1 \equiv \left[\frac{1}{2\pi} R \right] \in H^2(X)$ (first Chern-class)
 \uparrow
analytic invariant

one can easily prove that $R_{mn} = 0 \implies c_1 = 0$

1957, E Calabi: conjectured that c_1 is the only topological obstruction for there to exist a Ricci flat metric (ie $c_1 = 0 \implies \exists g_{mn}$ with $R_{mn} = 0$)

1977, S-T Yau proved this conjecture (he won the Fields medal)

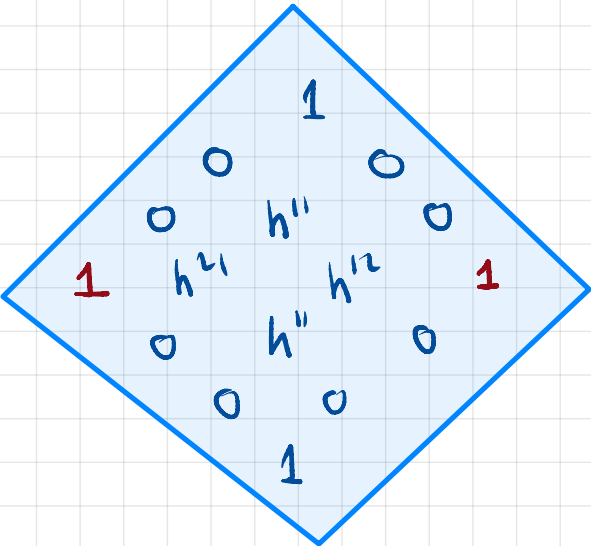
CY condition ($c_1=0$) : $h^{3,0} = h^{0,3} = 1$

($g=0$)

\exists well defined nowhere vanishing holomorphic (3,0)-form $\bar{\partial}\Omega=0$

$h^{1,0} = h^{0,1} = 0 \Rightarrow h^{2,0} = h^{0,2} = 0$ if X_6 has $4h = 543$

$b_2 = h''$
 $b_3 = 2(1+h^{12})$



$h'' = \#$ (Kähler class) deformations

[recall $[\omega] \in H^2$ in fact $H^{1,1}$
 $\Rightarrow \omega = \sum_{i=1}^{h''} t^i e_i$ Kähler class params]

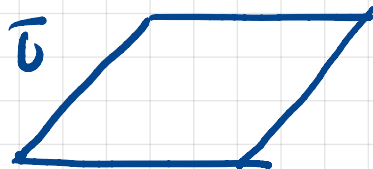
$h^{12} = \#$ of \mathbb{C} -structure deformations

$\chi = \sum (-1)^r b_r = 2(h'' - h^{12})$

Lower dims

► T^2

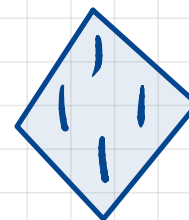
1 dim CY



T^2 / Λ

$$\Lambda = \mathbb{Z} + \bar{v}\mathbb{Z}$$

$$\dim_{\mathbb{C}} = 1$$



$$\chi = 0$$

Kähler structure: changing the volume but keeping the shape \bar{v} fixed

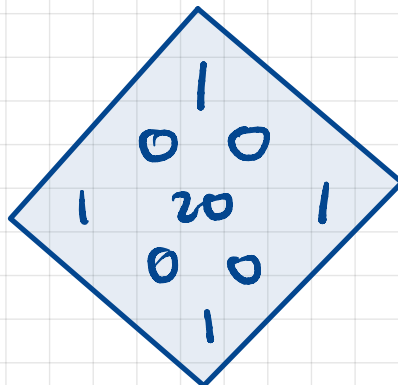
complex structure \bar{v} "shape parameter"

► $K3$

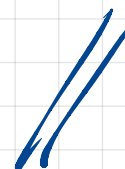
2 dim CY

$$\dim_{\mathbb{C}} = \text{SU}(2)$$

Kähler + Kähler + Kodaira



$$\chi = 24$$



Mathematical objects of interest: CY 3fold

algebraic varieties with certain special properties

set of solutions of

$P(\underline{q}, \underline{x}) = 0, \underline{x} \in \mathbb{A}^n$

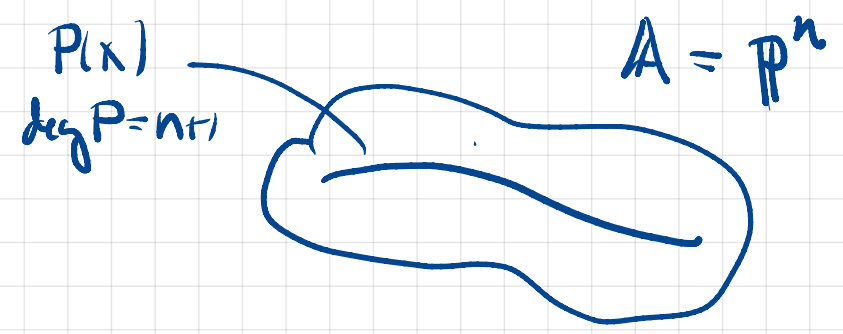
Calabi-Yau manifolds



polynomials with complex coefficients q

CY condition encoded in the properties of P

Examples : (the simplest!)



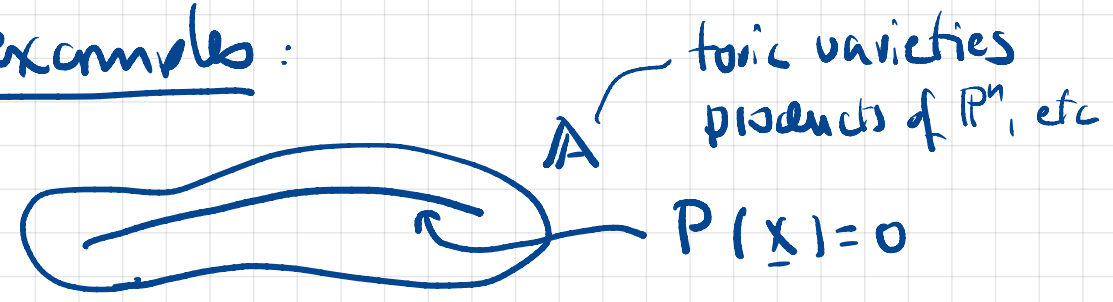
$\mathbb{P}^n [n+1]$ has $c_1 = 0$ and inherits its Kähler class from \mathbb{P}^n
 \uparrow degree $n+1$ polynomial $\Leftrightarrow c_1 = 0$

3-fold $\mathbb{P}^4 [5]$ quintic 3-fold

eg
$$P(x, \psi) = \sum_{i=1}^5 x_i^5 - 5\psi x_1 x_2 x_3 x_4 x_5 + \dots$$

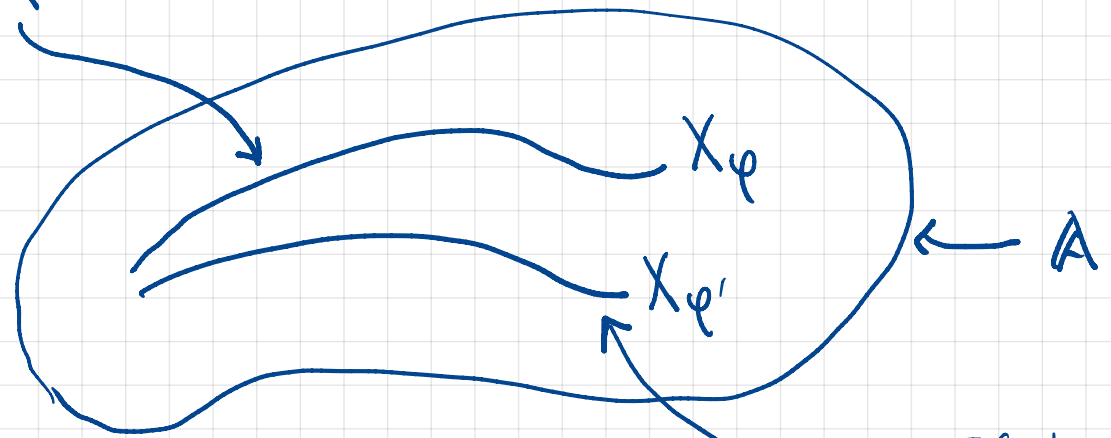
$$(x'_1, \dots, x'_5) \in \mathbb{P}^4 = \{ (x_1, \dots, x_5) \in \mathbb{C}^5, (x_1, \dots, x_5) \approx (\lambda x'_1, \dots, \lambda x'_5), \lambda \in \mathbb{C}^* \}$$

More sophisticated examples:



► complex structure parameters → coefficients of P

$$X_\varphi: P(\varphi, \underline{x}) = 0$$



$$X_{\varphi'}: P(\varphi', \underline{x}) = 0$$

Quintic : $A = \mathbb{P}^4$

$$h^{1,2} = 0$$

of polynomial deformations
(modulo identifications in \mathbb{P}^4)

$$h^{1,1} = 1$$

← Kähler class induced from ambient \mathbb{P}^4

The number of Kähler class parameters and C-structure parameters are given by topological invariants

▶ Kähler class : $\omega = \sum_{i=1}^n t^i e_i$, $N = h^{(1,1)} = b_2$
 $d\omega = 0$, $\{e_1, \dots, e_n\}$ basis of $H^{1,1}$

▶ C-structure :

of complex structure parameters = $h^{(2,1)}$
 $\rightarrow = \frac{1}{2} b_3 - 1$
for a CY

Mirror symmetry

24

Mirror symmetry was uncovered by String Theory. We say that a pair of CY manifolds (X, Y) is a mirror pair if

$$\Gamma(\text{IIA}[X]) = \Gamma(\text{IIB}[Y])$$

where $\Gamma(\dots)$ = effective QFT of a Type II string compactified on a CY

The mirror symmetry **conjecture** means that the physical theory of IIA strings propagating in the spacetime $M_4 \times X$ is **indistinguishable** from that of strings propagating in $M_4 \times Y$.

Indistinguishable means that the spectrum of particles as well as the quantum physical quantities (correlation functions) are the same in both theories.

MS was conjectured in the late 80's by L. Dixon (87) & W. Lerche, C. Vafa, N. Warner (89) based on the fact that in the CFT associated to CY compactifications, it is a matter of convention which parameters are associated to the \mathbb{C} -structure or the Kähler class

The most basic consequence of mirror symmetry is the equivalence of the massless spectrum.

This implies that for a mirror pair (X, Y)

massless scalar fields
↔ cohomology classes

$$h^{(1,1)}(X) = h^{(2,1)}(Y)$$
$$h^{(2,1)}(X) = h^{(1,1)}(Y)$$

topological numbers!

(and therefore $\chi(X) = -\chi(Y)$ so $|\chi(X)| = |\chi(Y)|$)

This simple fact is the first glimpse at how non-trivial MS is: X & Y are topologically different!

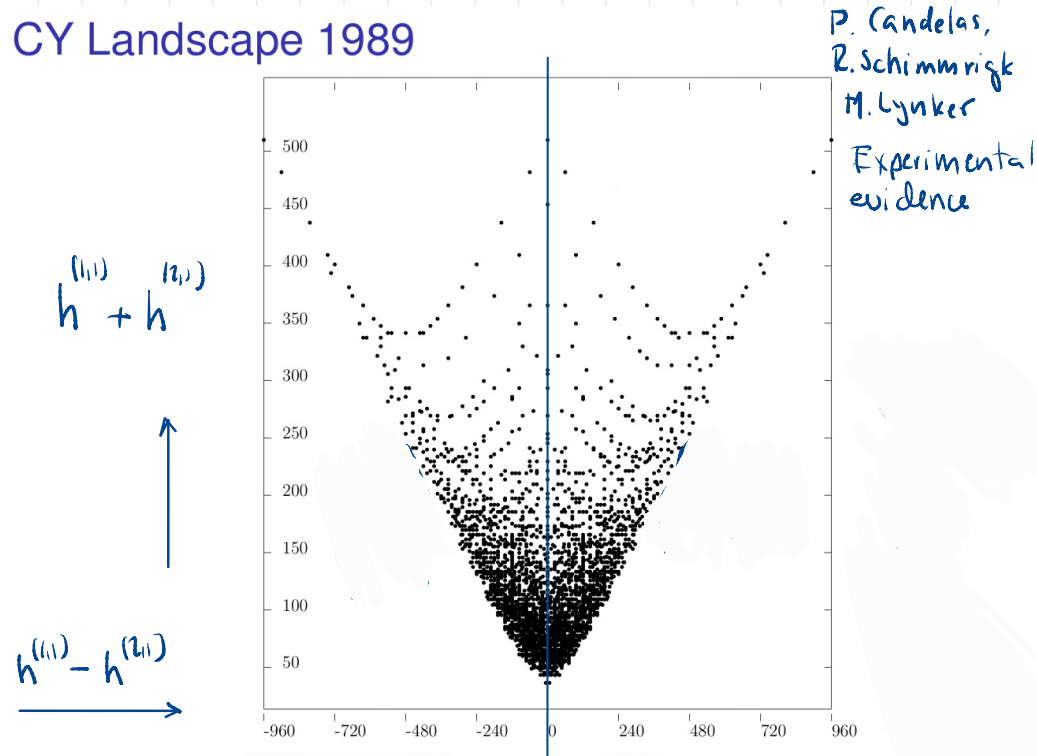
From the point of view of classical geometry, MS seems very mysterious.

The mathematical community was very skeptical originally and there weren't many examples of CY manifolds in the 80's.

In the late 1980's evidence for mirror symmetry started to accumulate.

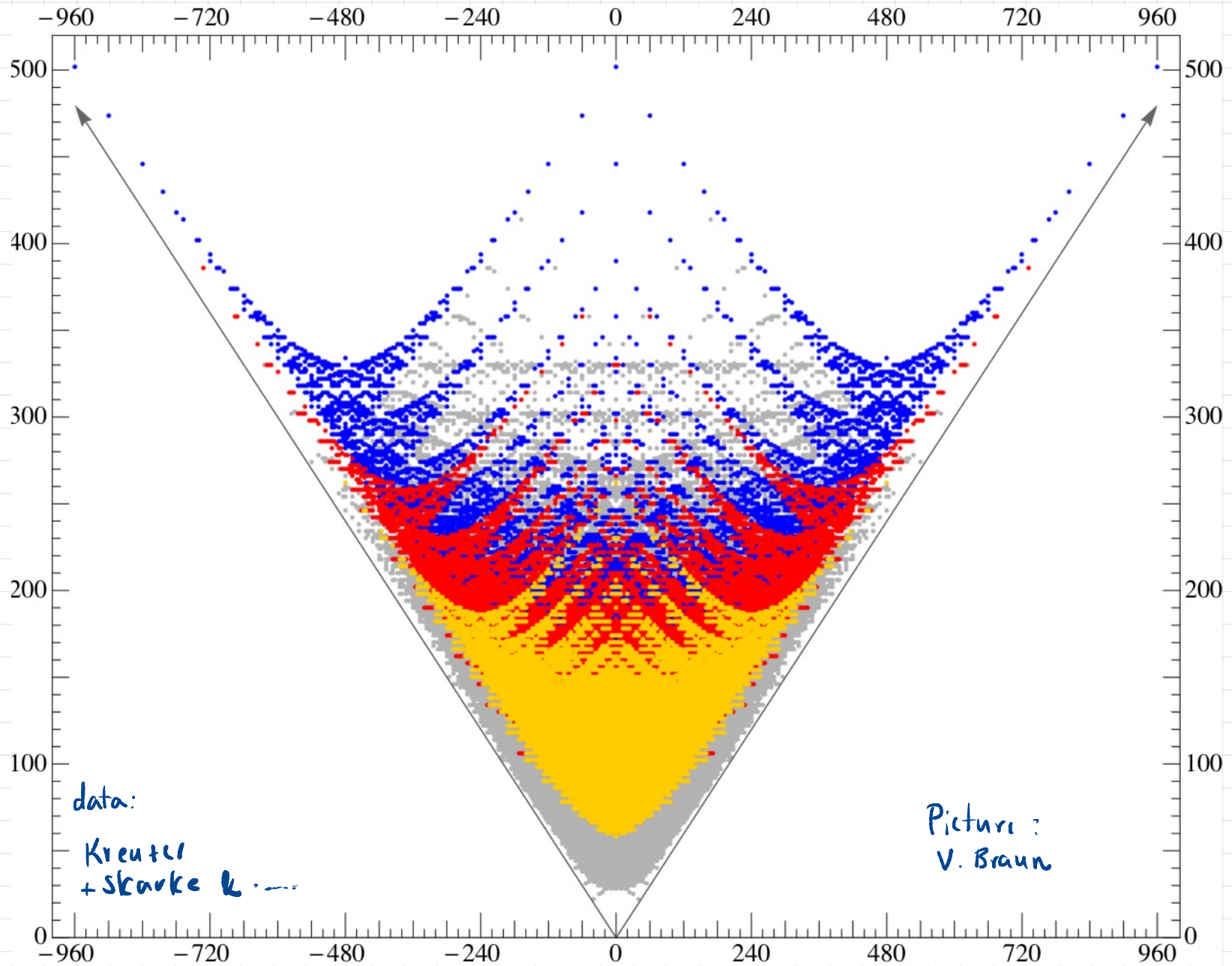
- Experimental evidence was obtained by constructing thousands of examples as hypersurfaces in projective spaces & products of projective spaces.

CY Landscape 1989



- B. Greene & R. Plesser: constructive evidence where the mirror of a CY was obtained explicitly for some examples (eg the mirror of the quintic 3-fold)

landscape today



Parameter spaces and mirror symmetry

Evidence of a deeper structure in relation to MS came from the study of the geometry of the moduli space of IIA and IIB string theories compactified on CY manifolds.

[XD & P. Candelas, A Strominger 90]

The understanding of this geometry is crucial to calculate the necessary physical quantities to obtain the 4 dim effective theory

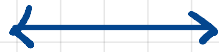
For example,

dimension of
the moduli space



number of massless
particles in the effective
theory

metric on the
moduli space




kinetic terms in the
4 dim effective action

certain natural
cubic forms



Yukawa couplings

⋮


 (3-point correlation
function)

⋮

As we have said, CY manifolds have two types of parameters. One can show that

$$M = M_{OS} \times M_{KC}$$

\swarrow
 $\mathbb{C} \cdot \dim M_{OS} = h^{(2,1)}$

\nwarrow
 parameters of the complexified Kähler class
 B+iw
 $\mathbb{C} \cdot \dim M_{KC} = h^{(1,1)}$

The complexification of the Kähler class comes from string theory. (the massless spectrum includes a closed 2-form B)

This is the first step away from the classical geometry of CY manifolds.

Both M_{cs} & M_{kc} are Kähler with a holomorphic prepotential ("special geometry")

The mirror symmetry conjecture implies that for a mirror pair (X, Y)

$$M_{cs}(Y) = M_{kc}(X)$$

$$\dim M_{cs}(X) = h^{(2,1)}(Y) = \dim M_{kc}(X) = h^{(1,2)}(X)$$

isomorphism map = mirror map

$$\begin{array}{ccc} \psi & \longmapsto & t(\psi) \\ \uparrow \text{cs of } Y & & \uparrow \text{kc of } X \end{array}$$

Complex structure parameters

[B[Y]

Let Y be a CY manifold.

$M_{cs}(Y)$ is Kähler with "holomorphic prepotential" \mathcal{G}

let $\{z^1, \dots, z^{h^{(2,1)}}\}$ be complex projective coordinates on $M_{cs}(Y)$. The Kähler metric on $M_{cs}(Y)$

$$G_{a\bar{b}} = \frac{\partial}{\partial z^a} \frac{\partial}{\partial \bar{z}^b} \mathcal{K} \quad a, b = 1, \dots, h^{(2,1)}$$

where the Kähler potential \mathcal{K} is given by

$$e^{-\mathcal{K}} = i \left(z^a \frac{\partial \bar{\mathcal{G}}}{\partial \bar{z}^a} - \bar{z}^a \frac{\partial \mathcal{G}}{\partial z^a} \right)$$

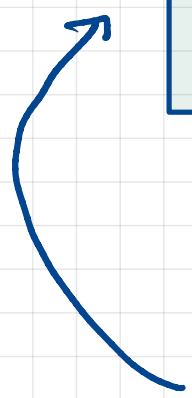
G_f also determines the Yukawa couplings of the 4 dim effective theory of IIB strings on Y

Mathematically: these are natural cubic forms on Y

$$y: H^{1,2} \times H^{1,2} \times H^{1,2} \longrightarrow \mathbb{C}$$

One finds that

$$y_{abc} = \partial_a \partial_b \partial_c G_f$$



complicated functions of the α -structure parameters. But calculable

In general correlation functions receive quantum corrections.

However in [B[4] these couplings are exact (Distler & Greene 88).

In fact: the classical geometry of $M_{0,3}(Y)$ is exact.

→ example of a "non renormalisation" theorem for supersymmetric theories

Kähler class parameters

[A [X]

Let X be a CY manifold. The surprising fact is that similar considerations apply to $M_{Kc}(X)$

$M_{Kc}(X)$ is also Kähler with a holomorphic prepotential.

But there are differences.

The main one is that the classical geometry of $M_{Kc}(X)$ is not enough: one has to compute the quantum corrections.

let $\{e_i\}_{i=1, \dots, h''}$ be a basis of $H^2(X)$ Then

$$B + iw = \sum_{i=1}^{h''} t^i e_i$$

where $\{t^1, \dots, t^{h''}\}$ are complex coordinates of $M_{\mathbb{C}}(X)$

let $\{w^0, w^1, \dots, w^{h''}\}$ be complex projective coordinates on $M_{\mathbb{C}}(X)$ with $t^i = \frac{w^i}{w^0} \quad i=1, \dots, h''$

Then there is a classical prepotential

$$F^{cl} = \frac{1}{3!} \hat{y}_{ijk}^a \frac{w^i w^j w^k}{w^0}$$

such that

$\hat{y}_{ijk}^a : \underbrace{H^{1,1} \times H^{1,1} \times H^{1,1}}_{\text{intersection number}} \longrightarrow \mathbb{Z}$
 \hat{y}_{ijk}^a is \mathbb{Z} -valued

and (trivially): $\hat{y}_{ijk}^a = \partial_i \partial_j \partial_k F^{cl}$

$G_{i,j}^a = \partial_i \partial_j K^a$
 $e^{-K^a} = i (w^I \bar{F}_I^a - \bar{w}^I F_I^a)$

the surprise is that this F^{cl} also determines the classical metric on $M_{K^a}(X)$

RECAP

▶ $\Gamma(\Pi A[X]) = \Gamma(\Pi B[Y])$

▶ $M_{KC}(X) = M_{CS}(Y)$

$t \mapsto \psi(t)$ mirror map

▶ classical computations are not exact

vs classical computations are exact

$\hat{y}(t) = \hat{y}^{cl}(t) + \dots$
 ↑ simple

$y(\psi)$ is exact but a complicated function of the \mathbb{G} -structure (but calculable)



$\hat{y}(t) = \hat{y}^{cl}(t) + \dots \overset{MS}{\longleftrightarrow} y(\psi)$

↑ typically hard to compute

(P Candlish & KD)

More on $M_{Kc}(X)$

To appreciate the power of MS, let's try to understand better $M_{Kc}(X)$ without using MS. In particular we want to understand where the "quantum corrections" come from.

For simplicity

let $X \in \mathbb{P}^4 [5]$

$$\begin{aligned} h^{(1,1)} &= 1 \\ (h^{(2,1)} &= 101) \end{aligned}$$

so $B + iw = t e$

↑
σ-Kähler class parameter

Yukawa couplings in physics are calculated using a path integral:

Let $\Sigma = 2$ -dim surface swept out by the string moving in X

and $x: \Sigma \longrightarrow X$ embedding of Σ into X

$$g_{ij} = \frac{\int [dx] e_i e_j e_k e^{-S[x]}}{\int [dx] e^{-S[x]}}$$

← action for a string moving in X

where

$$S[x] = t \int_{\Sigma} e \quad (\text{pull back of Kähler class to } \Sigma)$$

Today: $\Sigma = \mathbb{P}^1 = S^2$

To compute the PI:

- 1) expand PI around a classical solution (minimum of the action)
- 2) compute quantum corrections

Supersymmetry \Rightarrow quantum corrections to \hat{y}^a can only come from saddle points of the action: these are called instantons (Distler & Greene)

(saddle points: stationary points of S)

|| Distler & Greene proved that the result is then exact
 || What is this mathematically?

Stationary points of the action are

$$\frac{\partial \mathcal{X}^M}{\partial \bar{z}} = 0$$

$$\mathcal{X} : \Sigma = \mathbb{P}^1 \longrightarrow X$$

(z, \bar{z}) coordinates on \mathbb{P}^1

that is, it is a holomorphic embedding

$$\mathbb{P}^1 \longrightarrow X$$

Then, $\mathcal{X}(\Sigma)$ can be

- a point in X ($\mathcal{X} = \text{constant}$, classical contribution)
 - an algebraic curve in X
 - multiple cover of an algebraic curve in X
- } rational curves

A rational curve of degree k is a holomorphic embedding of degree k

► For example a rational curve of degree 2 would be

$$\mathbb{P}^1 \ni (u, v) \longmapsto (u^2, v^2, uv, 0, 0)$$

$$\xrightarrow{\text{or}} (u^k, v^k, 0, 0, 0)$$

↗ double cover of rational curve of deg 1

$$(u, v) \longmapsto (u, v, 0, 0, 0)$$

► Example of a rational curve of degree 1 in X -quintic

$$P = \sum x_i^5 - 5\psi x_1 x_2 x_3 x_4 x_5$$

$$(u, v) \longrightarrow (u, -\alpha^k u, v, -\alpha^k v, 0), \quad \alpha^5 = 1$$

Evaluating the PI

$$\hat{y}_{g_{ttt}} = \hat{y}_{g_{ttt}}^{\text{cl}} + \sum_k \dots$$

classical
contribution

sum over all inequivalent
isomorphic embeddings
of \mathbb{P}^1 into X

$$y_{g_{ttt}} = 5 + \sum_k n_k k^n \frac{q^k}{1-q^k}, \quad q = e^{mit} \quad \text{for the quintic}$$

of (irreducible) rational curves of degree k
in X (hard)

The problem in the late 80's was that the numbers N_k were extremely hard to compute using traditional mathematics.

By 1991 only n_1 & n_2 were known correctly.

→ 1984 n_1 (Schubert) 2875

→ 1986 n_2 (Katz) 609250

→ Ellingsrud + Strømme calculated n_3 incorrectly due to an error in their computing code; in 1992 they gave the right number.

M.S. → give a generating function for the N_k !

Mirror symmetry and the numbers n_k

49

For $X = \text{quintic 3-fold}$ & Y its mirror
(Candelas, de la Ossa, Green, Parkes 91)

$$y = 5 \frac{\psi^2}{\omega_0(\psi)^2(1-\psi^5)} \left(\frac{d\psi}{dt} \right)^3 = 5 + \sum_{k=0}^{\infty} n_k \frac{k^3 q^k}{1-q^k},$$

$$q = e^{2\pi i t}$$

mirror map: $\lambda = (5\psi)^{-5}$

$$\begin{aligned} \lambda &= q + 154 q^2 + 179139 q^3 \\ &+ 313195944 q^4 + 657313805125 q^5 + 1531113959577750 q^6 \\ &+ 3815672803541261385 q^7 + 9970002717955633142112 q^8 + \dots \end{aligned}$$

why
integer
coeffs?

(Rigorous proof: Givental 96; Lian, Liu & Yau 97)

k	n_k
1	2875
2	6 09250
3	3172 06375
4	24 24675 30000
5	22930 58888 87625
6	248 24974 21180 22000
7	2 95091 05057 08456 59250
8	3756 32160 93747 66035 50000
9	50 38405 10416 98524 36451 06250
10	70428 81649 78454 68611 34882 49750

Schubert (84) →
 Katz (86) →
 Ellingsrud & Strømme (92)
 Ekedahl & Xiao
 ⋮

Table 1 The numbers of rational curves of degree k for $1 \leq k \leq 10$.

CONCLUSIONS

- ▶ There are many generalisations:
the generating function presented today was the first example of a more general class of identities involving Gromov-Witten invariants
- ▶ MS is still a conjecture: much concerted effort has gone into its deep mathematical structure
- ▶ Mirror symmetry is **but one** example of a duality symmetry in string theory.

In each case, these give a deep relationship between different string theories and invariably involve very interesting connections to mathematics.

► Given a CY X , how do you find its mirror Y ?

1994 Batyrev: mirror symmetric class of CY manifolds which are hypersurfaces in a toric variety

1996 Strominger, Yau & Zaslow \rightarrow "T-duality"

► 1994 Kontsevich.

homological mirror symmetry conjecture

(D-branes: objects in categories)

Fukaya A_∞ -category
of Lagrangian submanifolds of X



Bounded category
of coherent sheaves on Y

► Mark Gross & Bernd Siebert programme

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-

► Other dualities!

THANKS!