Further Partial Differential Equations

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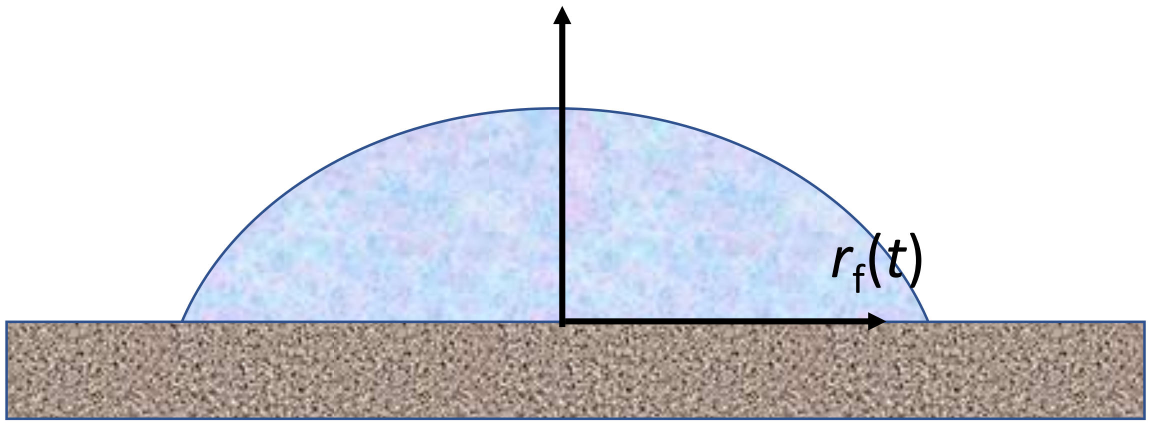
Problem sheet classes

- Problem sheet classes will take place on Tuesday of Week 3, 5, 7 and 9 at 4–5pm and 5–6pm with Dominic Vella (tutor) and Gabriel Cairns (TA).

- Hand-in time: Monday 9am Weeks 3, 5, 7, and 9.

Similarity solutions

- Consider the position of the front r_f as a function of time t



- The spreading behaviour is the same for all liquids.

- We see similar kinds of behaviour in other scenarios, e.g., trapping CO_2 underground or spreading of honey on porous toast.

- Can we understand the shape this evolves to more easily than solving the full problem?

Spreading of liquid on a vertical liquid-coated wall

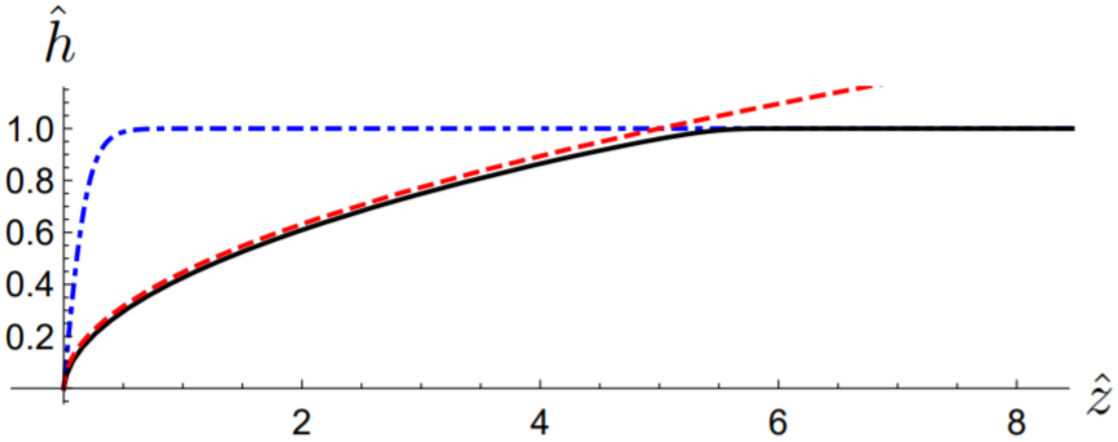
$$\frac{\partial \hat{h}}{\partial \hat{t}} + \frac{\partial \hat{Q}}{\partial \hat{z}} = 0 \qquad \hat{Q} = \frac{\rho g \hat{h}^3}{3\mu}$$

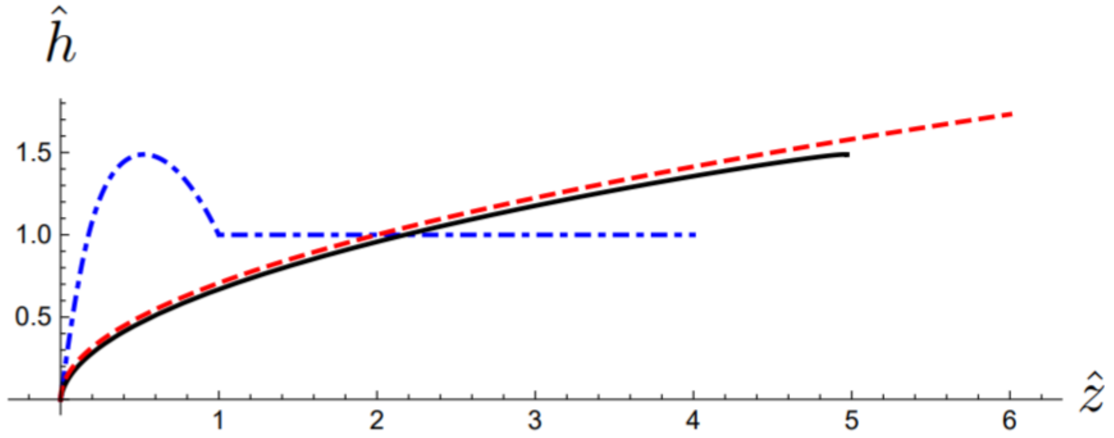
$$\frac{\partial \hat{h}}{\partial \hat{t}} + \frac{\rho g}{3\mu}\frac{\partial}{\partial \hat{z}}\left(\hat{h}^3\right) = 0$$

- What type of equation is this?
- First order hyperbolic equation
- What extra conditions do we need to solve this?
- One initial condition and one boundary condition

$$\frac{\partial \hat{h}}{\partial \hat{t}} + \frac{\rho g}{3\mu}\frac{\partial}{\partial \hat{z}}\left(\hat{h}^3\right) = 0 \qquad \hat{h}(\hat{z},0) = \hat{h}_0(\hat{z}) \qquad \hat{h}(0,\hat{t}) = 0$$

- We can solve this problem analytically (see problem sheet 1).
- For any initial condition, we appear to evolve to the same final shape.





- Can we extract what this universal shape is?
- First, non-dimensionalize:

$$\hat{z} = \hat{z}_0 z, \qquad \hat{t} = \hat{t}_0 t, \qquad \hat{h} = \hat{h}_0 h,$$

- There is no natural length or height scale.
- We could chose these using the initial conditions, but there is nothing inherent in the problem to specify these.

$$\frac{\partial h}{\partial t} + \frac{1}{3}\frac{\partial}{\partial z}\left(h^3\right) = 0$$

- Try a solution of the form

$$h(z,t) = f(\eta) \qquad \eta = \frac{z}{t^{\alpha}}$$

- Substituting in gives

$$f^2 f' - \frac{z}{t} f' = 0$$

- If we choose α=1 then we turn the equation into one solely in terms of η

$$(f^2 - \eta) f' = 0$$

$$f = \eta^{1/2} = \left(\frac{z}{t}\right)^{1/2}$$

$$\hat{h} = \left(\frac{\mu}{\rho g}\right)^{1/2} \left(\frac{\hat{z}}{\hat{t}}\right)^{1/2}$$

- This is precisely the shape that we see all the profiles evolving towards.

- However, in getting this solution, we did not use the initial condition to find this similarity solution. This highlights the fact that our similarity solution 'forgets' the initial information.

- Although in our case we could obtain an analytic solution, in many cases we cannot.

- This similarity solution provides key information about the system behaviour without having to solve the full equation.

- Before we move onto the next example, we will show how we can predict the form of the similarity variables.

- We approximate derivatives as $$\frac{\partial y}{\partial x} \sim \frac{Y}{X}$$

- This is called a scaling law approach.

- Substitute this into our partial differential equation:

$$\frac{\partial h}{\partial t} + \frac{1}{3}\frac{\partial}{\partial z}\left(h^3\right) = 0$$

$$H \sim \left(\frac{3\mu}{\rho g}\right)^{1/2}\left(\frac{Z}{T}\right)^{1/2}$$

- Rearranging gives $$H \sim \left(\frac{3\mu}{\rho g}\right)^{1/2} \left(\frac{Z}{T}\right)^{1/2}$$

- This has given us almost as much information as the similarity solution.
  - It gives us the right functional form and parametric dependence.
  - It doesn't give us the right prefactor.

- The scaling law gives the long time functional $(\hat{z}/\hat{t})^{1/2}$ and parametric $(\mu/\rho g)^{1/2}$ dependence.
- The similarity solution gives the long time solution.
- The analytic solution gives the correct behaviour for all time.

Summary of lecture 1

- Scaling laws can give the long time functional and parametric dependence
- Similarity solutions give the long time solution
- Analytic solutions give the correct behaviour for all time

- Hyperbolic first-order PDEs

The heat equation

$$\frac{\partial \hat{T}}{\partial \hat{t}} + \frac{\partial \hat{Q}}{\partial \hat{x}} = 0 \qquad \hat{Q} = -D\frac{\partial \hat{T}}{\partial \hat{x}}$$

$$\frac{\partial \hat{T}}{\partial \hat{t}} = D\frac{\partial^2 \hat{T}}{\partial \hat{x}^2}$$

- What kind of equation is this?
- Second order parabolic equation.
- What extra conditions do we need to solve this?
- One initial condition and one boundary condition.

- We non-dimensionalize the equation:

$$\hat{x} = \hat{x}_0 x, \qquad \hat{t} = \frac{\hat{x}_0^2}{D} t, \qquad \hat{T} = \left(\hat{T}_{+\infty} - \hat{T}_{-\infty}\right) T + \hat{T}_{-\infty}$$

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

$$T(x,t) \to 0 \ \text{ as } \ x \to -\infty$$
$$T(x,t) \to 1 \ \text{ as } \ x \to +\infty$$

- First let's try the scaling argument:

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

- This gives us that our similarity variable is $\eta = \dfrac{x}{t^{1/2}}$ but nothing more.
- So this time we try a similarity solution of the form $T = T(\eta)$ with $\eta = \dfrac{x}{t^{\alpha}}$

$$f'' + \alpha \frac{x}{t^{\alpha}} t^{2\alpha - 1} f' = 0$$

- Our governing equation and boundary conditions then become:

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

$$T(x,t) \to 0 \quad \text{as} \quad x \to -\infty$$
$$T(x,t) \to 1 \quad \text{as} \quad x \to +\infty$$

- This has solution $f = \frac{1}{2}\left(1 + \operatorname{erf}\left(\eta/2\right)\right)$ where $\operatorname{erf}(\xi) = \frac{2}{\sqrt{\pi}} \int_0^{\xi} \mathrm{e}^{-s^2}\, \mathrm{d}s$

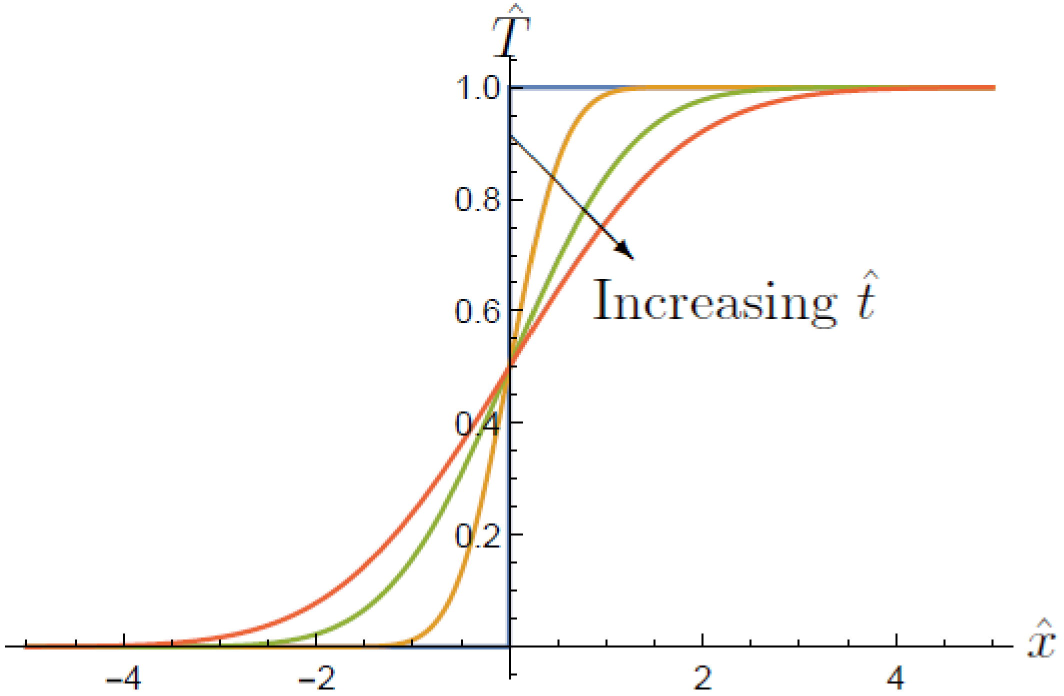
- In terms of dimensional variables, this is

$$\hat{T} = \frac{\left(\hat{T}_{+\infty} - \hat{T}_{-\infty}\right)}{2} \left(1 + \operatorname{erf}\left(\frac{\hat{x}}{2\sqrt{D\hat{t}}}\right)\right) + \hat{T}_{-\infty} \qquad (*)$$

$$\hat{T} = \frac{\left(\hat{T}_{+\infty} - \hat{T}_{-\infty}\right)}{2}\left(1 + \mathrm{erf}\left(\frac{\hat{x}}{2\sqrt{D\hat{t}}}\right)\right) + \hat{T}_{-\infty} \quad (*)$$

- As in the previous example for flow down a vertical wall, we did not use the initial condition to obtain this solution, so the result only holds for long time…

- …unless the initial condition happens to satisfy (*) at t=0



- This example was fairly similar to the previous example. However, now let's suppose that $\hat{T}_{-\infty} = \hat{T}_{+\infty}$

- In this case we must non-dimensionalize $\hat{T}$ differently.

- The dimensionless equation is the same as before but the boundary conditions are different:

- Question: what happens if we try the same similarity ansatz as before?

- While this is seemingly similar to the previous problem, the similarity solution we tried before now simply gives the trivial solution.

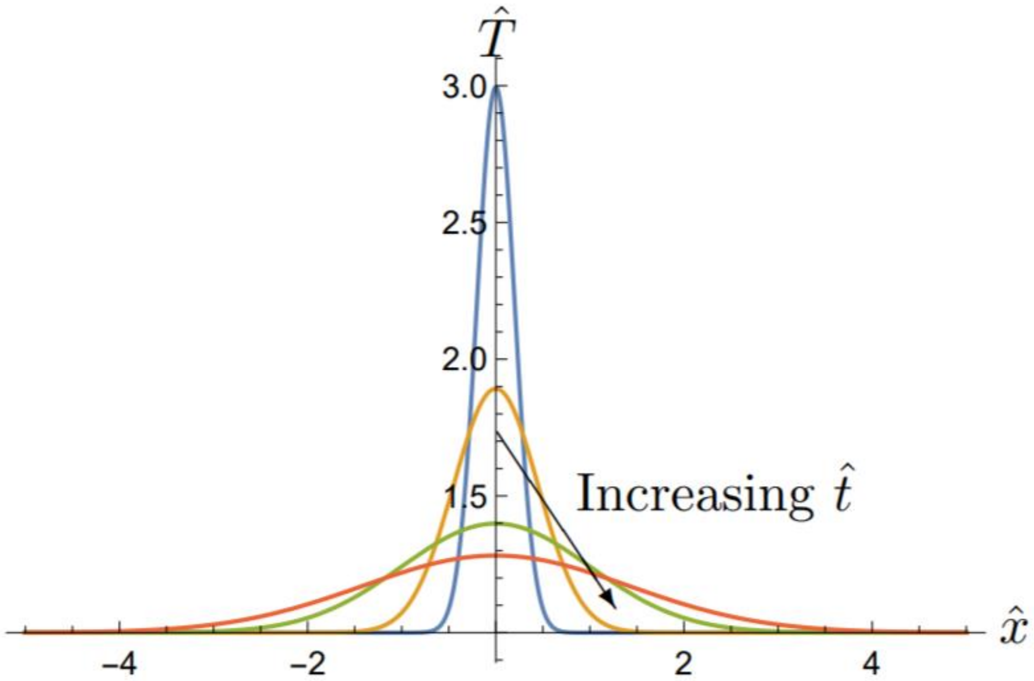
- We can see why by integrating the equation over the entire domain. This gives

- If we substitute our similarity ansatz into this result we get

- This means we need to seek a more general similarity ansatz if we are to obtain a non-trivial solution.
- We try $T = t^{\beta} f(\eta)$ with $\eta = \dfrac{x}{t^{\alpha}}$
- Substituting into the governing equation and the energy constraint tells us that we need $\alpha=1/2$ and $\beta=1/2$.
- Then, we get
- This has solution $f = \dfrac{E}{2\sqrt{\pi}} \mathrm{e}^{-\eta^2/4}$

- In original dimensional terms, this solution is

$$\hat{T} = \frac{\hat{E}}{2\sqrt{\pi D\hat{t}}}\exp\left(-\frac{\hat{x}^2}{4D\hat{t}}\right) + \hat{T}_{+\infty} \qquad (*)$$



- Again, we have not imposed an initial condition to obtain this result, but if our initial condition satisfied (*) for t=0 then this would be the solution for all time.

- But where did the similarity form $T = t^{-1/2} f\left(\frac{x}{t^{1/2}}\right)$ come from?

- If we make a transformation

$$x = a\tilde{x}, \qquad t = b\tilde{t}, \qquad T = c\tilde{T}.$$

and substitute into the equation and boundary condition then we find that if

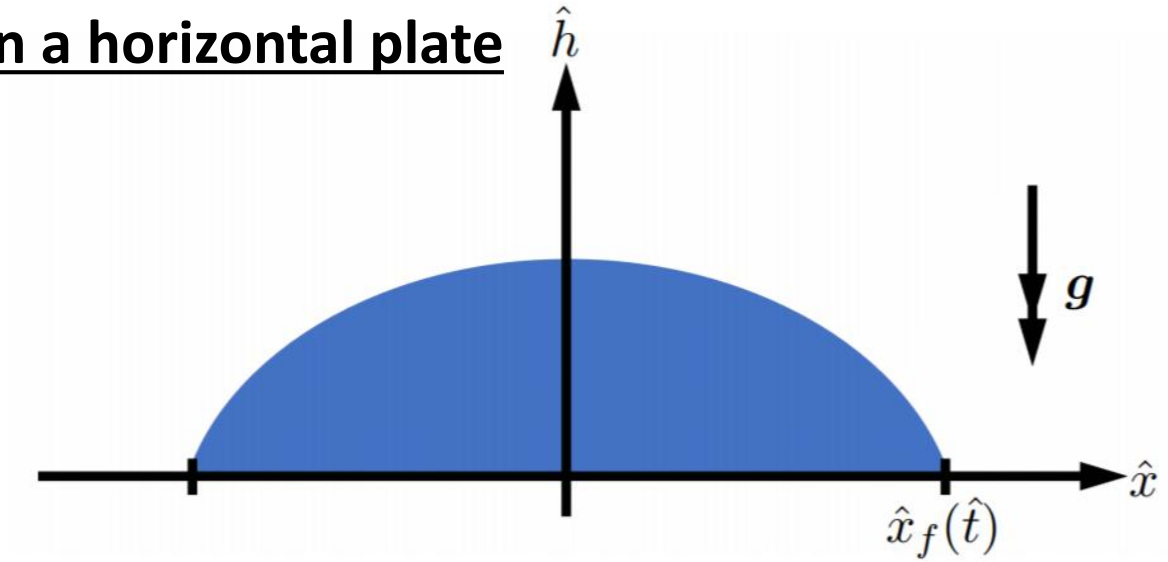
$$\frac{x}{t^{1/2}} = \frac{\tilde{x}}{\tilde{t}^{1/2}} \qquad Tt^{1/2} = \tilde{T}\tilde{t}^{1/2}$$

then the equations in terms of the new variables are identical.

- This indicates that $\frac{x}{t^{1/2}}$ and $Tt^{1/2}$ are invariants of the system.

- This tells us that the solution depends on $x/t^{1/2} = \eta$ and has the form $f(\eta) = t^{1/2}T(x,t)$ which is exactly what we have seen.

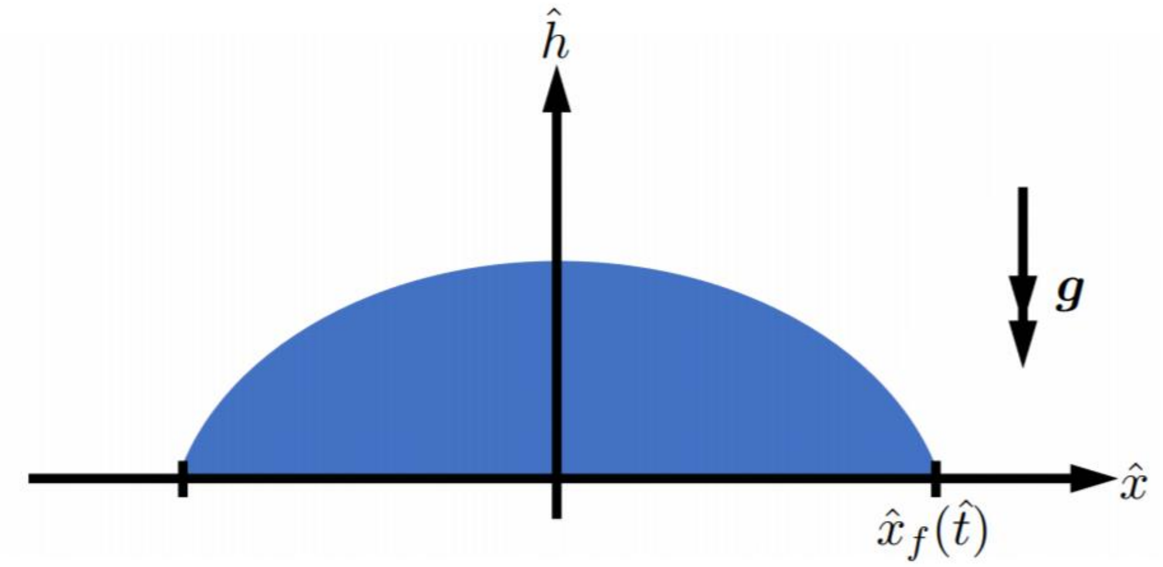
Spreading of liquid on a horizontal plate



$$\frac{\partial \hat{h}}{\partial \hat{t}} + \frac{\partial \hat{Q}}{\partial \hat{x}} = 0 \qquad \hat{Q} = -\frac{\Delta \rho g}{3\mu} \hat{h}^3 \frac{\partial \hat{h}}{\partial \hat{x}}$$

$$\frac{\partial \hat{h}}{\partial \hat{t}} - \frac{\Delta \rho g}{3\mu} \frac{\partial}{\partial \hat{x}} \left(\hat{h}^3 \frac{\partial \hat{h}}{\partial \hat{x}} \right) = 0$$

$$\frac{\partial \hat{h}}{\partial \hat{t}} - \frac{\Delta \rho g}{3\mu} \frac{\partial}{\partial \hat{x}} \left(\hat{h}^3 \frac{\partial \hat{h}}{\partial \hat{x}} \right) = 0 \qquad (1)$$



- What extra conditions do we need to solve this?
- One initial condition and two boundary conditions

- One boundary condition is $\hat{h}(\hat{x}_f(\hat{t}), \hat{t}) = 0 \qquad$ (A)

- Volume conservation: $$\hat{V}(\hat{t}) = \int_0^{\hat{x}_f(\hat{t})} \hat{h}(\hat{x},\hat{t})\,\mathrm{d}\hat{x}$$ (B)

- The final condition to close the system is $$\hat{h}^3 \frac{\partial \hat{h}}{\partial \hat{x}} \to 0 \quad \text{as } \hat{x} \to \hat{x}_f$$ (C)

- For numerics it can be easier to work with boundary conditions rather than integral constraints like (B) and regularity conditions like (C).

- But we can transform (B) and (C) into boundary conditions.

- We integrate (1) over $0 < \hat{x} < \hat{x}_f$ and apply (B) and (C) to get

$$-\frac{\Delta \rho g}{3\mu} \hat{h}^3 \frac{\partial \hat{h}}{\partial \hat{x}} = \hat{V}'(\hat{t}) \qquad \text{on } \hat{x} = 0, \qquad \text{(B1)}$$

- Now rescale into a local region near the moving front:

$$\hat{x} = \hat{x}_f(\hat{\tau}) + \epsilon\hat{\xi}, \qquad \hat{t} = \hat{\tau}$$

- And rescale the height:

$$\hat{h}(\hat{x}, \hat{t}) = \epsilon^{1/3}\hat{H}(\hat{\xi}, \hat{\tau})$$

- Then (1) becomes

$$\epsilon\frac{\partial \hat{H}}{\partial \hat{\tau}} - \frac{\mathrm{d}\hat{x}_f}{\mathrm{d}\hat{\tau}}\frac{\partial \hat{H}}{\partial \hat{\xi}} - \frac{\Delta\rho g}{3\mu}\frac{\partial}{\partial \hat{\xi}}\left(\hat{H}^3\frac{\partial \hat{H}}{\partial \hat{\xi}}\right) = 0$$

- At leading order this gives

- Integrating gives

- In terms of the original variables this gives

$$\hat{h}^2 \frac{\partial \hat{h}}{\partial \hat{x}} \sim -\frac{3\mu}{\Delta \rho g} \frac{\mathrm{d}\hat{x}_f}{\mathrm{d}\hat{t}} \qquad \text{as} \quad \hat{x} \to \hat{x}_f \qquad \text{(C1)}$$

- (B1) and (C1) are easier boundary conditions for numerics than (B) and (C).

- Here we will study just $\hat{V} = \text{constant}$

- Analytical solutions are not possible so we proceed directly to the similarity solution.

- A scaling law analysis gives

- This scaling law gives the parametric dependence.

- Now consider the full similarity solution.

- First, non-dimensionalize:

$$\hat{x} = \hat{x}_0 x, \qquad \hat{x}_f = \hat{x}_0 x_f, \qquad \hat{t} = \hat{t}_0 t, \qquad \hat{h} = \hat{h}_0 h$$

- Dimensionless problem:

$$
\begin{aligned}
\frac{\partial h}{\partial t} - \frac{\partial}{\partial x}\left(h^3 \frac{\partial h}{\partial x}\right) &= 0, \\
h &= 0, && \text{on} \quad x = x_f(t) \\
\int_0^{x_f} h \,\mathrm{d}x &= 1, \\
h^2 \frac{\partial h}{\partial x} &\sim -\frac{\mathrm{d}x_f}{\mathrm{d}t} && \text{as} \quad x \to x_f,
\end{aligned}
$$

- Dimensionless problem:

$$
\begin{aligned}
(f^3 f')' + \frac{1}{5}(f + \eta f') &= 0, \\
f &= 0, \qquad \text{on} \quad \eta = \eta_f, \\
\int_0^{\eta_f} f \,\mathrm{d}\eta &= 1, \\
f^2 f' &\sim -\eta_f \qquad \text{as} \quad \eta \to \eta_f,
\end{aligned}
$$

with $\eta_f = \dfrac{x_f(t)}{t^{1/5}}$

- A final change of variable decouples the spreading domain:

$$s = \frac{\eta}{\eta_f}, \qquad k(s) = \frac{f(\eta_f s)}{\eta_f^{2/3}}$$

- This leads to the system

$$\begin{aligned}
\left(k^3 k'\right)' + \frac{1}{5}\left(k + s k'\right) &= 0, \\
k(1) &= 0, \\
\eta_f &= \left(\int_0^1 k(s)\,\mathrm{d}s\right)^{-3/5}, \\
k^2 k' &= -1 \qquad \text{as} \quad s \to 1.
\end{aligned}$$

- The solution to this system is

$$k = \left(\frac{3}{10}\right)^{1/3} \left(1 - s^2\right)^{1/3}$$

$$\eta_f = \left[\left(\frac{3}{10}\right)^{1/3} \frac{\pi^{1/2}}{5} \frac{\Gamma(1/3)}{\Gamma(5/6)}\right]^{-3/5}$$

$$\Gamma(z) = \int_0^\infty x^{z-1} \mathrm{e}^{-x} \, \mathrm{d}x$$

- In dimensional terms:

$$\hat{h}(\hat{x},\hat{t}) = \frac{\eta_f^{2/3}}{\hat{t}^{1/5}} \left(\frac{\Delta \rho g}{3\mu \hat{V}^2} \right)^{-1/5} k\left(\frac{\hat{x}}{\hat{x}_f} \right),$$

$$\hat{x}_f(\hat{t}) = \eta_f \hat{t}^{1/5} \left(\frac{\Delta \rho g \hat{V}^3}{3\mu} \right)^{1/5}.$$

- The scaling analysis provided the time dependence and the parametric dependence but lacked the shape that this similarity solution provides.

- In all the examples we have considered so far we have been able to determine the form of the required solution by a scaling argument.

- Examples of this kind are called similarity solutions of the first kind.

- We will now consider an example where scaling analysis alone does not determine the solution.

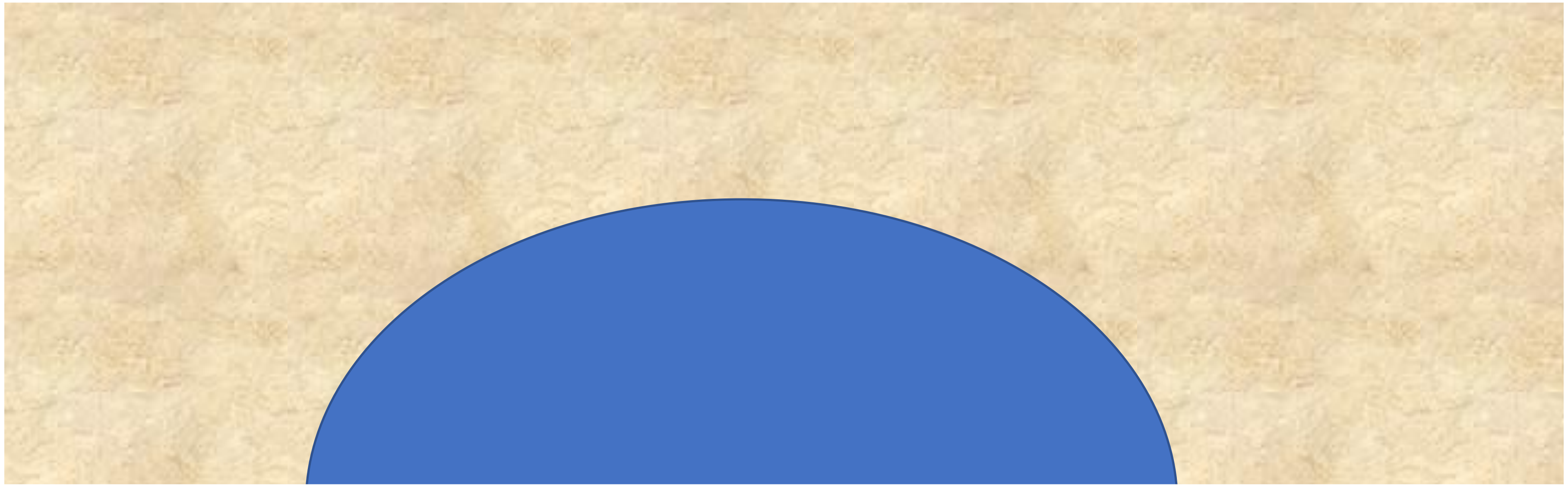
- These are called similarity solutions of the second kind.

Summary of lecture 3

- Spreading of liquid on a horizontal plate.
- We use the governing equation and mass conservation to determine the similarity solution.
- The position of the front is also an unknown we must solve for.
- We can transform the mass-conservation integral relation into a boundary condition.
- We can transform the regularity condition at the moving front into a boundary condition.
- We scale the variables so that we solve for the height of the profile first and determine the position of the front afterwards.

Spreading of a groundwater mound

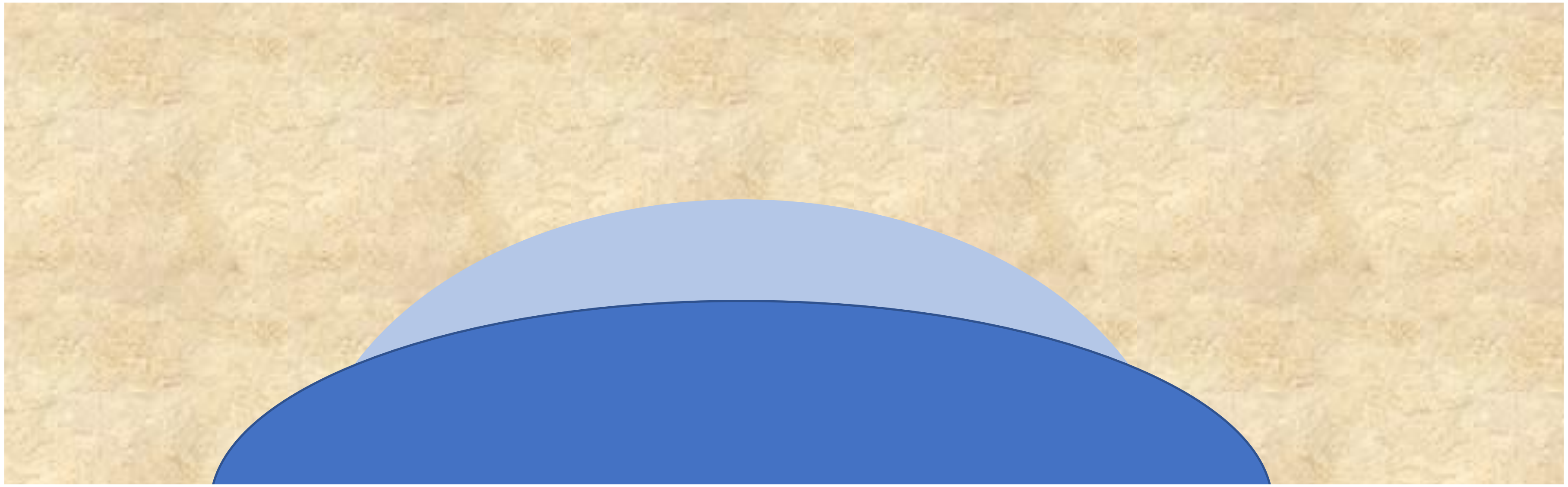
- Now let’s consider the spreading in a porous medium rather than in air:



- Suppose that the liquid leaves behind a residue.

Spreading of a groundwater mound

- Now let’s consider the spreading in a porous medium rather than in air:



- Suppose that the liquid leaves behind a residue.

- We now split up the problem into two regions.

- **Region 1: Liquid invading dry porous medium** $\partial\hat{h}/\partial\hat{t} > 0$

- Here the flow is governed by the porous medium equation

$$\phi\frac{\partial\hat{h}}{\partial\hat{t}} + \frac{\partial\hat{Q}}{\partial\hat{x}} = 0, \qquad \hat{Q} = -\frac{\Delta\rho g K}{\mu}\hat{h}\frac{\partial\hat{h}}{\partial\hat{x}}$$

- **Region 2: Liquid draining** $\partial \hat{h}/\partial \hat{t} < 0.$

- Here the flow is governed by a modified porous medium equation due to the partially saturated residue region:

$$(1-s)\phi\frac{\partial \hat{h}}{\partial \hat{t}} + \frac{\partial \hat{Q}}{\partial \hat{x}} = 0, \qquad \hat{Q} = -\frac{\Delta\rho g K}{\mu}\hat{h}\frac{\partial \hat{h}}{\partial \hat{x}}$$

- s = fraction of space occupied by liquid in drained region.
  s = 0 recovers the original porous medium equation. As s increases, $\partial \hat{h}/\partial \hat{t}$ gets larger due to the liquid in the drained region pushing down.

- Denote the position of the front by $\hat{x}_f(\hat{t})$ and the position of the joint separating Regions 1 and 2 by $\hat{x}_s(\hat{t})$.

- We have two second-order partial differential equations with two additional unknowns ($\hat{x}_f$ and $\hat{x}_s$) so we require initial conditions for the two regions plus six boundary conditions (four for the two equations and two for the unknowns).

- We apply:

$$\hat{h}(\hat{x}_f, \hat{t}) = 0$$

Definition of $\hat{x}_f$

$$\frac{\partial \hat{h}}{\partial \hat{t}}(\hat{x}_s, \hat{t}) = 0$$

Definition of $\hat{x}_s$

$$\hat{h}\frac{\partial \hat{h}}{\partial \hat{x}} = 0 \quad \text{at} \quad \hat{x} = 0,$$

No flux at the centre

$$\hat{h}\frac{\partial \hat{h}}{\partial \hat{x}} = 0 \quad \text{at} \quad \hat{x} = \hat{x}_f,$$

No flux at the front

Continuity in $\hat{h}$ and $\hat{h}\partial\hat{h}/\partial\hat{x}$ at the joint $\hat{x} = \hat{x}_s$

- As for the case of a spreading liquid, (D) doesn't look like it gives any more information than (A). However, it does enforce a regularity condition on $\hat{h}$ at the front.

- We can transform this into a stronger condition by performing a local analysis near the front. We rescale:

$$\hat{x} = \hat{x}_f(\hat{\tau}) + \epsilon\hat{\xi}, \qquad \hat{t} = \hat{\tau}, \qquad \hat{h}(\hat{x}, \hat{t}) = \epsilon \hat{H}(\hat{\xi}, \hat{\tau}).$$

- Substituting into (1) and considering the result at leading order gives

$$\frac{\partial \hat{H}}{\partial \hat{\xi}} = -\frac{\mu\phi}{\Delta\rho g K}\frac{\mathrm{d}\hat{x}_f}{\mathrm{d}\hat{\tau}}$$

which in terms of the original variables gives

$$\frac{\partial \hat{h}}{\partial \hat{x}}(\hat{x}_f, \hat{t}) = -\frac{\mu\phi}{\Delta\rho g K}\frac{\mathrm{d}\hat{x}_f}{\mathrm{d}\hat{t}}$$

- We non-dimensionalize via

$$\hat{x} = \hat{x}_0 x, \qquad \hat{x}_s = \hat{x}_0 x_s, \qquad \hat{x}_f = \hat{x}_0 x_f, \qquad \hat{t} = \hat{t}_0 t, \qquad \hat{h} = \hat{h}_0 h,$$

- The resulting dimensionless system is then

$$\frac{\partial h}{\partial t} - \kappa \frac{\partial}{\partial x}\left(h \frac{\partial h}{\partial x}\right) = 0, \qquad 0 \le x \le x_s(t), \qquad (1)$$

$$\frac{\partial h}{\partial t} - \frac{\partial}{\partial x}\left(h \frac{\partial h}{\partial x}\right) = 0, \qquad x_s(t) \le x \le x_f(t), \qquad (2)$$

$$h(x_f, t) = 0, \qquad (A)$$

$$\frac{\partial h}{\partial t}(x_s, t) = 0, \qquad (B)$$

$$\frac{\partial h}{\partial x}(0, t) = 0, \qquad (C)$$

$$\frac{\partial h}{\partial x}(x_f, t) = -\frac{\mathrm{d}x_f}{\mathrm{d}t} \qquad (D)$$

plus continuity in h and $h\partial h/\partial x$ at $x = x_s$ (E)

- Since the liquid leaves behind a residue we can't apply a simple global conservation law like in previous cases.

- We try a similarity solution of the form

$$h = t^{\beta} f(\eta) \quad \text{with} \quad \eta = \frac{x}{t^{\alpha}}.$$

This gives

$$\kappa(ff')' + \alpha\eta f' - (2\alpha - 1)f = 0, \qquad 0 \leq \eta \leq \eta_s,$$
$$(ff')' + \alpha\eta f' - (2\alpha - 1)f = 0, \qquad \eta_s \leq \eta \leq \eta_f,$$

where $\eta_f = x_f/t^{\alpha}$ and $\eta_s = x_s/t^{\alpha}$ are defined by

$$f(\eta_f) = 0,$$
$$(2\alpha - 1)f(\eta_s) - \alpha\eta_s f'(\eta_s) = 0,$$

- We also apply

$$f'(0) = 0,$$
$$f'(\eta_f) = -\alpha\eta_f,$$

as well as continuity in *f* and *f'* at η_s.

- Unlike in our previous examples, we have no further information to determine *α*. So, we press on.

- Make a change of variable $z = \eta/\eta_f$ and $k(z) = f(\eta_f z)/\eta_f^2$.

$$\kappa\left(kk'\right)' + \alpha z k' - (2\alpha - 1)k = 0, \qquad 0 \le z \le z_s,$$
$$\left(kk'\right)' + \alpha z k' - (2\alpha - 1)k = 0, \qquad z_s \le z \le 1,$$

$$k(1) = 0,$$
$$(2\alpha - 1)k(z_s) - \alpha z k'(z_s) = 0,$$
$$k'(0) = 0,$$
$$k'(1) = -\alpha,$$

And continuity in *k* and *k'* at $z=z_s$.

- This is two second-order ordinary differential equations plus one unknown (z_s).
- How many boundary conditions does this require?

- We have six boundary conditions. The sixth condition determines α.

- This is an eigenvalue problem.

- For a given value of κ there is an associated α that gives a solution.

- The natural similarity variables did not emerge from a scaling argument but instead as part of the solution.

- Such problems are called similarity solutions of the second kind.

- Importantly, in this case, the similarity solution only provides the shape of the interface up to a constant. Our final scaling moved the unknown η_s to determine α so we are still missing one pieced of information. (In our previous example of spreading liquid on a horizontal plate, this came from mass conservation.)

- To determine the appropriate scaling we must compare our similarity solution with the full numerical solution at one point in time.

Summary

- Scaling laws can provide a significant amount of information about the solution of a partial differential equation system.

- Similarity solutions of the first kind provide the full behaviour for long time but does not (always) capture the early-time behaviour (unless this happens to satisfy the initial condition).

- Similarity solutions of the second kind arise when we cannot fully determine the form of the similarity solution from the governing equation and boundary conditions. This forms an eigenvalue problem. The similarity solution provides the shape up to a scaling constant.

Summary of lecture 4

- Similarity solutions of the second kind arise when we cannot fully determine the form of the similarity solution from the governing equation and boundary conditions.

- This forms an eigenvalue problem.

- The similarity solutions provides the shape up to a scaling constant.

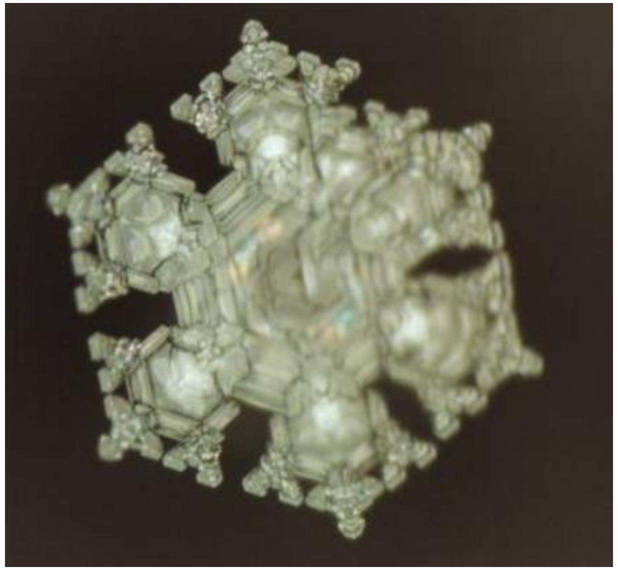
Free boundary problems

- A free boundary problem occurs when the region in which the problem is to be solved is unknown in advance and must be found as part of the solution.

- One example concerns the freezing or melting of ice. Here, we have to solve for the free boundary between the two phases as well as for the temperature field in each phase.

- The shape of a free boundary as it is formed can be quite exotic.

Masaru Emoto



Shimanto River, Japan

Mount Cook Glacier,
New Zealand

Playing Imagine by John Lennon to the water

Showing the water a photo of an elephant

Industrial problem: electric welding

- Steel plates are welded together by attaching electrodes to the edges and then passing a current through.

- This heats the metal, causing the plates to melt. When the current is switched off, the liquid metal solidifies and the two plates are welded together.

- How long should the current be applied to ensure the metal melts without melting the entire plates?

- How long should the plates be left once the current has been switched off to ensure that they have solidified completely?

One-dimensional Stefan problem

- Consider a slab of initially solid material occupying the region $0 < x < a$. The temperature $T(x,t)$ in the slab satisfies the heat equation:

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right)$$

- Suppose the slab is initially at a uniform temperature T_0 and the face $x = a$ is insulated, while the temperature of the face at $x = 0$ is suddenly increased to a temperature $T_1 > T_0$:

$$T(x,0) = T_0, \quad 0 < x < a; \qquad T(0,t) = T_1, \quad t > 0; \qquad \frac{\partial T}{\partial x}(a,t) = 0, \quad t > 0.$$

- If T_1 is high enough the solid will melt and become liquid in some neighbourhood of $x = 0$.

- If the slab melts, we must seek a solution in which the slab is liquid in $0 < x < s(t)$ and solid in $s(t) < x < a$, where $x = s(t)$ is the free boundary separating the liquid and solid phases.

- $s(t)$ is unknown in advance and must be found as part of the solution.

- We have to solve the heat equation in both solid and liquid phases, with different values of the parameters ρ, c and k in each phase.

- We also need to impose conditions at the free boundary $x = s(t)$.

- First we assume that the phase change happens at a known melting temperature so

$$T(s(t), t) = T_{\mathrm{m}}$$

- Since the temperature is continuous, this applies on both sides of the free boundary.

- The second condition comes from an energy balance. At a fixed boundary, the heat flux coming in from one side must equal the heat flux exiting from the other side, i.e., $q = -k\partial T/\partial x$ must be continuous.

- However, at a moving interface, we must also consider the energy associated with the phase change, known as the latent heat *L*.

- The latent heat acts as an energy source or sink at a moving solid–liquid interface, and the resulting boundary condition

$$k\frac{\partial T}{\partial x}\bigg|_{x=s(t)+} - k\frac{\partial T}{\partial x}\bigg|_{x=s(t)-} = \rho L\frac{\mathrm{d}s}{\mathrm{d}t}$$

- This is called the Stefan condition.

Assumptions

- We assumed that the heat flux is purely due to thermal conductivity. If the liquid flows at all, then there will also be a contribution due to convection.

- We assumed that the solid and liquid have the same density. If there is a change in density as the material melts, this would induce a velocity.

Summary of lecture 5

- A free boundary problem occurs when the region in which the problem is to be solved is unknown in advance and must be found as part of the solution.

- Stefan problems describe melting or freezing. At the interface we apply a Stefan condition:

$$k\frac{\partial T}{\partial x}\bigg|_{x=s(t)+} - k\frac{\partial T}{\partial x}\bigg|_{x=s(t)-} = \rho L\frac{\mathrm{d}s}{\mathrm{d}t}$$

Jožef Stefan
(1835-1893)
Slovene physicist

One-phase Stefan problem

- Assume that ρ, c and k are all constant on either side of the solid–liquid interface.

- Suppose that the initial temperature T_0 is equal to the melting temperature T_m. This means that the solid region $s(t) < x < a$ is always at temperature T_m and we only need to solve the heat equation in the liquid phase $0 < x < s(t)$.

- This is a one-phase Stefan problem.

- Non-dimensionalize:

$$x = ax', \qquad t = \left(\frac{\rho L a^2}{k(T_1 - T_{\mathrm{m}})}\right) t', \qquad T = T_{\mathrm{m}} + (T_1 - T_{\mathrm{m}})u.$$

$$\begin{aligned}
\mathrm{St}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} & 0 < x < s(t), \quad t > 0, \\
u &= 1 & x = 0, \quad t > 0 \\
u = 0, \quad -\frac{\partial u}{\partial x} &= \frac{\mathrm{d}s}{\mathrm{d}t} & x = s(t), \quad t > 0 \\
s(0) &= 0.
\end{aligned}$$

- Seek a similarity solution:

$$s(t) = \beta\sqrt{t} \qquad \text{and} \quad u(x,t) = f(\eta), \qquad \text{where} \quad \eta = \frac{x}{\sqrt{t}}$$

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2} + \frac{\mathrm{St}}{2}\,\eta\frac{\mathrm{d}f}{\mathrm{d}\eta} = 0 \qquad 0 < \eta < \beta,$$

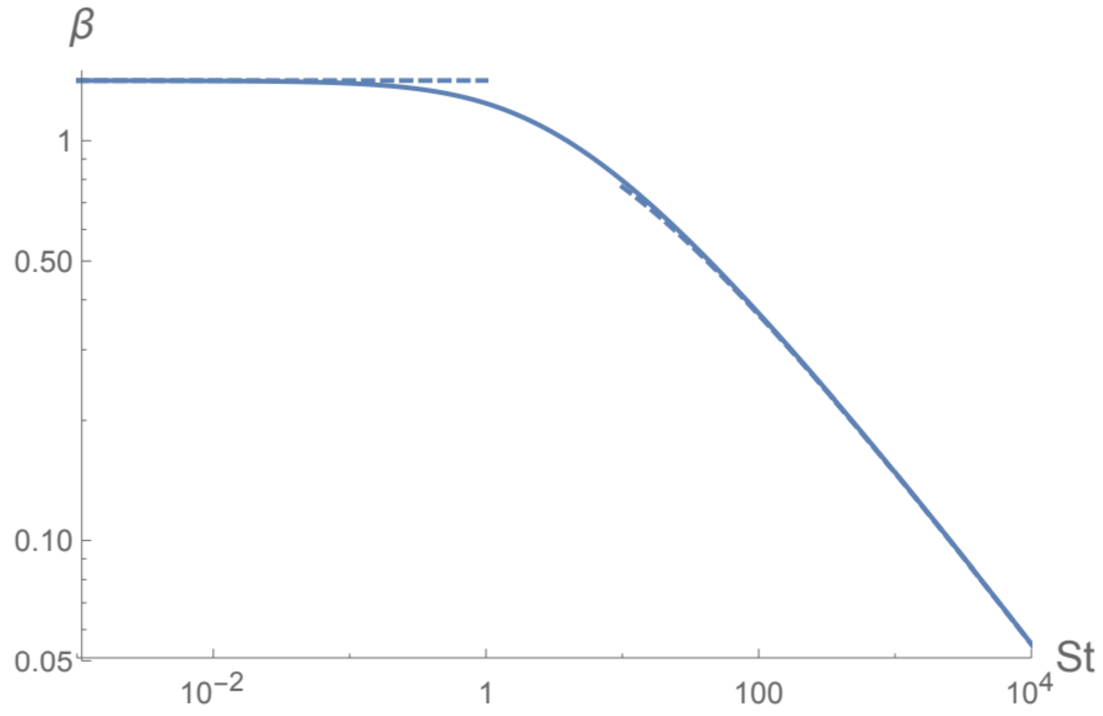
$$f = 1 \qquad \eta = 0,$$

$$f = 0, \quad \frac{\mathrm{d}f}{\mathrm{d}\eta} = -\frac{\beta}{2} \qquad \eta = \beta.$$

- The solution is

$$f(\eta) = 1 - \frac{\mathrm{erf}\left(\eta\sqrt{\mathrm{St}}/2\right)}{\mathrm{erf}\left(\beta\sqrt{\mathrm{St}}/2\right)}$$

$$\frac{\sqrt{\pi}\beta e^{\mathrm{St}\beta^2/4}\,\mathrm{erf}\left(\beta\sqrt{\mathrm{St}}/2\right)}{2\sqrt{\mathrm{St}}} = 1$$



$$\beta \sim \begin{cases} \sqrt{2} & \text{as } \mathrm{St} \to 0, \\ \dfrac{2}{\sqrt{\mathrm{St}}}\sqrt{\log\left(\dfrac{\mathrm{St}}{\sqrt{\pi}}\right)} & \text{as } \mathrm{St} \to \infty \end{cases}$$

- The entire slab has melted when s(t) = 1, and the dimensionless time taken is thus given by $1/\beta^2$.

- By reversing the non-dimensionalization we find approximations for the melting time t_m:

$$t_{\mathrm{m}} \sim \begin{cases} \dfrac{\rho L a^2}{2k(T_1 - T_{\mathrm{m}})} & \mathrm{St} \ll 1, \\ \dfrac{\rho c a^2}{4k \log\left(\mathrm{St}/\sqrt{\pi}\right)} & \mathrm{St} \gg 1. \end{cases}$$

- When St<<1, the melting of the slab is limited by the latent heat required.
- When St>>1, the main barrier is the energy needed to heat up the slab, and the melting time depends only weakly on the latent heat *L*.

Two-phase Stefan problem

$$\mathrm{St}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < s(t), \quad t > 0,$$

$$\frac{\mathrm{St}}{\kappa}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad s(t) < x < 1, \quad t > 0,$$

$$u = 1 \qquad x = 0, \quad t > 0,$$

$$\frac{\partial u}{\partial x} = 0 \qquad x = 1, \quad t > 0,$$

$$u = 0, \quad K\left[\frac{\partial u}{\partial x}\right]^{+} - \left[\frac{\partial u}{\partial x}\right]^{-} = \frac{\mathrm{d}s}{\mathrm{d}t} \qquad x = s(t), \quad t > 0$$

$$u = -\theta, \quad s = 0 \qquad t = 0,$$

$$\kappa = \frac{c_1 k_2}{c_2 k_1}, \qquad K = \frac{k_2}{k_1}, \qquad \theta = \frac{T_\mathrm{m} - T_0}{T_1 - T_\mathrm{m}}$$

- When St<<1:

$$u(x,t) = \begin{cases} 1 - \dfrac{x}{s(t)} & 0 < x < s(t) < 1 \\ 0 & 0 < s(t) < x < 1 \end{cases} \qquad (*)$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{s} \qquad \Rightarrow \qquad s(t) = \sqrt{2t},$$

- This is equivalent to the one-phase small St Stefan problem ($\beta = \sqrt{2}.$)

- The approximate solution (*) does not satisfy the initial condition (except in the special case when $\theta = 0$, i.e., $T_0 = T_m$).

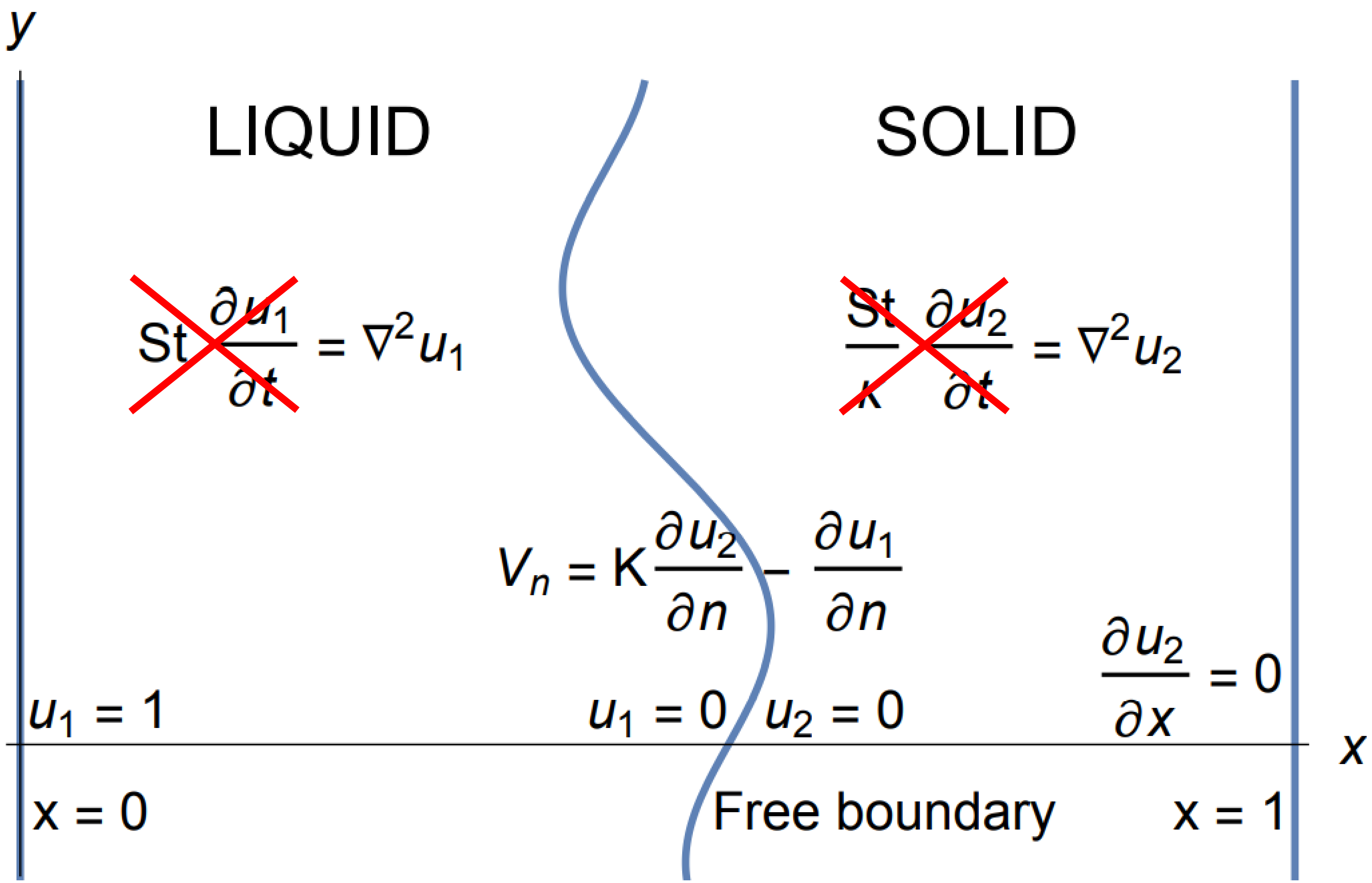
- We resolve this by considering a small-t boundary layer in which u adjusts from the initial value $-\theta$ to the outer solution.

MMSC project: Mathematical modelling to improve tumble-dryer filters

Ian Griffiths and Yixuan Sun

- How can we design and manufacture the tumble-dryer filter to maximize removal efficiency while minimizing energy usage?

Two-dimensional Stefan problem



- We study the small Stefan number limit again: St<<1.
- We then have a Hele-Shaw problem.

- We examine the linear stability.
- For simplicity, consider the one-phase Stefan problem again so u_2=0.
- Assume that the interface is moving at speed V and there is a temperature gradient -λ. (We expect λ>0 if the left region is liquid and the right is solid.) The solution is then

$$u(x, y, t) = -\lambda(x - Vt) \qquad\qquad x = Vt$$

- Consider a small perturbation about this solution

$$u(x, y, t) = -\lambda(x - Vt) + \epsilon\tilde{u}(x, y, t) \qquad x = Vt + \epsilon\xi(y, t) \qquad \epsilon \ll 1$$

- Substitute into the Stefan condition:

$$V = \lambda.$$

- Substitute into the governing equations

$$\begin{aligned} \nabla^2 \tilde{u} &= 0 & x &< Vt, \\ \tilde{u} - \lambda\xi &= 0 & x &= Vt, \\ -\frac{\partial \tilde{u}}{\partial x} &= \frac{\partial \xi}{\partial t} & x &= Vt. \end{aligned}$$

- Try a wave-like solution:

$$\tilde{u}(x,y,t) = A\mathrm{e}^{\sigma t + \mathrm{i}ky + k(x - Vt)}, \qquad \xi(y,t) = B\mathrm{e}^{\sigma t + \mathrm{i}ky},$$

Here, *k* is the wavenumber and *σ* is the linear growth rate.

- Substituting into the governing equations gives the linear system

$$\begin{pmatrix} 1 & -\lambda \\ k & \sigma \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \mathbf{0}.$$

- Non-trivial solutions require the determinant of the matrix to be zero, giving

$$\sigma = -\lambda k = -Vk$$

$$\sigma = -\lambda k = -Vk$$

If V>0

- The solid is melting.
- $\sigma<0$ and so the perturbations decay with time – the interface is stable.

If V<0

- The liquid is freezing.
- $\sigma>0$ and so the perturbations grow with time – the interface is unstable.
- The smallest wavelength modes grow the fastest – this is an ill-posed problem.
    - The solution does not depend continuously on the data.
    - an arbitrarily small perturbation of the initial conditions may cause an arbitrarily large change in the solution.
    - This is physically unacceptable, and such ill posed problems are also effectively impossible to solve numerically.
- $\lambda<0$ which means that the temperature in the liquid is lower than the melting temperature. The liquid is supercooled.

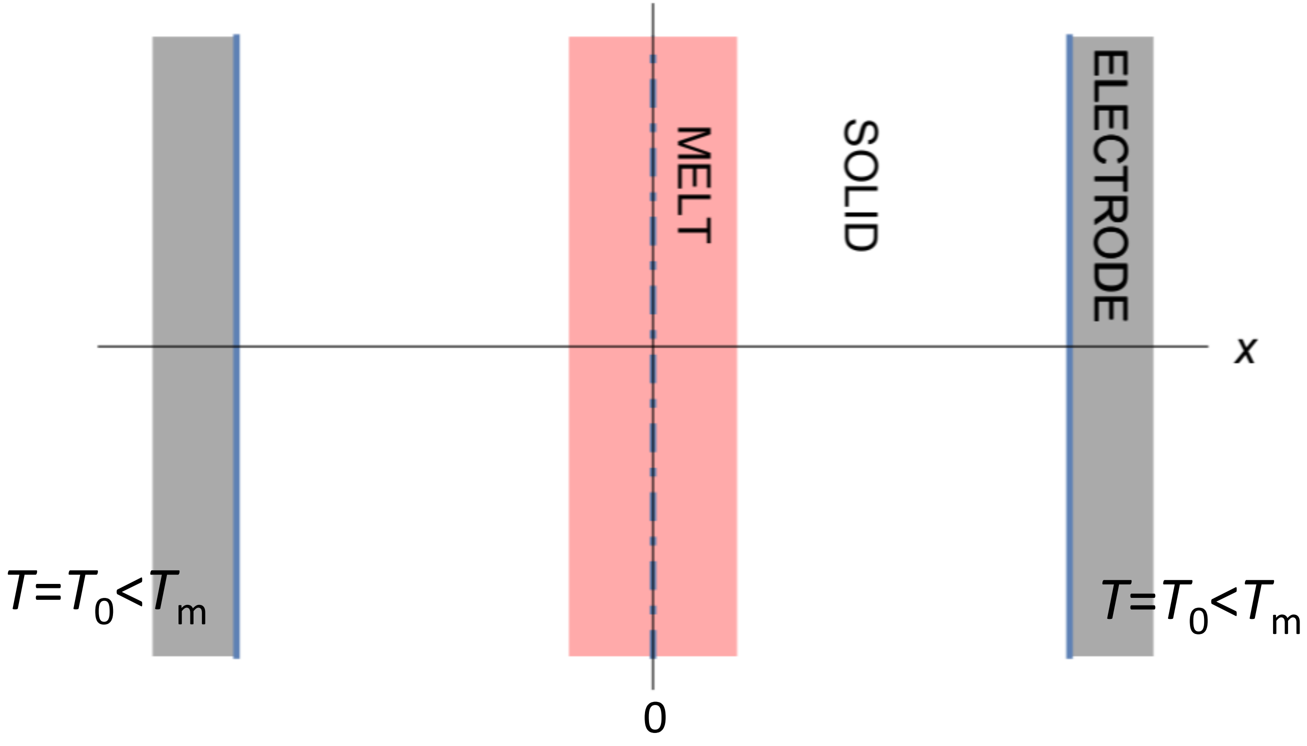
- The results we have found tie in with what we observe physically
  - An ice cube melts smoothly in a drink.
  - But as water freezes, dendrites form whose structure are unpredictable.

- It is reasonable for a freezing boundary to be unstable, but it is not acceptable for the problem to be ill posed.
- Some additional physics must become important to regularize the problem by suppressing arbitrarily high wavenumbers, e.g.,:
  - Surface energy effects penalise high curvature variations in the free boundary.
  - Kinetic undercooling – the temperature need not be continuous if the free boundary evolves rapidly enough to be out of thermodynamic equilibrium.
  - Model the complex dendrites.

- But we shall choose a simpler strategy
  – to model the dendritic region as a "mushy region"

One-dimensional welding problem

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{J^2}{\sigma}$$



- Assume that the parameters *ρ*, *c*, *k* and *σ* are constant and the same in both phases.
- Assume symmetry about *x*=0.

- Let’s draw how we expect the temperature to evolve:

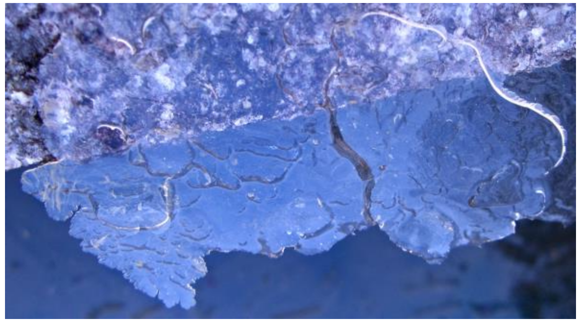
- This is an ill-posed problem.

Summary of lecture 6

Two-dimensional Stefan problem

- Linear stability demonstrates that:

  - solid melting is stable



  - liquid freezing is unstable – this is an ill-posed problem.

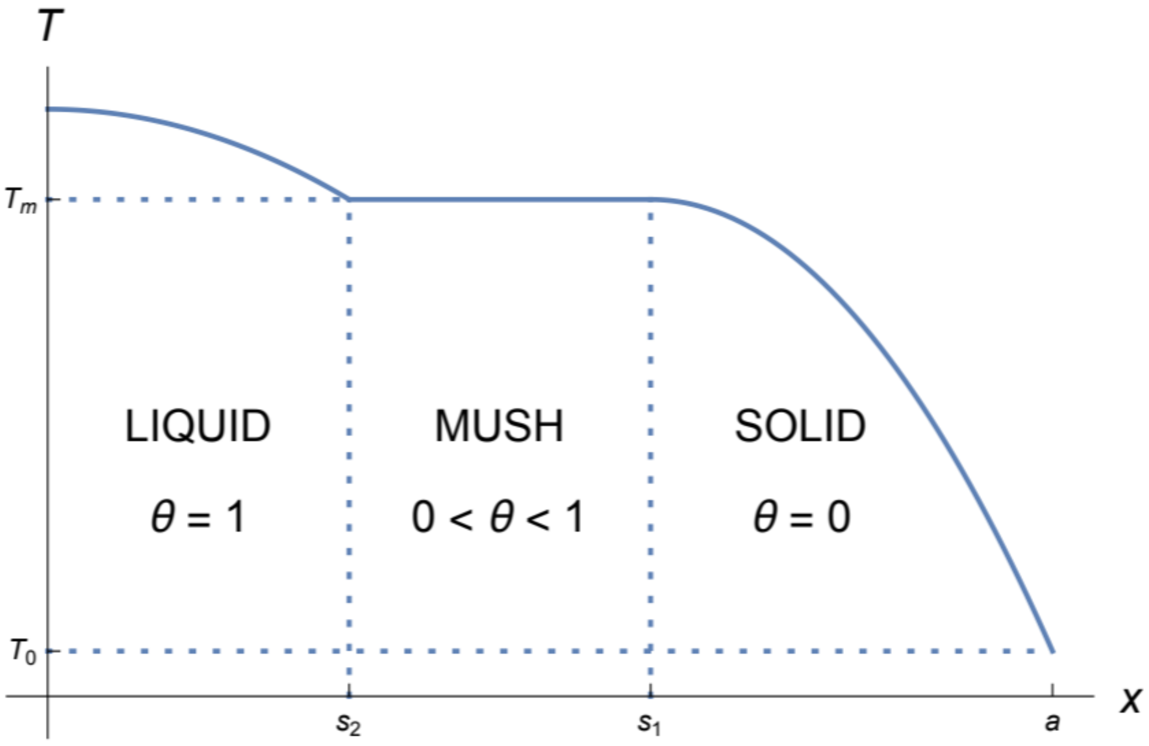
- We need a model for liquid freezing.

- What happens instead?

- There is not a simple free boundary between pure liquid and pure solid phases.

- Instead, there is a “mushy region” in which both solid and liquid phases coexist, with the solid existing in a very fine dendritic crystalline structure.

- Rather than try to resolve this highly complicated structure, we construct a homogenised model that describes the net macroscopic behaviour of the mixture as a whole.



- We have two free boundaries $x = s_1(t)$ and $x = s_2(t)$,
  with solid material in $s_1 < x < a$,
  mush in $s_2 < x < s_1$
  liquid in $0 < x < s_2$.

- It just remains to construct a model for the mush.

- The first observation is that, for both phases to coexist, the temperature must everywhere be close to the melting temperature, so we may set $T = T_m$ in the mush.
- The energy source does not heat up the mixture but supplies the latent heat needed to melt the solid.
- The energy equation in the mush is $$\rho L \frac{\partial \theta}{\partial t} = \frac{J^2}{\sigma} \quad (*)$$

  where θ is the liquid fraction in the mixture: θ = 0 in a pure solid and θ = 1 in pure liquid.

- We have two free boundaries. The equation for conservation of energy at each of these is

$$\left[\rho L\theta\frac{\mathrm{d}s_j}{\mathrm{d}t} + k\frac{\partial T}{\partial x}\right]_-^+ = 0 \quad \text{at } s = s_j(t) \qquad j=1,2$$

- Finally, we require one initial condition for the PDE (*), namely that θ is continuous across the advancing melting boundary $x = s_1(t)$ (so that the liquid fraction is zero when the mush first forms.

- The dimensionless problem we must work with is thus:

$$\mathrm{St}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + q \qquad 0 < x < 1, \quad t > 0,$$

$$\frac{\partial u}{\partial x} = 0 \qquad x = 0, \quad t > 0,$$

$$u = -1 \qquad x = 1, \quad t > 0,$$

$$u = -1 \qquad 0 < x < 1, \quad t = 0,$$

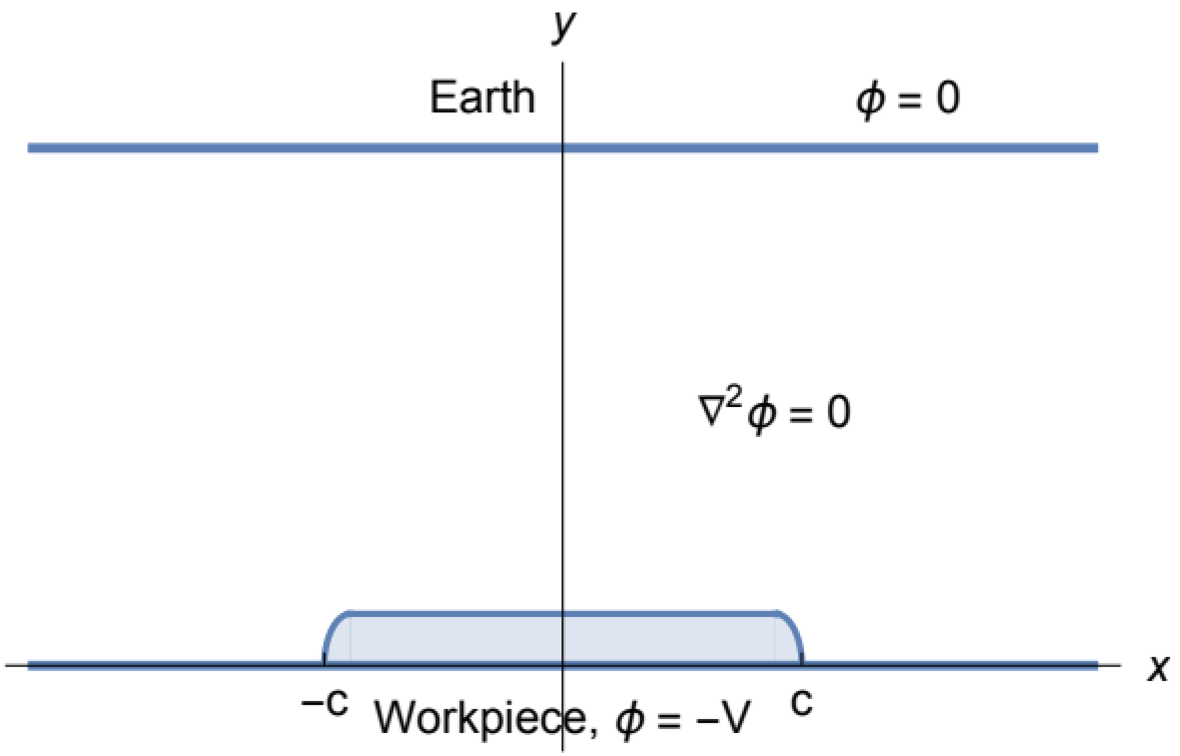
$$u = 0, \quad \frac{\partial \theta}{\partial t} = q \qquad s_2(t) < x < s_1(t), \quad t > 0,$$

$$u = 0, \quad \left[\frac{\mathrm{d}s_j}{\mathrm{d}t} + k\frac{\partial u}{\partial x}\right]_-^+ = 0 \qquad \text{at } s = s_j(t), \quad t > 0$$

- This modified free boundary problem is well posed.
- In general, numerical solution is required. A very useful approach is to use the enthalpy, which measures the total internal energy in the material, including thermal energy and latent heat.
- However, the initial stages at least may be analysed analytically, and in the quasi-steady limit St $\rightarrow$ 0 the problem may be solved completely.

Co-dimension two free boundary problems

- In electrochemical painting, metal objects are coated using electrochemistry.



- The object (“workpiece”) is immersed in an electrolyte solution, across which a potential difference V is imposed.
- This drives a current j of ions which attach themselves as a layer of paint on the workpiece.
- A model is needed to determine when (and if) the entire surface of the workpiece is covered in paint, and the thickness of the paint layer achieved.

Modelling assumptions

- The concentration of ions in solution is sufficiently large that their depletion during the process is negligible. We may then treat the concentration as a constant.

- The flux of ions in the solution due to the imposed electric field is then given by

$$\boldsymbol{j} = -\sigma \boldsymbol{\nabla} \phi$$

  where ϕ is the electric potential and *σ* is the conductivity.

- Conservation of charge then gives

$$\boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$$

- We apply boundary conditions ϕ=-*V* on the workpiece and ϕ=0 on the outer surface.

- Let $h(x,t)$ denote the thickness of the paint layer (where it exists).

- Assume the paint acts as a resistive layer with conductivity $\bar{\sigma}$ much lower than that of the solution. If the layer is thin, then it will act as a classical resistor with surface conductivity $\bar{\sigma}/h$.

- We can relate the potential difference across the layer to the current across it:

$$\sigma \frac{\partial \phi}{\partial y} = \frac{\bar{\sigma}}{h}(V + \phi) \quad \text{at } y = h(x,t)$$

- It is observed experimentally that, when the current is switched off, the paint dissolves back into solution at some rate d. So,

$$\frac{\partial h}{\partial t} = -d + \alpha\sigma \frac{\partial \phi}{\partial y} \quad \text{at } y = h(x,t)$$

where $\alpha = V_M/F$, where V_M is the molar volume of solid paint and F is Faraday's constant.

- We non-dimensionalize via:

$$(x, y) = L(\tilde{x}, \tilde{y}), \qquad \phi = -V + V\tilde{\phi} \qquad h = \epsilon L \tilde{h}, \qquad t = \frac{\epsilon L^2}{\alpha \sigma V}\tilde{t}, \qquad \epsilon = \frac{\bar{\sigma}}{\sigma} \ll 1$$

- The boundary conditions then read

$$\frac{\partial \phi}{\partial y} = \frac{\phi}{h}, \quad \frac{\partial h}{\partial t} = \frac{\partial \phi}{\partial y} - \delta \quad \text{at } y = \epsilon h(x,t) \qquad \delta = \frac{Ld}{\alpha \sigma V}$$

- We exploit the smallness of *ε* and expand:

$$\phi(x, \epsilon h, t) = \phi(x, 0, t) + \epsilon h \frac{\partial \phi}{\partial y}(x, 0, t) + O(\epsilon^2)$$

- We can then replace the boundary condition above to apply on *y*=0.

- In places where there is no paint (h=0) we apply

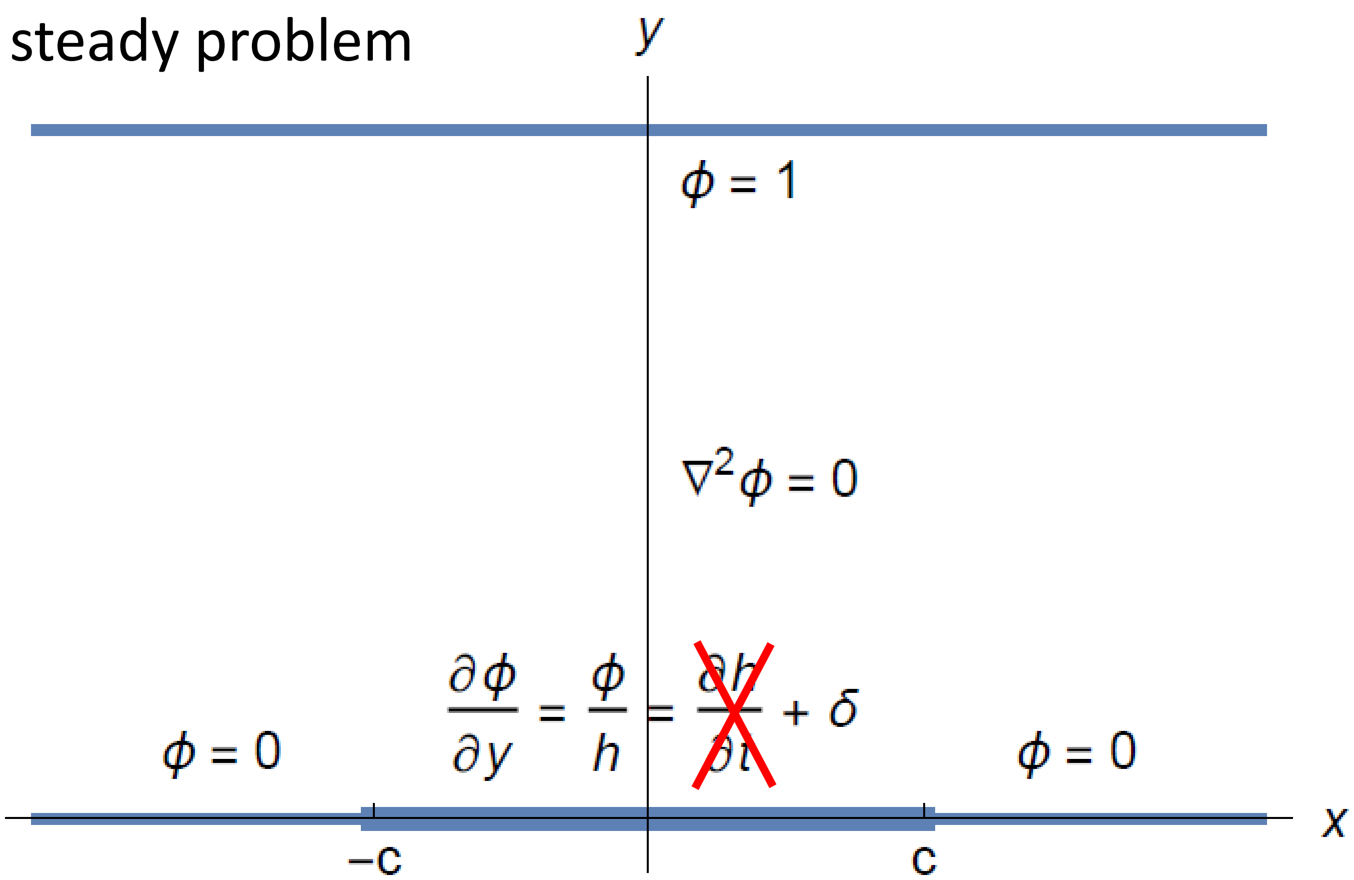
$$\phi = 0 \quad \text{at } y = 0$$

Summary of lecture 7

- Solid melting is stable – a well-posed problem.
- Liquid freezing is unstable – an ill-posed problem.
- To model freezing we introduce a mushy region composed of liquid fraction θ and solid fraction $1-\theta$.
- We solve for the temperature T in the solid and liquid regions and for θ in the mushy region.

- The geometry to solve Laplace's equation is specified in advance.
- But the points $x= \pm c$ where the boundary conditions switch from Dirchlet to Neumann are not known in advance and must be found as part of the solution.
- These *points* are the free boundaries.
- This is called a codimension two free boundary problem because two is the codimension between the space in which the problem is posed (two dimensions) and the free boundaries (points, i.e., zero dimensions).

- Consider the steady problem



- We can solve for ϕ and then determine *h* after using the remaining boundary condition

$$h(x) = \frac{\phi(x,0)}{\delta}$$

- Note that since $h \geq 0$ this means that $\phi(x,0) \geq 0$.

- Suppose that the earth is a single point at (0,1). Then the potential near this point is a point singularity with unit strength:

$$\phi \sim -\frac{1}{2\pi}\log\big|(x,y)-(0,1)\big| = -\frac{1}{4\pi}\log\left(x^2+(y-1)^2\right) \quad \text{as } (x,y)\to(0,1)$$

- We subtract off the singularity, an image at (0,-1) and δy to satisfy the boundary condition $\partial\phi/\partial y=\delta$ on $y=h$:

$$\phi(x,y) = \delta y - \frac{1}{4\pi}\log\left(x^2+(y-1)^2\right) + \frac{1}{4\pi}\log\left(x^2+(y+1)^2\right) + \Phi(x,y)$$

- The problem then reads:

$$\begin{aligned}
\nabla^2\Phi &= 0 && y > 0, \\
\frac{\partial \Phi}{\partial y} &\to -\delta && x^2 + y^2 \to \infty, \\
\Phi = 0, \quad \frac{\partial \Phi}{\partial y} &\leq -f(x) && y = 0, \ |x| > c, \\
\Phi \geq 0, \quad \frac{\partial \Phi}{\partial y} &= -f(x) && y = 0, \ |x| < c,
\end{aligned}$$

where $f(x) = \dfrac{1}{\pi\left(1 + x^2\right)}$

- This is a mixed boundary value problem. We expect the solution to have singularities at $(\pm c,0)$ where the boundary conditions switch from Dirichlet to Neumann.

- This problem can be solved by taking a Fourier transform in x, in terms of the function Φ at the surface of the workpiece, $\Phi_0(x)=\Phi(x,0)$. Doing this, we find that

$$\frac{\partial \Phi}{\partial y} = \frac{1}{\pi}\int_{-c}^{c} \frac{(x-s)\Phi_0'(s)}{(x-s)^2+y^2}\,\mathrm{d}s$$

- Carefully taking the limit y->0 and applying the boundary condition

$$\frac{\partial \Phi}{\partial y} = -f(x) \qquad y=0,\ |x|<c$$

we find that Φ_0 satisfies the singular integral equation:

$$-f(x) = \frac{1}{\pi} ⨍_{-c}^{c} \frac{\Phi_0'(s)}{x-s}\,\mathrm{d}s$$

- Here we take the principal value of the integral:

$$⨍_{-c}^{c} \frac{\Phi_0(s)}{x-s}\,\mathrm{d}s = \lim_{\epsilon\to 0}\left(\int_{-c}^{x-\epsilon} + \int_{x+\epsilon}^{c}\right)\frac{\Phi_0(s)}{x-s}\,\mathrm{d}s$$

- We solve for Φ_0(x), and find the value of *c* such that is differentiable at (±*c*, 0).
- The thickness of the painted layer is then recovered from $h(x) = \frac{\Phi_0(x)}{\delta}$.

Solution of problem

- The most elegant way to solve this problem is using complex variable theory.

- We write $\Phi(x,y) + \mathrm{i}\Psi(x,y) = w(z)$

  where w(z) is holomorphic in Im(*z*) > 0.

  (The imaginary part Ψ is the harmonic conjugate to Φ – they are related through the Cauchy–Riemann equations.)

- We differentiate with respect to *z*:

$$\frac{\partial \Phi}{\partial x} - \mathrm{i}\frac{\partial \Phi}{\partial y} = w'(z)$$

- Our task is then to find a $w'(z)$ that is holomorphic on the upper half plane and satisfies the conditions

$$
\begin{aligned}
w'(z) &\to \delta \mathrm{i} && z \to \infty, \\
\operatorname{Re}\left[w'(z)\right] &= 0, && y = 0,\ |x| > c, \\
\operatorname{Im}\left[w'(z)\right] &= f(x) && y = 0,\ |x| < c.
\end{aligned}
$$

- The mixed character of the problem is reflected in the fact that we specify the real part of w' in one place and the imaginary part in another.

- But we can use a trick to transform this into a problem for just the real part by letting

$$w'(z) = \sqrt{z^2 - c^2}\, W(z)$$

where $\sqrt{z^2 - c^2} = \sqrt{z - c}\,\sqrt{z + c} = \sqrt{r_1 r_2}\, \mathrm{e}^{\mathrm{i}(\theta_1 + \theta_2)/2}$

- The key point is that $\sqrt{z^2 - c^2}$ is real when *x*>*c* and imaginary when x<c as y->0 so *W*(*z*) stays real everywhere.

- The new function W satisfies a classical Dirichlet problem, satisfying Laplace's equation with its real part specified everywhere on the boundary

$$W(z) \sim \frac{\mathrm{i}\delta}{z} \qquad z \to \infty, \quad \text{(A)}$$

$$\operatorname{Re}\left[W(z)\right] = \begin{cases} 0 & |x| > c, \\ \dfrac{f(x)}{\sqrt{c^2 - x^2}} & |x| < c \end{cases} \qquad y = 0. \quad \text{(B)}$$

- The solution is given using Poisson's formula:

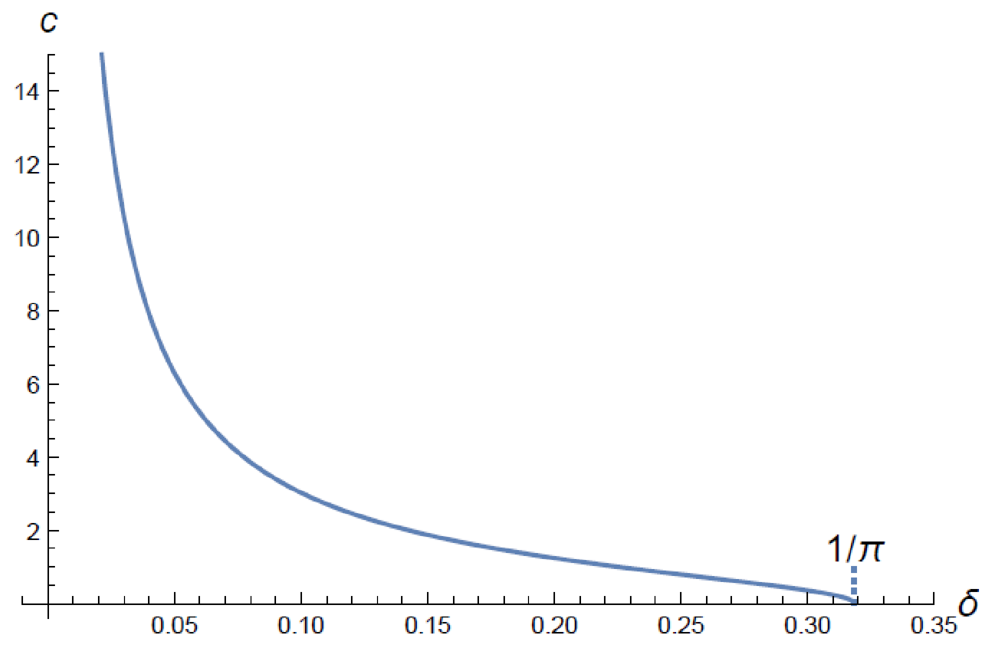
$$W(z) = \frac{1}{\pi \mathrm{i}} \int_{-c}^{c} \frac{f(t)\,\mathrm{d}t}{(t - z)\sqrt{c^2 - t^2}}$$

- Comparing (*) with (A) gives a relation between δ and *c.*

- Taking the limit of our integral (*) as *z*->∞ and comparing with (A) gives

$$\delta = \frac{1}{\pi\sqrt{1+c^2}} \qquad \Longrightarrow \qquad c = \sqrt{\frac{1}{\delta^2\pi^2} - 1}.$$

- This tells us how the size of the painted region, 2*c*, depends on the control parameter δ

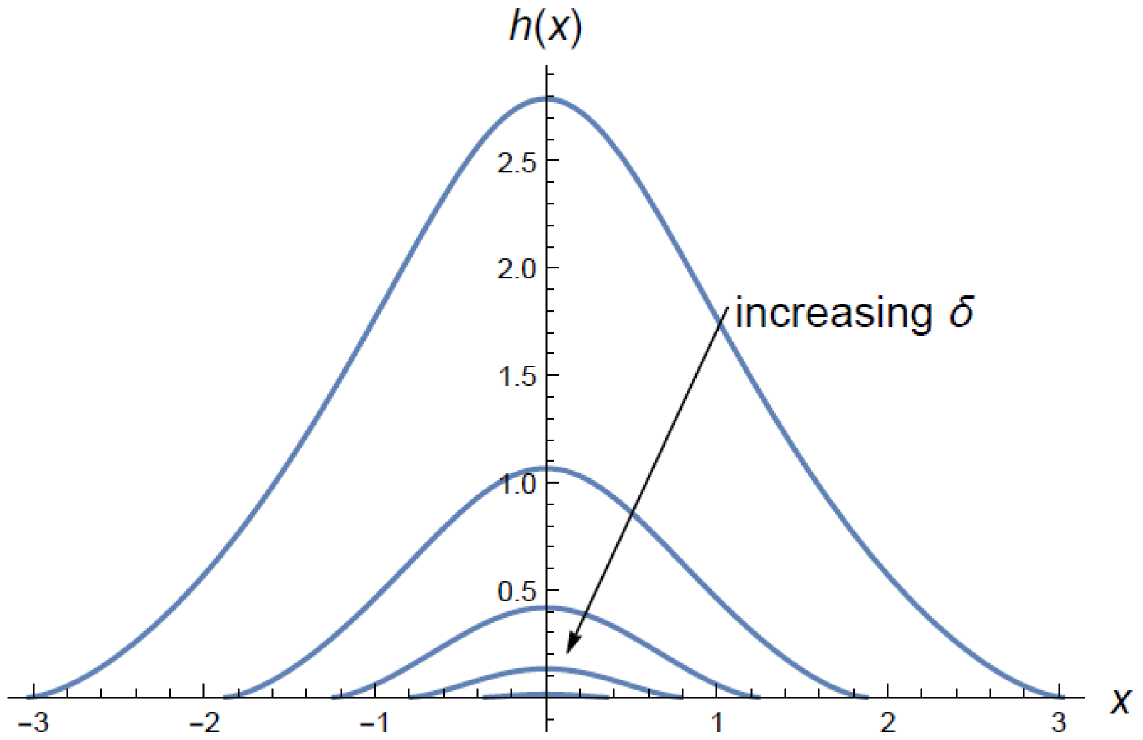


$$\delta = \frac{Ld}{\alpha\sigma V}$$

- Using our result for *W* and the definition of *W*, we can relate back to Φ_0 to get

$$\Phi_0(x) = \frac{1}{\pi}\left[\tanh^{-1}\left(\sqrt{\frac{c^2-x^2}{1+c^2}}\right) - \sqrt{\frac{c^2-x^2}{1+c^2}}\right]$$

- The thickness profile is then given by $h(x) = \Phi_0(x)/\delta$



Useful texts for more reading and examples

1) I. Barenblatt, 1979, *Similarity, Self-similarity and Intermediate Asymptotics*, Consultants Bureau.

2) A.B. Tayler, 2001, *Mathematical Models in Applied Mechanics*. Oxford University Press.

3) S.C. Gupta, 2017, *The Classical Stefan Problem: Basic Concepts, Modelling and Analysis with Quasi-analytical Solutions and Methods*. Elsevier.