3. SEQUENCES AND CONVERGENCE

How do we handle a specific real number in practice? One option is to look at successive approximations. For example, we could have the following approximations for $\sqrt{2}$:

$$1, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \ldots$$

– namely the truncated decimal expansions for $\sqrt{2}$ – or we could approaximate π with the sequence

$$3, \frac{22}{7}, \frac{333}{106}, \frac{103993}{33102}, \ldots$$

which are the 'continued fraction convergents' of π. Our first task is to make precise the idea that these approximations approach the real numbers that they represent.

Definition 3.1 *A **sequence of real numbers** or, more simply, a **real sequence** is a function $a\colon \mathbb{N} \to \mathbb{R}$.*

Definition 3.2 *A **sequence of complex numbers** or, more simply, a **complex sequence** is a function $a\colon \mathbb{N} \to \mathbb{C}$.*

In these definitions we typically take $\mathbb{N}$ to be the set $\{0, 1, 2, \ldots\}$ or $\{1, 2, 3, \ldots\}$.

Definition 3.3 *Given a natural number n, the n**th term** of the sequence a is $a(n)$ and we denote this a_n.*

Example 3.4 *Here are some sequences:*

- $n \mapsto \alpha(n) = (-1)^n$,
- $n \mapsto \beta(n) = 0$,
- $n \mapsto \gamma(n) = n$.

Note, in practice, we often just give the sequence's values, and say 'the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$' if it is clear what the function 'must be'. Or we may be more explicit and write 'the sequence $(a_n)_{n=1}^{\infty}$' or 'the sequence (a_n)' where a_n is a formula in n.

Note also that although n determines the nth value of a sequence, the nth value does not determine n because the defining function need not be injective. Consider the sequences α and β above for example.

The space of real (or complex) sequences is naturally a vector space; in fact it is naturally an algebra where elements can be multiplied. Suppose that (a_n) and (b_n) are sequences of real (or complex) numbers and $c \in \mathbb{R}$ (or $\mathbb{C}$). We define the sequences

$$(a_n + b_n), \qquad (ca_n), \qquad (a_n b_n), \qquad (a_n/b_n)$$

in the obvious, termwise way. All are well defined except possibly the quotient, where we must insist on all the terms of (b_n) being non-zero.

Example 3.5 $a_n = (-1)^n$ *and* $b_n = 1$ *for all* $n \geqslant 0$.

$$(a_n + b_n) = (0, 2, 0, 2, 0, 2, 0, \ldots); \qquad (-a_n) = (-1)^{n+1};$$
$$(a_n b_n) = (a_n); \qquad (a_n^2) = (b_n).$$

3.1 Convergence

Definition 3.6 *Let* (a_n) *be a real sequence and* $L \in \mathbb{R}$. *We say that* (a_n) ***converges to*** L *if*

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geqslant N \quad |a_n - L| < \varepsilon.$$

We also say that (a_n) ***tends*** *to* L. *We write this as*

$$(a_n) \to L \qquad \text{or} \qquad a_n \to L \quad \text{as} \quad n \to \infty \qquad \text{or just} \qquad a_n \to L.$$

Note than N *can, and typically will, depend on* ε. *The smaller* ε *is, the larger* N *will typically need to be.*

Definition 3.7 *If* $(a_n) \to L$ *then we say that* L *is a* ***limit*** *of* (a_n) *and we write*

$$L = \lim_{n\to\infty} a_n \qquad \text{or just} \qquad L = \lim a_n.$$

Definition 3.8 *We say that* (a_n) ***converges*** *or is* ***convergent*** *if there exists* $L \in \mathbb{R}$ *such that* $(a_n) \to L$. *In full*

$$(a_n) \text{ converges} \quad \iff \quad \exists L \in \mathbb{R} \quad \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geqslant N \quad |a_n - L| < \varepsilon.$$

Definition 3.9 *We say that* (a_n) ***diverges*** *or is* ***divergent*** *if it does not converge. In full*

$$(a_n) \text{ diverges} \quad \iff \quad \forall L \in \mathbb{R} \quad \exists \varepsilon > 0 \quad \forall N \in \mathbb{N} \quad \exists n \geqslant N \quad |a_n - L| \geqslant \varepsilon.$$

Remark 3.10 *In the above,* ε *is an arbitrary positive number though instinctively we usually think of* ε *as being very small. The smaller the value of* ε *the further into the sequence we will usually have to look to find a value of* N *that will suffice.*

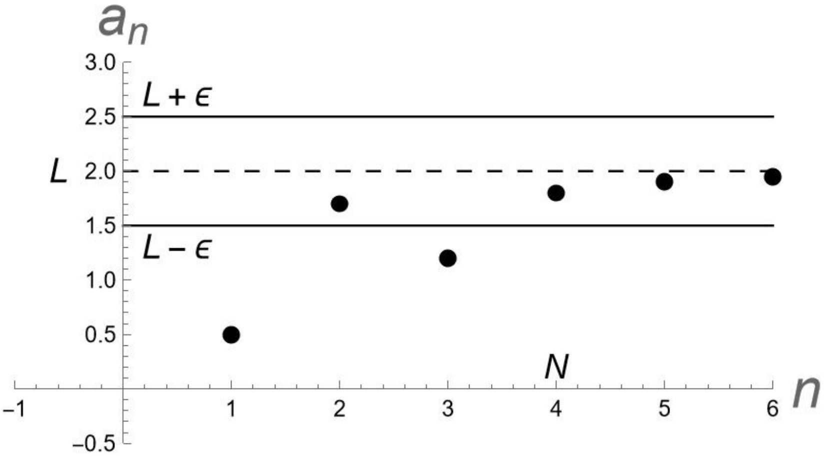

Fig. 3.1 – Graphing a Real Sequence

In Figure 3.1 we have $L = 2$ and $\varepsilon = 0.5$. Note that a_2 lies in the range $(L - \varepsilon, L + \varepsilon)$ though 2 cannot act as N here as a_3 is not in the required range. It seems, from the figure, that $N = 4$ would suffice as each of $x_4, x_5, x_6, \ldots$ appears to lie in $(L - \varepsilon, L + \varepsilon)$. In fact any $N \geqslant 4$ would be satisfactory, it doesn't have to be first such N. For ε much smaller than 0.5 then the larger N will need to be.

Looking then at the definition of $a_n \to L$, we need to find some N, not necessarily the smallest, such that $a_N, a_{N+1}, a_{N+2}, \ldots$ lies in $(L - \varepsilon, L + \varepsilon)$ and we need to be able to do this for all $\varepsilon > 0$.

The definition of '(a_n) converges' makes no specific mention of the limit L, and so to demonstrate this the first task is to determine the candidate limit L and then to show $a_n \to L$.

Remark 3.11 *We also note from the above that showing*

$$\exists L \in \mathbb{R} \quad \forall \varepsilon \in (0,1) \quad \exists N \in \mathbb{N} \quad \forall n \geqslant N \quad |a_n - L| < \varepsilon$$

is sufficient to show convergence. That is, WLOG, we can assume $0 < \varepsilon < 1$. Previously we had to concern ourselves with, say, finding the sequence eventually in $(L - 2, L + 2)$. But as we can find the sequence eventually in $(L - 0.5, L + 0.5)$ then the sequence is eventually in $(L - 2, L + 2)$ as well.

And there's nothing special about assuming $\varepsilon < 1$ here. If it suited us we could assume $0 < \varepsilon < \varepsilon_0$ for any positive ε_0.

Definition 3.12 ***(Tails and Neighbourhoods)*** *Let (a_n) be a sequence, and let k be a natural number. Then the **kth tail** of (a_n) is the sequence $n \mapsto a_{n+k}$ i.e. it equals the sequence*

$$(a_k, a_{k+1}, a_{k+2}, a_{k+3}, \ldots)$$

which we will also write as $(a_{n+k})_{n=0}^{\infty}$ or $(a_n)_k^{\infty}$.

*For $L \in \mathbb{R}$ and $\varepsilon > 0$, we refer to the set $(L - \varepsilon, L + \varepsilon)$ as a **neighbourhood** of L (or sometimes a **basic neighbourhood** of L).*

So we can rephrase '(a_n) converges to L' as 'any neighbourhood of L contains a tail of (a_n).'

In practice, we will not be interested in a specific kth tail so much as in some (unspecified) tail or all tails past a certain point in the sequence. The tails give a way of focusing on the *long-term behaviour* of a sequence ignoring any short-term aberrant behaviour at the start of a sequence. Whether or not a sequence converges purely depends on the long-term behaviour of the sequence as we see in the next proposition.

Proposition 3.13 *Let (a_n) be a real sequence and let $L \in \mathbb{R}$. Then the following three statements are equivalent.*

(a) (a_n) converges (to L);

(b) some tail of (a_n) converges (to L);

(c) all tails of (a_n) converge (to L).

Proof. We shall demonstrate the implications as (a) implies (c), (c) implies (b) and (b) implies (a).

(a) $\implies$ (c): Suppose that (a_n) converges to L and let $k \in \mathbb{N}$, $\varepsilon > 0$. As $(a_n) \to L$ then there exists N such that

$$\forall n \geqslant N \quad |a_n - L| < \varepsilon.$$

For such n, we have $n + k \geqslant n \geqslant N$ and so

$$\forall n \geqslant N \quad |a_{n+k} - L| < \varepsilon.$$

Hence, for any $k \in \mathbb{N}$, the kth tail of (a_n) converges to L.

(c) $\implies$ (b): (c) clearly implies (b).

(b) $\implies$ (a) : Suppose that the kth tail of (a_n) converges to L. Let $\varepsilon > 0$. Then there exists N such that

$$\forall n \geqslant N \quad |a_{n+k} - L| < \varepsilon.$$

Hence

$$\forall n \geqslant N + k \quad |a_n - L| < \varepsilon$$

and we see that (a_n) converges to L. ■

Remark 3.14 *(Intersection of tails) We will often find ourselves in a situation where we know something is true of a sequence (a_n) for $n \geqslant N_1$ and a second thing is true for $n \geqslant N_2$. Note that both facts will apply for the tails' intersection, which is when $n \geqslant \max(N_1, N_2)$, which is itself a tail of the sequence.*

This argument can be applied finitely many times, but only finitely many. The intersection of infinitely many tails can be empty – e.g. when $N_k = k^2$ say.

Before giving some examples, we show that a limit, if it exists, is unique. So we are justified in the use of the language 'the limit'.

Theorem 3.15 *(**Uniqueness of Limits**) Let (a_n) be a real sequence and suppose that $a_n \to L_1$ and $a_n \to L_2$ as $n \to \infty$. Then $L_1 = L_2$.*

Thoughts: Proofs of uniqueness usually begin by assuming non-uniqueness and obtaining a contradiction or assuming there are two such elements and showing they're equal. Our proof is by contractions. If there were two limits, $L_1 \neq L_2$, then the would be tails of the sequence in a neighbourhood of each. Provided these neighbourhoods are disjoint, there is nowhere for the tails' intersection to be.

Proof. Suppose not and set $\varepsilon = |L_1 - L_2| > 0$. Then $\varepsilon/2 > 0$ and there exists N_1 such that

$$n \geqslant N_1 \quad \implies \quad |a_n - L_1| < \varepsilon/2$$

Likewise there exists N_2 such that

$$n \geqslant N_2 \quad \implies \quad |a_n - L_2| < \varepsilon/2.$$

Then for $n \geqslant \max(N_1, N_2)$ both inequalities hold and

$$\begin{aligned} |L_1 - L_2| &= |(L_1 - a_n) + (a_n - L_2)| \\ &\leqslant |L_1 - a_n| + |a_n - L_2| \qquad \text{by the triangle law} \\ &< \varepsilon/2 + \varepsilon/2 \\ &= |L_1 - L_2| \end{aligned}$$

which is the required contradiction. ■

Example 3.16 *Let*

$$a_n = \frac{2^n - 1}{2^n} \qquad \text{for } n \geqslant 1.$$

Then $(a_n) \to 1$.

Thoughts: Here the limit is given, namely $L = 1$, so we don't have to put any thought into deciding what the limit is (as in the next example). The statement of $(a_n) \to 1$ is

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geqslant N \quad |a_n - L| < \varepsilon.$$

To address the first quantifier all we need do is introduce a positive ε. Given this ε, our task is to find a suitable N. In the example below we include the necessary 'back of the envelop' calculation as part of the proof.

Solution. Note

$$|a_n - 1| = \left|1 - 2^{-n} - 1\right| = 2^{-n}.$$

Let $\varepsilon > 0$. We need to find N such that

$$n \geqslant N \quad \Longrightarrow \quad 2^{-n} < \varepsilon.$$

But note that $2^n > n$ for all $n \in \mathbb{N}$ and so if $N > 1/\varepsilon$ (which we know to exist by the Archimedean Property) and $n \geqslant N$ we have

$$|a_n - 1| = 2^{-n} = \frac{1}{2^n} < \frac{1}{n} \leqslant \frac{1}{N} < \varepsilon.$$

■

Example 3.17 *The sequence*

$$a_n = \frac{n^2 + n + 1}{3n^2 + 4} \qquad (n \geqslant 1)$$

is convergent.

Thoughts: Because the statement for convergence is

$$\exists L \in \mathbb{R} \quad \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geqslant N \quad |a_n - L| < \varepsilon$$

our first work is in deciding what the limit L is. Note the limit was given to us in the previous example. Our 'back of the envelop' argument might go: l for large positive n,

$$a_n = \frac{n^2 + n + 1}{3n^2 + 4} = \frac{1 + \frac{1}{n} + \frac{1}{n^2}}{3 + \frac{4}{n^2}} \approx \frac{1}{3}$$

We give no exact definition of $\approx$ (approximately equal to) but none of the above is part of our, rather informal first thoughts. So $\frac{1}{3}$ seems the obvious candidate for our limit. To begin the proof then, looking at the quantifiers we need to address:

Solution. Let $\varepsilon > 0$. Note

$$\begin{aligned}\left|a_n - \frac{1}{3}\right| &= \left|\frac{n^2+n+1}{3n^2+4} - \frac{1}{3}\right| \\ &= \frac{3n-1}{3(3n^2+4)} \\ &\leqslant \frac{3n}{3(3n^2+4)} \\ &\leqslant \frac{3n}{3\times 3n^2} \\ &< \frac{1}{n}.\end{aligned}$$

By the Archimedean Property, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$. Then, for any $n \geqslant N$, we have

$$\left|a_n - \frac{1}{3}\right| < \frac{1}{n} \leqslant \frac{1}{N} < \varepsilon$$

to complete the proof. ■

Example 3.18 *Let*

$$a_n = \frac{(-1)^n n^2}{n^2+1} \qquad (n \geqslant 1)\,.$$

Then (a_n) *diverges.*

Thoughts: The quantified definition of divergence is

$$\forall L \in \mathbb{R} \quad \exists \varepsilon > 0 \quad \forall N \in \mathbb{N} \quad \exists n \geqslant N \quad |a_n - L| \geqslant \varepsilon,$$

so we need to show that any limit real L cannot be a limit.

Looking at the sequence we can see that for large even n

$$a_n = \frac{(-1)^n n^2}{n^2+1} = \frac{1}{1+n^{-2}} \approx 1,$$

whilst for large odd n we have

$$a_n = \frac{(-1)^n n^2}{n^2+1} = \frac{-1}{1+n^{-2}} \approx -1.$$

A natural way forward seems to be to suppose, for a contradiction, that a limit exists and argue (carefully!) that this limit would need to be both near 1 and -1; this would be the required contradiction. In fact, if we look in more detail at the sequence we see that $a_{2n} \geqslant \frac{4}{5}$ for all n and $a_{2n-1} \leqslant -\frac{1}{2}$, so we will take ε in such a way that $2\varepsilon < \frac{4}{5} + \frac{1}{2} = \frac{13}{10}$ which is the closest the even and odd terms get. Our proof thus begins:

Solution. Suppose, for a contradiction, that a limit L exists and set $\varepsilon = \frac{1}{2}$. Then there exists N such that for $n \geqslant N$

$$\left|\frac{(-1)^n n^2}{n^2+1} - L\right| < \frac{1}{2}.$$

In particular, for even $n \geqslant N$ we have

$$\frac{n^2}{n^2+1} - L < \frac{1}{2},$$
$$\implies L > \frac{1}{1+n^{-2}} - \frac{1}{2} \geqslant \frac{1}{5/4} - \frac{1}{2} = \frac{3}{10}.$$

Similarly, for odd $n \geqslant N$ we have

$$L + \frac{n^2}{n^2+1} < \frac{1}{2},$$
$$\implies L < \frac{1}{2} - \frac{1}{1+n^{-2}} \leqslant \frac{1}{2} - \frac{1}{2} = 0.$$

The necessary inequalities $L > \frac{3}{10}$ and $L < 0$ give us our required contradiction. ■

Corollary 3.19 *Let a be a real number with $a > 1$, and k be a positive integer. Then*

$$\frac{n^k}{a^n} \to 0 \quad \textit{as} \quad n \to \infty.$$

Proof. This is a corollary to Proposition 1.46. There we showed that There is some $c > 0$ such that

$$\frac{a^n}{n^k} \geqslant c$$

for all $n \geqslant 1$. Replacing k with $k+1$ there exists $c > 0$ such that $a^n/n^{k+1} \geqslant c$ for all $n \geqslant 1$; hence

$$0 < \frac{n^k}{a^n} \leqslant \frac{1}{cn}.$$

Given $\varepsilon > 0$, by the Archimedean property there exists N such that $\left|n^k/a^n\right| < \varepsilon$ for all $n \geqslant N$. That is $n^k/a^n \to 0$ as $n \to \infty$. ■

Proposition 3.20 ***(Convergent sequences are bounded)*** *Let (a_n) be a real convergent sequence. Then (a_n) is bounded.*

Thoughts: Pick any neighbourhood of the limit and a tail of the sequence of the sequence will be in that neighbourhood. Only finitely many terms of the sequence aren't in that tail.

Proof. Say that $(a_n) \to L$ and set $\varepsilon = 1$. Then $|a_n - L| < 1$ for some tail $n \geqslant N$ and, in particular, $|a_n| < |L| + 1$ by the triangle inequality. Then $|a_n| < M$ for all n where

$$M = \max\{|a_0|, |a_1|, \ldots, |a_{N-1}|, |L|+1\} + 1.$$

■

3.2 Complex Sequences

Definition 3.21 *Let (z_n) be a sequence of complex numbers and let $w \in \mathbb{C}$. We say that (z_n) converges L and write $(z_n) \to L$ or $\lim z_n = L$ if*

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geqslant N \quad |z_n - L| < \varepsilon.$$

That is $|z_n - L| \to 0$ as $n \to \infty$.

Corollary 3.22 *(**Uniqueness of Limits**) Let (a_n) be a complex sequence and suppose that $a_n \to L_1$ and $a_n \to L_2$ as $n \to \infty$. Then $L_1 = L_2$.*

Proof. The proof of uniqueness is identical to the previous proof for real sequences. ∎

Corollary 3.23 *Let (a_n) be a convergent complex sequence. Then the sequence is bounded.*

Proof. The proof of boundedness is identical to the previous proof for real sequences. ∎

Remark 3.24 *(**Graphing complex sequences**) We can represent the behaviour of complex sequences in $\mathbb{C}$ by plotting the terms in the Argand diagram. In Figure 3.2 below, the sequence's limit is $L = 1+i$ and $\varepsilon = 0.3$. Rather than an open interval $(L - \varepsilon, L + \varepsilon)$, the region $|z - L| < \varepsilon$ is an open disc with centre L and radius ε. That is the (basic) neighbourhoods of L are discs centred at L. Again $z_n \to L$ if every neighbourhood of L contains a tail of (z_n). In the figure it appears that any tail of the sequence from $N \geqslant 5$ is inside the sketched disc.*

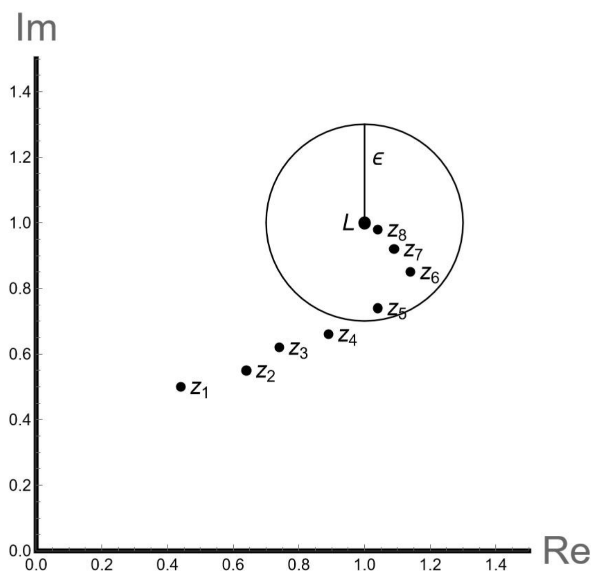

Fig 3.2 – graphing convergence in $\mathbb{C}$

Theorem 3.25 *Let $z_n = x_n + iy_n$. Then (z_n) converges if and only if (x_n) and (y_n) both converge.*

Thoughts: Visually this result is not surprising. For z_n to be within a radius ε disc of $L_1 + iL_2$ means $x_n + iy_n$ is within a square of side 2ε. For a tail of x_n to be within $\varepsilon/2$ of L_1 and a tail of y_n to be within $\varepsilon/2$ of L_2 means the tails' interection of $x_n + iy_n$ is within a quare of side ε which itself is within a radius ε disc of $L_1 + iL_2$.

Proof. Suppose that x_n and y_n both converge and that $\varepsilon > 0$. Set $x = \lim x_n$, $y = \lim y_n$ and $L = x + iy$. By the Triangle Inequality

$$|z_n - L| = |(x_n - x) + i(y_n - y)| \leqslant |x_n - x| + |y_n - y| .$$

As $x_n \to x$ and $y_n \to y$ then we can find N_1 and N_2 such that

$$\begin{aligned} |x_n - x| &< \varepsilon/2 \quad \text{whenever } n \geqslant N_1, \\ |y_n - y| &< \varepsilon/2 \quad \text{whenever } n \geqslant N_2. \end{aligned}$$

So if $n \geqslant \max(N_1, N_2)$ we have $|z_n - L| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ and we see that $z_n \to L$.

Conversely suppose that z_n converges to L and let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $|z_n - L| < \varepsilon$ whenever $n \geqslant N$. As $|\operatorname{Re} w| \leqslant |w|$ and $|\operatorname{Im} w| \leqslant |w|$ for any $w \in \mathbb{C}$ then

$$\begin{aligned} |x_n - x| &= |\operatorname{Re}(z_n - L)| \leqslant |z_n - L| < \varepsilon \quad \text{whenever } n \geqslant N, \\ |y_n - y| &= |\operatorname{Im}(z_n - L)| \leqslant |z_n - L| < \varepsilon \quad \text{whenever } n \geqslant N. \end{aligned}$$

Hence $x_n \to x$ and $y_n \to y$ as required. ■

Example 3.26 *Let*

$$z_n = \left(\frac{1}{1+\imath}\right)^n .$$

Then $z_n \to 0$.

Proof. Let $\varepsilon > 0$. Note

$$|z_n - 0| = \left|\left(\frac{1}{1+\imath}\right)^n\right| = \left|\frac{1}{1+\imath}\right|^n = \frac{1}{|1+\imath|^n} = \left(\frac{1}{\sqrt{2}}\right)^n .$$

We have already shown that $2^{-k} < \varepsilon$ for $k > 1/\varepsilon$ and so $\left(\sqrt{2}\right)^n < \varepsilon$ for $n > 2/\varepsilon$. ■

Remark 3.27 *Note that in the above example Theorem 3.25 is not particularly useful. It is often simpler to work with a complex sequence as such rather than as a sequence made up of its real and complex parts. By De Moivre's Theorem, the real and imaginary parts of z_n are*

$$\begin{aligned} x_n &= \operatorname{Re}\left(\frac{\cos(\pi/4) - i\sin(\pi/4)}{\sqrt{2}}\right)^n = \frac{1}{2^{n/2}} \cos\left(\frac{n\pi}{4}\right); \\ y_n &= \operatorname{Im}\left(\frac{\cos(\pi/4) - i\sin(\pi/4)}{\sqrt{2}}\right)^n = \frac{(-1)^n}{2^{n/2}} \sin\left(\frac{n\pi}{4}\right), \end{aligned}$$

and it only makes for more work to show that both of these tend to 0.

Notation 3.28 ***(Asymptotic notation)*** *Let a_n, b_n be sequences.*

(a) We write $a_n = O(b_n)$ if there exist c such that for some N

$$n \geqslant N \quad \Longrightarrow \quad |a_n| < cb_n.$$

*This is referred to as **big O**.*

(b) We write $a_n = o(b_n)$ if a_n/b_n is defined and

$$\frac{a_n}{b_n} \to 0.$$

*This is referred to as **little o**.*

(c) We write

$$a_n \sim b_n$$

*if $a_n/b_n \to 1$ as $n \to \infty$. We say that a_n and b_n are **asymptotically equal**.*

Example 3.29 *As examples*

- $n = O(n^2)$
- $n = o(n^2)$
- $\sin n = O(1)$
- $\sin n = o(n)$
- $(n+1)^2 = n^2 + O(n)$
- $(n+1)^2 \sim n^2$.

3.3 Infinity

Definition 3.30 ***(Real Infinities)*** *Let a_n be a sequence of real numbers. We say 'a_n tends to infinity' and write $a_n \to \infty$ as $n \to \infty$ to mean*

$$\forall M \in \mathbb{R} \quad \exists N \in \mathbb{N} \quad \forall n \geqslant N \quad a_n > M.$$

Similarly we write $b_n \to -\infty$ if

$$\forall M \in \mathbb{R} \quad \exists N \in \mathbb{N} \quad \forall n \geqslant N \quad b_n < M.$$

(Here we tend to think of M as being a very large positive/negative number.)

Definition 3.31 ***(Complex Infinity)*** *Let z_n be a complex sequence. We say that $z_n \to \infty$ if*

$$\forall M \in \mathbb{R} \quad \exists N \in \mathbb{N} \quad \forall n \geqslant N \quad |z_n| > M.$$

That is $|z_n| \to \infty$ as a real sequence.

- **Note that the real infinities $\pm\infty$ are not real numbers and complex infinity ∞ is not a complex number** and should not be treated as such.
- Certainly you should **never** be writing anything like the following:

$$\lim_{n\to\infty} \frac{n}{n+1} = \frac{\infty}{\infty} = 1, \qquad \text{or} \qquad \lim_{n\to\infty} \frac{n}{n^2} = \frac{\infty}{\infty} = 1.$$

The first limit, by a fluke, is correct and the second is false; if properly argued it would be seen that the second limit exists and equals 0.

Remark 3.32 *(**Indeterminate forms**) If $a_n \to \infty$ and $b_n \to \infty$ then there is nothing that can be said about the long term behaviour of a_n/b_n as seen from the examples above. This can be expressed as '$\frac{\infty}{\infty}$ is an indeterminate form'. The other indeterminate forms are*

$$\frac{\infty}{\infty}, \qquad \frac{0}{0}, \qquad 0\times\infty, \qquad \infty-\infty, \qquad 0^0, \qquad 1^\infty, \qquad \infty^0.$$

It can be useful to talk about '$\frac{\infty}{\infty}$' type limits but this is only informal, shorthand language to describe a family of such sequences. For a specific example, a limit might be found using careful analysis, but there is no single answer for limits of such sequences.

Note that $\infty+\infty$ and $\infty\times\infty$ are not on the list of indeterminates because if $a_n \to \infty$ and $b_n \to \infty$ then $a_n + b_n \to \infty$ and $a_n b_n \to \infty$.

Below is a list of examples to show that the other examples above are indeed indeterminates.

Type	a_n	b_n	*form*	*long term*	*Type*	a_n	b_n	*form*	*long term*
$\frac{0}{0}$	$\frac{1}{n}$	$\frac{1}{n}$	1	$\to 1$	0^0	$\frac{1}{n}$	$\frac{1}{n}$	$1/\sqrt[n]{n}$	$\to 1$
$\frac{0}{0}$	$\frac{1}{n}$	$\frac{(-1)^n}{n}$	$(-1)^n$	*no limit*	0^0	2^{-n}	$\frac{1}{n}$	$1/2$	$\to \frac{1}{2}$
$0\times\infty$	$\frac{1}{n}$	n	1	$\to 1$	1^∞	$1+\frac{1}{n}$	n	$\left(1+\frac{1}{n}\right)^n$	$\to e$
$0\times\infty$	$\frac{1}{n}$	n^2	n	$\to \infty$	1^∞	$1+\frac{1}{n^2}$	n	$\left(1+\frac{1}{n^2}\right)^n$	$\to 1$
$0\times\infty$	$(-2)^{-n}$	2^n	$(-1)^n$	*no limit*	1^∞	$1+\frac{1}{n}$	n^2	$\left(1+\frac{1}{n}\right)^{n^2}$	$\to \infty$
$\infty-\infty$	n	$2n$	$-n$	$\to -\infty$	∞^0	n	$\frac{1}{n}$	$\sqrt[n]{n}$	$\to 1$
$\infty-\infty$	n	$n+\sin n$	$-\sin n$	*no limit*	∞^0	2^n	$\frac{1}{n}$	2	$\to 2$

Remark 3.33 *(**Hyperreals – off-syllabus**) There are ways to formally treat infinities and infinitesimals. One such approach is the **hyperreals** which were first studied by Edwin Hewitt in 1948 and greatly extended by Abraham Robinson in 1966. The use of hyperreals is called 'non-standard analysis'. For more see Sheet 3, Exercises 10 and 11.*

Remark 3.34 *(**Neighbourhoods of Infinity – off-syllabus**) Note that a real sequence (a_n) converges to $L \in \mathbb{R}$ if every $(L-\varepsilon, L+\varepsilon)$ contains a tail of (a_n). The interval $(L-\varepsilon, L+\varepsilon)$ is called a **neighbourhood** of L.*

Now $(a_n) \to \infty$ if every interval (M,∞) contains a tail of (a_n) and we call (M,∞) a neighbourhood of ∞. Note that $a_n \neq \infty$ for all n as (a_n) is a sequence of real numbers.

By comparison, when we have a real sequence (a_n) in the interal $(-\infty, r]$ with $a_n \neq r$ for all n, then $(a_n) \to r$ if every interval (M,r) contains a tail of (a_n).

So when we include ∞ and $-\infty$ to make an 'extended real line' then we essentially make the closed interval $[-\infty,\infty]$.

The situation is rather different with the 'extended complex plane'. There is only one complex infinity which is 'out there' in all directions. A neighbourhood of ∞ *is a set*

$$\{z \in \mathbb{C} \mid |z| > M\}.$$

Effectively we are wrapping up the complex plane with a single point at infinity and, topologically, this creates a sphere, commonly known as the ***Riemann sphere****. There are actually quite detailed connections between the geometry of the sphere and that of the extended complex plane, which can be made explicit via* ***stereographic projection****.*

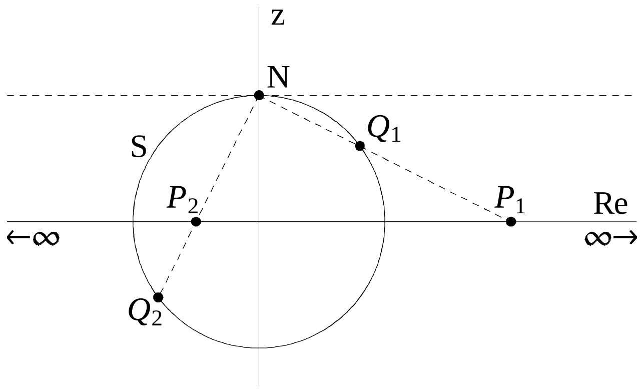

Fig. 3.3 – Stereographic Projection

Let S denote the unit sphere in $\mathbb{R}^3$. Thinking of $\mathbb{C}$ as the xy-plane, every complex number $P = X + YI$ can be identified with a point Q on S by drawing a line from $(X, Y, 0)$ in the xy-plane to the sphere's north pole $N = (0, 0, 1)$; this line intersects the sphere at two points Q and N. We define a map π from the sphere $S\backslash\{N\}$ to $\mathbb{C}$ by setting $\pi(Q)) = P$. The points Q that are near N are mapped to P with large moduli, so it makes sense to extend π by setting $\pi(N) = \infty$ and then we have a bijection from S to $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ which is called stereographic projection.

Specifically this defines

$$\pi(x, y, z) = \begin{cases} \frac{x+yi}{1-z} & z \neq 1 \\ \infty & z = 1 \end{cases}$$

with inverse

$$\pi^{-1}(X + iY) = \left(\frac{2X}{1 + X^2 + Y^2}, \frac{2Y}{1 + X^2 + Y^2}, \frac{X^2 + Y^2 - 1}{1 + X^2 + Y^2}\right)$$

But π is much more than a simple bijection. It has important geometric properties.

- *Under stereographic projection, circles on S which pass through N correspond to lines in $\mathbb{C}$, and circles on S which don't pass through N correspond to circles in $\mathbb{C}$.*

- *The map π is conformal, meaning it is angle-preserving.*

- *The Möbius transformations $z \to (az+b)/(cz+d)$, where $ad \neq bc$, are bijections of $\mathbb{C}_\infty$, which correspond to the conformal bijections of the sphere.*

Returning now to real and complex sequences:

Proposition 3.35 *(a) Let (a_n) be a sequence of positive real numbers. The following are equivalent:*
(i) $a_n \to \infty$ as $n \to \infty$;
(ii) $1/a_n \to 0$ as $n \to \infty$.
The equivalence fails if the (a_n) are simply non-zero.
(b) Let (a_n) be a sequence of non-zero complex numbers. The following are equivalent:
(i) $a_n \to \infty$ as $n \to \infty$;
(ii) $1/a_n \to 0$ as $n \to \infty$.

Proof. (a): (i) $\implies$ (ii) Let $\varepsilon > 0$ and set $M = 1/\varepsilon$. As $a_n \to \infty$ then there exists N such that $a_n > M$ for all $n \geqslant N$. But then $0 < 1/a_n < \varepsilon$ for all $n \geqslant N$ and $1/a_n \to 0$.

(a): (ii) $\implies$ (i): Let $M > 0$ and $\varepsilon = 1/M$. As $1/a_n \to 0$ then there exists N such that $1/a_n < \varepsilon$ for all $n \geqslant N$. But then $a_n > 1/\varepsilon = M$ for all $n \geqslant N$ and $a_n \to \infty$.

(a): If we set $a_n = (-1)^n n$ then $1/a_n = (-1)^n /n \to 0$ but $a_n \nrightarrow \infty$ as $a_{2n} \to \infty$ yet $a_{2n+1} \to -\infty$.

(b) : (i) $\implies$ (ii) Let $\varepsilon > 0$ and set $M = 1/\varepsilon$. As $a_n \to \infty$ then there exists N such that $|a_n| > M$ for all $n \geqslant N$. But then $|1/a_n| < \varepsilon$ for all $n \geqslant N$ and $1/a_n \to 0$.

(b): (ii) $\implies$ (i): Let $M > 0$ and $\varepsilon = 1/M$. As $1/a_n \to 0$ then there exists N such that $|1/a_n| < \varepsilon$ for all $n \geqslant N$. But then $|a_n| > 1/\varepsilon = M$ for all $n \geqslant N$ and $a_n \to \infty$. ■

Example 3.36 *Let (a_n) be a real sequence such that $a_n \to \infty$ as $n \to \infty$. Prove, or disprove with a counter-example, each of the following statements.*
(a) If (b_n) is a bounded, non-zero sequence then $a_n/b_n \to \infty$.
(b) If (b_n) is a bounded, positive sequence then $a_n/b_n \to \infty$.
(c) If b_n is a non-zero sequence which converges to $L > 0$ then $a_n/b_n \to \infty$.

Solution. (a) False. We can see this by taking $a_n = n$ and $b_n = (-1)^n$. Then $a_n/b_n = (-1)^n n$ does not tend to ∞. [Note that part (a) is a trivial consequence of (b) and so it would have made for an odd question if (a) had been true.]

(b) True. As b_n is bounded then there exists $K > 0$ such that $0 < b_n < K$ for all n. Let $M \in \mathbb{R}$. As $a_n \to \infty$ then there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$ we have $a_n > MK$. So for all $n \geqslant N$ we have $a_n/b_n > (MK)/K = M$ and we see $a_n/b_n \to \infty$.

(c) True. Taking $\varepsilon = L/2 > 0$ we see that there exists N with $|b_n - L| < L/2$ for all $n \geqslant N$. In particular, $0 < L/2 < b_n < 3L/2$ for $n \geqslant N$. By the previous part, the tail of $(a_n/b_n)_N^\infty$ tends to ∞ and hence so does the whole sequence (a_n/b_n). ■

Example 3.37 *Let (a_n) be a real sequence.*
(a) If $a_n \to \infty$ as a real sequence, need $a_n \to \infty$ as a complex sequence?
(b) If $a_n \to \infty$ as a complex sequence, need $a_n \to \infty$ as a real sequence?

Solution. (a) *True: As* $a_n \to \infty$ then $|a_n| \to \infty$ which is equivalent to $a_n \to \infty$ as a complex sequence.

(b) *False*: A counter-example is $a_n = (-1)^n n$. ■

Example 3.38 ***(Harmonic numbers)*** *The* n*th harmonic number is*

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

where $n \geqslant 1$. *Show that* $H_n \to \infty$ *as* $n \to \infty$.

Solution. Note that

$$\begin{aligned} H_{2^k} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{k-1}} + \cdots + \frac{1}{2^k}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^k} + \cdots + \frac{1}{2^k}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} \\ &= 1 + \frac{k}{2}. \end{aligned}$$

Given $M > 0$ there is a positive integer k such that $H_{2^k} > 1 + k/2 \geqslant M$. Hence $H_n > M$ for all $n \geqslant 2^k$ as H_n is increasing. ■