B2.2 Commutative Algebra

Sheet 5 — HT26

Section 11 (for the interested readers)

Section C

1. Let A, B be integral domains and suppose that $A \subseteq B$. Suppose that A is integrally closed and that B is integral over A. Let

   $$\mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_n$$

   be a descending chain of prime ideals in A. Let $k \in \{0, \ldots, n-1\}$ and let

   $$\mathfrak{q}_0 \supset \mathfrak{q}_1 \supset \cdots \supset \mathfrak{q}_k$$

   be a descending chain of prime ideals in B, such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for all $i \in \{0, \ldots, k\}$. Then there is a descending chain of prime ideals

   $$\mathfrak{q}_k \supset \mathfrak{q}_{k+1} \supset \cdots \supset \mathfrak{q}_n$$

   such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for all $i \in \{k+1, \ldots, n\}$. This is the "Going-down Theorem", see AT, Th. 5.16, p. 64.

   Let L (resp. K) be the fraction field of B (resp. A). Prove the Going-down Theorem when L is a (finite) Galois extension of K.

2. Let R be a Dedekind domain. Let I be a non zero ideal in R. Show that every ideal in R/I is principal. Deduce that every ideal in a Dedekind domain can be generated by two elements.

3. Let R be a PID. Suppose that $2 = 1 + 1$ is a unit in R. Let $c_1, \ldots, c_t \in R$ be distinct irreducible elements with $t \geqslant 1$, and let $c = c_1 \cdots c_t$. Show that the ring $R[x]/(x^2 - c)$ is a Dedekind domain. Use this to show that $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$ is a Dedekind domain.

4. Let R be a Dedekind domain. Show that R is a PID if and only if it is a UFD.

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