# 4. THE ALGEBRA OF LIMITS

Reassuringly limits respect important relations and algebraic operations that mean we can don't need to go back to first principle definitions of convergence and divergence to analyze more complicated sequences.

**Theorem 4.1** ***(Limits respect weak inequalities)*** *Let* $(a_n)$ *and* $(b_n)$ *be real sequences such that* $(a_n) \to L$ *and* $(b_n) \to M$. *If* $a_n \leqslant b_n$ *for all* $n$, *then* $L \leqslant M$.

**Thoughts:** A proof by contradiction. If $L > M$ then there would be a tail of the $a_n$ in a neighbourhood of $L$ and a tail of the $b_n$ near $M$. If these neighbourhoods are small enough to be disjoint, then $a_n > b_n$ in the tails' intersection. Note $\varepsilon$ is chosen in the proof below so that $(L-\varepsilon, L+\varepsilon)$ is disjoint from and to the right of $(M-\varepsilon, M+\varepsilon)$.

**Proof.** Suppose, for a contradiction, that $L > M$. Set $\varepsilon = (L-M)/2 > 0$.

$$\begin{aligned} &\text{As } a_n \to L \text{ then there exists } N_1 \text{ such that } n \geqslant N_1 \implies |a_n - L| < \varepsilon; \\ &\text{as } b_n \to M \text{ then there exists } N_2 \text{ such that } n \geqslant N_2 \implies |b_n - M| < \varepsilon. \end{aligned}$$

So

$$\begin{aligned} n \geqslant N_1 &\implies \frac{L+M}{2} = L - \varepsilon < a_n \\ n \geqslant N_2 &\implies b_n < M + \varepsilon = \frac{L+M}{2}. \end{aligned}$$

Hence for $n \geqslant \max(N_1, N_2)$ we have

$$a_n > \frac{L+M}{2} > b_n$$

which contradicts $a_n \leqslant b_n$ for all $n$. ∎

**Remark 4.2** *Note* lim *does not respect strict inequalities: e.g.* $\frac{1}{n} > 0$ *for all* $n \geqslant 1$ *but* $0 = \lim \frac{1}{n} > \lim 0 = 0$ *is false.*

*Note in the above proof that* $n \geqslant \max(N_1, N_2)$ *is the intersection of both tails, so both inequalities hold there.*

A second important result that helps us ignore or bound unimportant expressions in a sequence is the following. This result is also referred to as the 'squeeze theorem'.

**Theorem 4.3** ***(Sandwich Rule)*** *Suppose that* $x_n \leqslant a_n \leqslant y_n$ *for all* $n$ *and that*

$$L = \lim x_n = \lim y_n.$$

*Then* $a_n \to L$ *as* $n \to \infty$.

**Thoughts:** Given any neighbourhood of $L$, there will be tails of $(x_n)$ and $(y_n)$ in that neighbourhood. These tails bound a tail of $(a_n)$.

**Proof.** Let $\varepsilon > 0$. Then there exist $N_1$ and $N_2$ such that

$$\begin{aligned} x_n - L &> -\varepsilon \quad \text{for all } n \geqslant N_1, \\ y_n - L &< \varepsilon \quad \text{for all } n \geqslant N_2. \end{aligned}$$

So for $n \geqslant \max(N_1, N_2)$ we have

$$-\varepsilon < x_n - L \leqslant a_n - L \leqslant y_n - L < \varepsilon,$$

which shows that $a_n \to L$ also. ■

**Example 4.4** *Show that the sequence*

$$a_n = \frac{2n + \cos(n^2)}{3n^2 - \sin(n^3)}$$

*converges.*

**Solution.** We note for all $n \geqslant 1$,

$$\frac{1}{3n} = \frac{n}{3n^2} \leqslant \frac{2n-1}{3n^2+1} \leqslant a_n = \frac{2n + \cos(n^2)}{3n^2 - \sin(n^3)} \leqslant \frac{2n+1}{3n^2-1} \leqslant \frac{3n}{2n^2} = \frac{3}{2n}.$$

As the LHS and RHS both tend to 0 then $a_n \to 0$ by sandwiching. ■

Most sequences can be built up from simpler ones using addition, multiplication, etc. The algebra of limits (AOL) tells us how the corresponding limits behave. Throughout the following $(a_n)$ and $(b_n)$ denote real or complex sequences.

**Proposition 4.5** ***(AOL: Constants)*** *If $a_n = a$ for all $n$, then $a_n \to a$.*

**Proof.** For any $\varepsilon > 0$, take $N = 1$; $n \geqslant N \Longrightarrow |a_n - a| = 0 < \varepsilon$. ■

**Proposition 4.6** ***(AOL: Sums)*** *If $a_n \to a$ and $b_n \to b$ then $a_n + b_n \to a + b$.*

**Thoughts**: We need to show that $|(a_n + b_n) - (a + b)|$ is eventually small given that $|a_n - a|$ and $|b_n - b|$ are each eventually small. The triangle inequality helps here by noting

$$|(a_n + b_n) - (a + b)| \leqslant |a_n - a| + |b_n - b|$$

In the following proof we use two standard techniques of analysis. We know two facts which hold in two tails of a sequence, so we take the tails' intersection where both are true – we've employed this idea before. The second issue is that we need a final inequality to hold within a margin of $\varepsilon$. But the final inequality relies on two previous inequalities. The idea is to achieve each of the first two inequalities with margins of $\varepsilon/2$ and then the triangle inequality, within the tails' intersection, shows the final inequality holds with a margin of $\varepsilon/2 + \varepsilon/2 = \varepsilon$.

**Proof.** Let $\varepsilon > 0$. Then $\varepsilon/2 > 0$ and so

$$\begin{array}{lcccl} \exists N_1 & n & \geqslant & N_1 & \Longrightarrow & |a_n - a| < \varepsilon/2, \\ \exists N_2 & n & \geqslant & N_2 & \Longrightarrow & |b_n - b| < \varepsilon/2. \end{array}$$

Put $N_3 = \max(N_1, N_2)$. Then

$$\begin{array}{rcccccl} n \geqslant N_3 \Longrightarrow & & |(a_n + b_n) & - & (a+b)| & & \\ & \leqslant & |a_n - a| & + & |b_n - b| & & \text{by the } \Delta \text{ law} \\ & < & \varepsilon/2 & + & \varepsilon/2 & & \\ & = & \varepsilon & & & & \end{array}$$

■

**Proposition 4.7** ***(AOL: Scalar Products)*** *If $a_n \to a$ as $n \to \infty$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) then $\lambda a_n \to \lambda a$.*

**Proof.** Let $\varepsilon > 0$. Then $\varepsilon/(|\lambda| + 1) > 0$ and so there exists $N$ such that $|a_n - a| < \varepsilon/(|\lambda| + 1)$ for all $n \geqslant N$. Hence

$$|\lambda a_n - \lambda a| = |\lambda| |a_n - a| \leqslant \frac{|\lambda| \varepsilon}{|\lambda| + 1} < \varepsilon$$

for all $n \geqslant N$. (Note that we use $\varepsilon/(|\lambda| + 1)$ rather than $\varepsilon/|\lambda|$ to avoid the possibility of dividing by zero.) ■

**Corollary 4.8** ***(AOL: Differences)*** *If $a_n \to a$ and $b_n \to b$ then $a_n - b_n \to a - b$.*

**Corollary 4.9** ***(AOL: Translations)*** *If $a_n \to a$ and $c \in \mathbb{R}$ (or $\mathbb{C}$) then $a_n + c \to a + c$.*

**Lemma 4.10** *If $x_n \to 0$ and $y_n \to 0$ then $x_n y_n \to 0$.*

**Proof.** Let $\varepsilon > 0$. By Remark 3.11, WLOG we can further assume that $\varepsilon < 1$. Then

$$\begin{array}{lcccl} \exists N_1 & n & \geqslant & N_1 & \Longrightarrow & |x_n| < \varepsilon_1, \\ \exists N_2 & n & \geqslant & N_2 & \Longrightarrow & |y_n| < \varepsilon_1. \end{array}$$

So if $n \geqslant \max(N_1, N_2)$ we have

$$|x_n y_n| \leqslant |x_n| |y_n| < \varepsilon^2 < \varepsilon,$$

which completes the proof. ■

**Proposition 4.11** ***(AOL: Products)*** *If $a_n \to a$ and $b_n \to b$ then $a_n b_n \to ab$.*

**Proof.** Note that

$$a_n b_n - ab = (a_n - a)(b_n - b) + b(a_n - a) + a(b_n - b),$$

that $(a_n - a)(b_n - b) \to 0$ by the previous lemma, that $b(a_n - a) \to 0$ and $a(b_n - b) \to 0$ by Proposition 4.7. Hence $a_n b_n \to ab$ by Proposition 4.6. ■

**Proposition 4.12** *(**AOL: Reciprocals**) If $a_n \to a \neq 0$ and $a_n \neq 0$ for all $n$, then $1/a_n \to 1/a$.*

**Thoughts**: Our aim is to show

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| = \frac{|a_n - a|}{|a_n||a|}$$

is arbitrary small in a tail, and we know $|a_n - a|$ is small. The $|a|$ in the denominator is non-zero and constant and so is not problematic. At first glance though, whilst $|a_n|$ is non-zero it might be arbitrarily small, which would be problematic. But remembering $a_n \to a \neq 0$ then we can focus on a tail of $a_n$ suitably close to $a$. If $a_n$ is within $|a|/2$ of $a$, then $a_n$ will be at least $|a|/2$ away from zero.

**Proof.** Let $\varepsilon > 0$. As $a \neq 0$ then $|a|/2 > 0$. So there exists $N_1$ such that for $n \geqslant N_1$ we have $|a_n - a| < |a|/2$. By the triangle inequality

$$|a| \leqslant |a_n| + |a - a_n| = |a_n| + |a_n - a|$$

and so $|a_n| > |a|/2$ and $|1/a_n| < 2/|a|$.

Further, as $|a|^2 \varepsilon/2 > 0$ then there exists $N_2$ such that for $n \geqslant N_2$

$$|a_n - a| < |a|^2 \frac{\varepsilon}{2}.$$

For $n \geqslant \max(N_1, N_2)$ we have

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| = \frac{|a_n - a|}{|a_n||a|} < \left(|a|^2 \frac{\varepsilon}{2}\right) \frac{2}{|a|} \frac{1}{|a|} = \varepsilon.$$

■

**Corollary 4.13** *(**AOL: Quotients**) If $a_n \to a$, $b_n \to b$, and $b_n \neq 0$ for all $n$ and $b \neq 0$, then $a_n/b_n \to a/b$.*

**Proof.** This follows from Propositions 4.11 and 4.12. ■

**Proposition 4.14** *(**AOL: Modulus**) If $a_n \to a$ then $|a_n| \to |a|$.*

**Proof.** By the reverse triangle inequality

$$0 \leqslant ||a_n| - |a|| \leqslant |a_n - a| \to 0$$

So $||a_n| - |a|| \to 0$ by sandwiching. ■

**Example 4.15** *Show*

$$a_n = \frac{n^2 + n + 1}{3n^2 + 4} \to \frac{1}{3}.$$

**Solution.** We write

$$\frac{n^2 + n + 1}{3n^2 + 4} = \frac{1 + \frac{1}{n} + \frac{1}{n^2}}{3 + 4\frac{1}{n^2}} \to \frac{1 + 0 + 0}{3 + 0} = \frac{1}{3}$$

by the algebra of limits, specifically noting

- $\frac{1}{n} \to 0$ by the Archimedean property;
- $\frac{1}{n^2} \to 0$ by Proposition 4.11;
- $1 \to 1$ by Proposition 4.5;
- $1 + \frac{1}{n} + \frac{1}{n^2} \to 1$ by Proposition 4.6;
- $3 + \frac{4}{n^2} \to 3$ by Proposition 4.6;
- $\frac{1}{3+\frac{4}{n^2}} \to \frac{1}{3}$ by Corollary 4.13;
- $a_n \to \frac{1}{3}$ by Proposition 4.11.

■

**Example 4.16** ***(Fibonacci numbers)*** *Suppose $F_1 = 1$, $F_2 = 1$, and we recuresively define*

$$F_{n+2} = F_{n+1} + F_n, \quad \text{for } n \geqslant 1.$$

*It is easy to prove by induction on $n$ that there is then a unique sequence of natural numbers satisfying these requirements. They are called the Fibonacci numbers.*

**Proposition 4.17** *$F_{n+1}/F_n$ is convergent.*

**Proof.** By induction, $F_n \geqslant 1$ for all $n$. So for $n \geqslant 1$

$$\left(\frac{F_{n+2}}{F_{n+1}}\right) = 1 + \left(\frac{F_{n+1}}{F_n}\right)^{-1}.$$

Write $x_n = F_{n+1}/F_n$ for $n \geqslant 1$. Note that $F_n > 0$ for all $n$. Then

$$x_1 = 2 \quad \text{and} \quad x_{n+1} = 1 + 1/x_n.$$

Suppose that we did have convergence and that $x_n \to L$ so that $x_{n+1} \to L$. Note $L \geqslant 1 > 0$ as $F_{n+1} > F_n$ and so $1 + \frac{1}{x_n} \to 1 + \frac{1}{L}$ by AOL. So

$$L = 1 + \frac{1}{L}$$

by the uniqueness of limits. Hence $L^2 - L - 1 = 0$ giving $L = \frac{1 \pm \sqrt{5}}{2}$. But $L \geqslant 1$ giving

$$L = \frac{1 + \sqrt{5}}{2} > 1.$$

All the above was based on the assumption that $x_n$ converged. We will show that $x_n$ is convergent to $\frac{1+\sqrt{5}}{2}$, which we will denote $\varphi$, and is called the **golden ratio**.

$$x_{n+1} - \varphi = 1 + \frac{1}{x_n} - \varphi = 1 + \frac{1}{x_n} - 1 - \frac{1}{\varphi} = \frac{1}{x_n} - \frac{1}{\varphi} = \frac{x_n - \varphi}{x_n \varphi}$$

as $\varphi^2 = \varphi + 1$. So

$$\left|\frac{x_{n+1} - \varphi}{x_n - \varphi}\right| = \frac{1}{|x_n||\varphi|} = \frac{1}{\varphi x_n} \leqslant \frac{1}{\varphi}$$

as $x_n > 1$ for all $n$. By induction we get

$$-\frac{1}{\varphi^n} \leqslant x_n - \varphi \leqslant \frac{1}{\varphi^n}$$

and are done by the sandwich rule, since $\varphi > 1$ and so $\pm\frac{1}{\varphi^n} \to 0$. ■

**Example 4.18** *Which of the following statements are true of the given non-zero real or complex sequence $(a_n)$? Provide a proof or a counter-example.*

*(a) If $(a_n)$ converges then $a_{n+1} - a_n \to 0$.*
*(b) If $a_{n+1} - a_n \to 0$ then $(a_n)$ converges.*
*(c) If $(a_n)$ converges then $a_{n+1}/a_n \to 1$.*
*(d) If $a_n \to L \neq 0$ then $a_{n+1}/a_n \to 1$.*
*(e) If $a_{n+1}/a_n \to 1$ then $(a_n)$ converges.*
*(f) If $a_{n+1}/a_n \to 1$ and $(a_n)$ is bounded then $(a_n)$ converges.*

**Solution.** Let $H_n$ denote the $n$th harmonic number.

(a) *True*: If $a_n \to L$ then by the algebra of limits $a_{n+1} - a_n \to L - L = 0$.
(b) *False*: Let $a_n = H_n$. Then $a_{n+1} - a_n = (n+1)^{-1} \to 0$ yet $H_n \to \infty$ (see Example 3.38).
(c) *False*: Let $a_n = (-1)^n/n$ so that $a_n \to 0$. However $a_{n+1}/a_n = -\frac{n}{n+1} \to -1$.
(d) *True*: If $a_n \to L \neq 0$ then by the algebra of limits $a_{n+1}/a_n \to L/L = 1$.
(e) *False*: Let $a_n = n$. Then $a_{n+1}/a_n = 1 + n^{-1} \to 1$ but $a_n \to \infty$.
(f) *False*: Let $a_n = e^{iH_n}$. Then

$$a_{n+1}/a_n = e^{i(H_{n+1} - H_n)} = e^{i/(n+1)} \to e^0 = 1,$$

as $n \to \infty$ but $e^{iH_n}$ does not converge as $H_n \to \infty$. ■

**Remark 4.19** *The necessary AOL properties to justify the answer to (f) won't be proven until Analysis II in Hilary Term. The notion of a continuous function will be defined there and we will see that if $a_n \to L$ and $f$ is continuous then $f(a_n) \to f(L)$. In fact, this property is an alternative definition of $f$ being continuous.*

As was commented in Remark 3.32, there are several indeterminate forms including $\infty$, so we cannot expect any AOL results re

$$\infty - \infty, \qquad \frac{\infty}{\infty}, \qquad 0 \times \infty.$$

But there are some cases where AOL-like results are true.

**Proposition 4.20** ***(AOL: Infinity)*** *Let* $(a_n)$ *and* $(b_n)$ *be real sequences.*

*(a) If* $a_n \to \infty$ *and* $b_n \to \infty$ *then* $a_n + b_n \to \infty$.
*(b) If* $a_n \to \infty$ *and* $b_n \to \infty$ *then* $a_n b_n \to \infty$.
*(c) If* $a_n \to \infty$ *and* $b_n \to -\infty$ *then* $a_n b_n \to -\infty$.
*(d) If* $a_n \to \infty$ *and and* $(b_n)$ *is bounded then* $a_n + b_n \to \infty$.
*(e) If* $a_n \to \infty$ *and and* $(b_n)$ *is bounded then* $b_n/a_n \to 0$.
*(f) If* $a_n \to \infty$ *and* $b_n \to L > 0$ *then* $a_n b_n \to \infty$.

**Solution.** These are left as exercises. ■

**Proposition 4.21** ***(AOL: Asymptotics)*** *Let* $(a_n)$, $(\alpha_n)$, $(b_n)$ *and* $(\beta_n)$ *be real sequences.*

*(a) If* $a_n = O(\alpha_n)$ *and* $b_n = O(\beta_n)$ *then* $a_n b_n = O(\alpha_n \beta_n)$.
*(b) If* $a_n = O(\alpha_n)$ *and* $b_n = O(\beta_n)$ *then* $a_n + b_n = O(\max(|\alpha_n|, |\beta_n|))$.
*(c) If* $a_n \sim \alpha_n$ *and* $b_n \sim \beta_n$ *then* $a_n b_n \sim \alpha_n \beta_n$.
*(d) If* $a_n \sim \alpha_n$ *and* $b_n \sim \beta_n$ *then* $a_n/b_n \sim \alpha_n/\beta_n$.

**Solution.** These are left as exercises. ■

**Remark 4.22** ***(The Relative Orders of Terms)*** *Our first thoughts, when considering the long term behaviour of a sequence which has various components to it, should be on which terms dictate the sequence's behaviour in the long term. Usually, for this, we need to appreciate the relative magnitudes of the terms as* $n$ *becomes large. As a rule of thumb, when it comes to the long term behaviour of functions*

*bounded trig functions and constants* $<$ *logarithms* $<$ *polynomials* $<$ *exponentials.*

*More precisely:*

- $|\cos n| \leqslant 1$ *and* $|\sin n| \leqslant 1$ *for all* $n$.
- *For any rational* $q > 0$, $\log n / n^q \to 0$ *as* $n \to \infty$.
- *For any* $a > 1$ *and polynomial* $p$ *then* $p(n)/a^n \to 0$ *as* $n \to \infty$.

*The third bullet point is a consequence of Corollary 3.19. The second bullet point is essentially the same result. If we write* $n = e^t$ *then*

$$\lim_{n\to\infty} \frac{\log n}{n^q} = \lim_{t\to\infty} \frac{t}{(e^q)^t} = 0$$

*as* $e^q > 1$.

**Example 4.23** *Qualitatively describe the long-term behaviour of the following sequences.*

- $$(-1)^n \left( \frac{n^6 + 7n^2}{2^n} \right)$$

  *This will tend to* $0$ *(albeit in an oscillatory way) as the dominant term is* $2^n$.

- $$\left(\frac{2n+3}{3n+8}\right)\cos n.$$

  *At first glance the polynomial terms seem dominant. But being of the same degree, and working to counter one another, we see* $(2n+3)\,/\,(3n+8) \to 2/3$. *So actually it is the oscillating behaviour of* $\cos n$ *which stops the sequence from converging.*

- $$\frac{\log n}{\sqrt{n}}\cos\left(\frac{2^n-n}{n^2+3n-6}\right).$$

  *As* $|\cos\theta| \leqslant 1$ *for all* $\theta$ *then the cosine takes the sting out of the term* $(2^n-n)\,/\,(n^2+3n-6)$ *which is just a red herring. In the long term* $\sqrt{n}$ *dominates* $\log n$ *and* $\log n/\sqrt{n} \to 0$. *The messy cosine term has no crucial effect on this behaviour.*

- *How would you make these first thoughts into rigourous proofs using the algebra of limits, sandwich rule, etc.?*