5. MORE ON SEQUENCES

5.1 Monotone Sequences

We now turn to a crucially important kind of sequence.

Definition 5.1 *Let (a_n) be a real sequence.*
*We say (a_n) is **increasing** if $a_n \leqslant a_m$ whenever $n < m$.*
*We say (a_n) is **decreasing** if $a_n \geqslant a_m$ whenever $n < m$.*
*We say (a_n) is **strictly increasing** if $a_n < a_m$ whenever $n < m$.*
*We say (a_n) is **strictly decreasing** if $a_n > a_m$ whenever $n < m$.*
*We say (a_n) is **monotone** if it is either decreasing or increasing.*

Example 5.2 *Let $a_n = n$. Then (a_n) is increasing. So is $a_n = (2n+1)^2$.*
The sequence $a_n = (-1)^n$ is not monotone as $a_1 < a_2$ and $a_2 > a_3$.

Theorem 5.3 *Let (a_n) be an increasing, bounded above sequence. Then (a_n) converges.*

Proof. Let $L = \sup\{a_n \mid n \in \mathbb{N}\}$; this exists by the completeness axiom as the set is bounded above and non-empty. Let $\varepsilon > 0$. By the approximation property there exists $N \in \mathbb{N}$ such that

$$L - \varepsilon < a_N \leqslant L.$$

As the sequence is increasing then for any $n \geqslant N$

$$L - \varepsilon < a_N \leqslant a_n \leqslant L,$$

and so

$$\forall n \geqslant N \quad |a_n - L| < \varepsilon.$$

That is $a_n \to L$. ■

Corollary 5.4 *An increasing real sequence either converges or tends to infinity.*

Proof. Let (a_n) be an increasing real sequence. If it is bounded above, then (a_n) converges. Otherwise for any $M > 0$ then M is not an upper bound to (a_n). Hence there exists $N \in \mathbb{N}$ such that $a_N \geqslant M$. Now as (a_n) is increasing $a_n \geqslant M$ for all $n \geqslant N$. That is $a_n \to \infty$. ■

Corollary 5.5 *Let (a_n) be a decreasing, bounded below sequence. Then (a_n) converges.*

Proof. $(-a_n)$ is increasing and bounded above so $-a_n \to L$ by the previous result. Hence $a_n \to -L$ by AOL. ■

Remark 5.6 *Theorem 5.3 is in fact equivalent to the completeness axiom. That is, the axioms of an ordered field together with Theorem 5.3 characterize the real numbers. Details are left to Sheet 4, Exercise 6.*

The next theorem, the Nested Intervals Theorem, together with the Archimedean property, form another alternative to the completeness axiom.

Theorem 5.7 ***(Nested Intervals Theorem – Cantor, 1872)*** *Let $I_n = [a_n, b_n]$ be a nested sequence of closed bounded intervals. (That is $I_{n+1} \subseteq I_n$ for all $n \geqslant 1$.)*

(a) Then $\bigcap_1^\infty I_n \neq \varnothing$.

(b) If $l(I_n) = b_n - a_n \to 0$ as $n \to \infty$ then $\bigcap_1^\infty I_n$ is a singleton.

(c) Note the theorem need not hold if the intervals are bounded but not closed, e.g. $I_n = (0, 1/n)$, or closed but not bounded, e.g. $I_n = [n, \infty)$.

Proof. (a) As $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ then $a_n \leqslant a_{n+1} \leqslant b_{n+1} \leqslant b_n$. So (a_n) is an increasing sequence which is bounded above, and (b_n) is a decreasing sequence bounded below. This means both sequences converge and set $\alpha = \lim a_n$ and $\beta = \lim b_n$.

As $a_m \leqslant \alpha \leqslant \beta \leqslant b_n$ for all m, n, then $[\alpha, \beta] \subseteq I_n$ for all n and so $\bigcap_1^\infty I_n \neq \varnothing$.

(b) As $b_n - a_n \geqslant \beta - \alpha$ for each n and $b_n - a_n \to 0$ then $\alpha = \beta$. Certainly $\alpha \in \bigcap_1^\infty I_n$. And if $x, y \in \bigcap_1^\infty I_n$ with $x < y$ then

$$0 < y - x < b_n - a_n$$

for all n, a contradiction as $b_n - a_n \to 0$. Hence $\bigcap_1^\infty I_n = \{\alpha\}$. ■

Example 5.8 *(a) Let $x \in \mathbb{R}$. Show that $x^n/n! \to 0$.*
(b) Deduce that $z^n/n! \to 0$ for $z \in \mathbb{C}$.

Solution. (a) If $x = 0$ this is clear. Otherwise set $a_n = |x|^n / n!$ and note

$$\frac{a_{n+1}}{a_n} = \frac{n!\,|x|^{n+1}}{(n+1)!\,|x|^n} = \frac{|x|}{n+1} \to 0 \quad \text{as } n \to \infty.$$

So in some tail $a_{n+1}/a_n < 1$ and (a_n) is eventually decreasing and bounded below by 0. Hence a_n converges to some limit L.

We have

$$a_{n+1} = \left(\frac{|x|}{n+1}\right) a_n.$$

Letting $n \to \infty$ and applying AOL, we have

$$L = 0 \times L = 0,$$

as required.

(b) Now take $z \in \mathbb{C}$. By (a) $|z^n/n!| = |z|^n / n! \to 0$ and hence $z^n/n! \to 0$. ■

Example 5.9 *(a) Let $a \geqslant 1$. By considering the iteration*

$$x_0 = a, \qquad x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \qquad \text{for } n \geqslant 0,$$

show the existence and uniqueness of $\sqrt{a}$.

(b) Deduce the existence and uniqueness of $\sqrt{a}$ for $0 \leqslant a < 1$.

Remark 5.10 *This iteration was known to the Babylonians for finding square roots. From a modern perspective it is an instance of the* ***Newton-Raphson method*** *applied to the function* $f(x) = x^2 - a$.

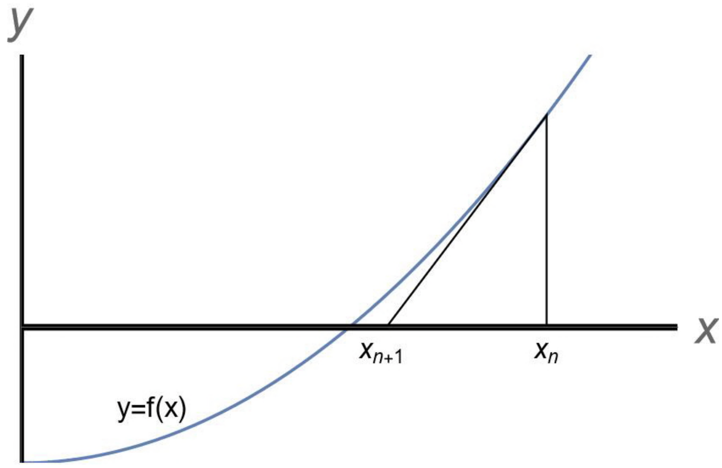

Fig. 5.1 – Newton-Raphson method

The Newton-Raphson iteration seeks to solve an equation $f(x) = 0$. *It takes an estimate* x_n *for a root and replaces it with*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

This estimate x_{n+1} *is achieved (as in Figure 5.1) by drawing the tangent to the curve* $y = f(x)$ *at the point* $(x_n, f(x_n))$ *and intersecting it with the* x*-axis.*

In this particular case $f(x) = x^2 - a$ *and so*

$$x_{n+1} = x_n - \left(\frac{x_n^2 - a}{2x_n}\right) = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

Solution. (a) I claim the following to be true of the sequence (x_n):

(i) $a \leqslant x_n^2$ for all n;

(ii) (x_n) is decreasing;

(iii) $L = \lim x_n$ satisfies $L^2 = a$.

(i) As $x_0 = a$ then (i) is true for $n = 0$ as $a^2 - a = a\,(a-1) \geqslant 0$. If $a \leqslant x_n^2$ then

$$\begin{aligned} x_{n+1}^2 - a &= \left[\frac{1}{2}\left(x_n + \frac{a}{x_n}\right)\right]^2 - a \\ &= \frac{1}{4x_n^2}\left[\left(x_n^2 + a\right)^2 - 4ax_n^2\right] \\ &= \frac{1}{4x_n^2}\left[x_n^4 - 2ax_n^2 + a^2\right] \\ &= \frac{1}{4x_n^2}\left[x_n^2 - a\right]^2 \geqslant 0 \end{aligned}$$

Hence (i) follows by induction.

(ii) Note that

$$x_n - x_{n+1} = x_n - \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) = \frac{x_n^2 - a}{2x_n} \geqslant 0$$

by (i).

(iii) So (x_n) is decreasing and bounded below and therefore converges. Let $L = \lim x_n$. Letting $n \to \infty$ in the iteration

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

we get

$$L = \frac{1}{2}\left(L + \frac{a}{L}\right)$$

by AOL and the uniqueness of limits. This rearranges to $L^2 = a$. As $x_n \geqslant 0$ for all n then $L = \sqrt{a}$ (as opposed to $-\sqrt{a}$).

Now L is a root of $x^2 = a$. As $x^2 - a = (x - L)(x + L)$ then we see that the two roots of $x^2 - a$ are $\pm L$. From this we also see that the two square roots of a are $\pm\sqrt{a}$, showing there is a unique positive square root of a.

(b) Clearly 0 is the only square root of 0. Say now that $0 < a < 1$ so that $a^{-1} > 1$. By (a)

$$x^2 = a \qquad \Longleftrightarrow \qquad \left(\frac{1}{x}\right)^2 = a^{-1} \qquad \Longleftrightarrow \qquad \frac{1}{x} = \pm\sqrt{a^{-1}},$$

only one root of which is positive. Hence $\sqrt{a}$ is uniquely defined with $\sqrt{a} = \left(\sqrt{a^{-1}}\right)^{-1}$. ■

Remark 5.11 ***(Cobwebbing)*** *The previous iteration can also be achieved via cobwebbing which aims to solve equations of the form $x = f(x)$. The previous Newton-Raphson iteration took the form*

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

which, if it converges, leads to a solution of $x = f(x)$ where

$$f(x) = \frac{1}{2}\left(x + \frac{a}{x}\right).$$

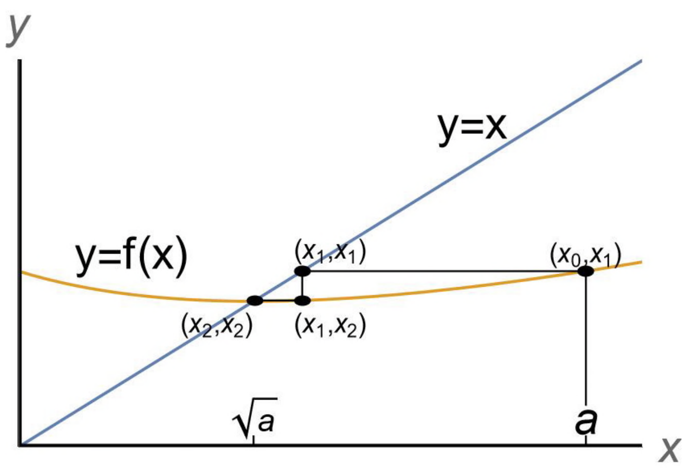

Fig. 5.2 – Cobwebbing

We sketch $y = x$ and $y = f(x)$ on the same axes. Given an initial estimate x_0 we draw a vertical line to the curve to get to (x_0, x_1) and then move horizontally to (x_1, x_1) and so on to $(x_1, x_2), (x_2, x_2), (x_2, x_3), \ldots$. If the sequence (x_n) converges to α, say, then α is a fixed point. That is $\alpha = f(\alpha)$; this essentially follows by AOL.

We can see from Figure 5.2 how any sequence beginning with $x_0 \geqslant \sqrt{a}$ will monotonically decrease to $\sqrt{a}$. Any sequence beginning with $0 < x_0 < \sqrt{a}$ will jump to $x_1 > \sqrt{a}$ and then decrease again to $\sqrt{a}$. Of course, the figure itself ***proves*** *nothing but provides useful qualitative information for what needs proving.*

In this particular case the iteration converges quickly. As $f(\alpha) = \alpha$ then

$$x_{n+1} - \alpha \approx f'(\alpha)(x_n - \alpha),$$

and for this particular iteration $-1 < f'(\alpha) < 1$ as

$$f'\left(\sqrt{a}\right) = \frac{1}{2}\left(1 - \frac{a}{\left(\sqrt{a}\right)^2}\right) = 0.$$

When $|f'(\alpha)| < 1$ the the fixed point α is said to be an ***attracting*** *fixed point.*

The convergence will be monotonic if $0 < f'(\alpha) < 1$ and will be oscillatory if $-1 < f'(\alpha) < 0$. When $|f'(\alpha)| > 1$ the fixed point is called ***repelling*** *and the iteration will not generally converge.*

We conclude this section by defining the decimal expansion (and more generally base expansions) for a real number. For uniqueness we do this in such a way that the truncated decimal expansions form a strictly convergent sequence converging to the real number in question.

Example 5.12 *(**Decimal Expansions**) Let $0 < x \leqslant 1$. Then there is a unique sequence of integers $a_1, a_2, a_3, \ldots$ such that*

(a) $0 \leqslant a_n \leqslant 9$ for each n;

(b) for each n,

$$x - \frac{1}{10^n} \leqslant \sum_{k=1}^{n} \frac{a_k}{10^k} < x;$$

(c)

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{10^k} = x.$$

Solution. We will proceed inductively. The integer a_1 needs to satisfy

$$x - \frac{1}{10} \leqslant \frac{a_1}{10} < x \qquad \Longrightarrow \qquad 10x - 1 \leqslant a_1 < 10x.$$

The interval $[10x-1, 10x)$ contains a unique integer a_1 and further, as

$$-1 < 10x - 1 \leqslant a_1 < 10x \leqslant 10$$

then $0 \leqslant a_1 \leqslant 9$.

Suppose now, as our inductive hypothesis, that $a_1, a_2, \ldots, a_N$ have been uniquely found satisfying (i) and (ii). Then

$$x - \frac{1}{10^{N+1}} \leqslant \sum_{k=1}^{N+1} \frac{a_k}{10^k} < x$$

which rearranges to

$$x - \frac{1}{10^{N+1}} - \sum_{k=1}^{N} \frac{a_k}{10^k} \leqslant \frac{a_{N+1}}{10^{N+1}} < x - \sum_{k=1}^{N} \frac{a_k}{10^k}$$

and then to

$$\left(10^{N+1}x - \sum_{k=1}^{N} 10^{N+1-k} a_k\right) - 1 \leqslant a_{N+1} < \left(10^{N+1}x - \sum_{k=1}^{N} 10^{N+1-k} a_k\right).$$

There is a unique integer in this range, and we set a_{N+1} to be this integer. Further, by hypothesis,

$$\begin{aligned} a_{N+1} &\geqslant 10^{N+1}\left(x - \textstyle\sum_{k=1}^{N} 10^{-k} a_k\right) - 1 > -1, \\ a_{N+1} &< 10^{N+1}\left(x - \textstyle\sum_{k=1}^{N} 10^{-k} a_k\right) \leqslant 10^{N+1} \times \frac{1}{10^N} = 10. \end{aligned}$$

So $0 \leqslant a_{N+1} \leqslant 9$ as required. Finally, letting $n \to \infty$ and applying the sandwich rule to

$$x - \frac{1}{10^n} \leqslant \sum_{k=1}^{n} \frac{a_k}{10^k} < x,$$

we find

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{10^k} = x.$$

This sequence is called the **decimal expansion** of x and we write

$$x = 0.a_1 a_2 a_3 \ldots.$$

■

Remark 5.13 *In the sense of the above example* $\frac{1}{5}$ *would have decimal expansion* $0.1999\ldots$ *rather than* $0.200\ldots$ *To avoid any ambiguity for those reals with two different decimal expansions (in the usual sense) the above example chooses decimal expansions whose terminating decimal expansions never equal the real in question.*

A similar argument to that above shows the uniqueness for any base $b \geqslant 2$ *expansions. As with the example of* $\frac{1}{5}$ *in decimal, in binary,* $b = 2$*, we would have* $\frac{1}{2} = 0.0111\ldots$ *rather than* $\frac{1}{2} = 0.1$.

5.2 Subsequences

Example 5.14 *Let* $a_n = \frac{1}{n^2}$ *so that*

$$(a_n)_1^\infty = \left(1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots\right).$$

We can get new sequences by selectively looking at

everything after second place	$\left(\frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots\right)$
all odd terms	$\left(1, \frac{1}{9}, \frac{1}{25}, \ldots\right)$
all prime terms	$\left(\frac{1}{4}, \frac{1}{9}, \frac{1}{25}, \frac{1}{49}, \ldots\right)$

etc.. These are examples of subsequences of (a_n).

Definition 5.15 *Let* (a_n) *be a sequence. We say that a sequence* (b_n) *is a* ***subsequence*** *of* (a_n) *if there is a strictly increasing sequence of natural numbers* $(f(n))$ *that* $(b_n) = \left(a_{f(n)}\right)$*. (There may be more than one such function* f*.) Often we write* n_r *for* $f(r)$ *and write a subsequence as* (a_{n_r}) *or* $(a_{n_r})_{r=1}^\infty$.

Example 5.16 *In the previous example* $n_r = r + 2$, $n_r = 2r - 1$ *and* $n_r = p_r$ *(the* r*th prime) respectively.*

Example 5.17 *Let*

$$\begin{aligned} (a_n) &= \left(n^2\right) = (1, 4, 9, 16, \ldots); \qquad (b_n) = (0) = (0, 0, 0, 0, \ldots); \\ (f(n)) &= (2n) = (2, 4, 6, 8, \ldots); \qquad (g(n)) = (2n - 1) = (1, 3, 5, 7, \ldots). \end{aligned}$$

Then

$$\begin{aligned} \left(a_{f(n)}\right) &= (a_{2n}) = (4, 16, 36, 64, \ldots); \qquad \left(a_{g(n)}\right) = (a_{2n+1}) = (1, 9, 25, 49, \ldots); \\ \left(b_{f(n)}\right) &= (b_{2n}) = (0, 0, 0, 0, \ldots); \qquad \left(b_{g(n)}\right) = (b_{2n+1}) = (0, 0, 0, 0, \ldots). \end{aligned}$$

Proposition 5.18 *Suppose that the sequence* (a_n) *converges to* L*. Then every subsequence* (a_{n_r}) *also converges to* L.

Proof. Let $\varepsilon > 0$. Then there exist N such that

$$n \geqslant N \quad \Longrightarrow \quad |a_n - L| < \varepsilon$$

As $r \mapsto n_r$ is strictly increasing then $n_r \geqslant r$ for all r and so

$$r \geqslant N \quad \Longrightarrow \quad n_r \geqslant N \quad \Longrightarrow \quad |a_{n_r} - L| < \varepsilon$$

and hence $a_{n_r} \to L$ as $r \to \infty$. ■

The converse in the form 'if all subsequences of (a_n) converge to L then $(a_n) \to L$' is true because the whole sequence is a subsequence of itself. However, just one subsequence converging is clearly not enough to guarantee convergence of the whole sequence. For example $a_n = (-1)^n$ which is divergent despite $a_{2n} \to 1$.

Theorem 5.19 *Let (a_n) be a real sequence. Then (a_n) has a montone subsequence.*

Proof. We consider the set

$$V = \{k \in \mathbb{N} \mid m > k \Longrightarrow a_m < a_k\}.$$

This is the set of 'scenic viewpoints' – were we to plot the points (k, a_k) in $\mathbb{R}^2$ then from a scenic viewpoint we could see all the way to ∞ with no greater a_n getting in the way. There are two cases to consider: the set V is either finite or infinite.

- V is infinite. Listing the elements of V in increasing order: $k_1 < k_2 < \ldots$ we see (a_{k_r}) is a subsequence with
$$r > s \quad \Longrightarrow \quad k_r > k_s \quad \Longrightarrow \quad a_{k_r} < a_{k_s}$$
  That is (a_{k_r}) is strictly decreasing.
- V is finite. Let m_1 be the last viewpoint and consider a_{m_1+1}.
  As $m_1 + 1$ is not a viewpoint then there exists $m_2 > m_1 + 1$ such that $a_{m_2} \geqslant a_{m_1}$.

1. As m_2 is not a viewpoint then there exists $m_3 > m_2$ such that $a_{m_3} \geqslant a_{m_2}$.

   . . .

   Continuing in this way and we can generate an increasing sequence (a_{m_k}).

■

Theorem 5.20 ***(Bolzano-Weierstrass Theorem, Bolzano 1817, Weierstrass c. 1861)*** *Let (a_n) be a real bounded sequence. Then (a_n) has a convergent subsequence.*

Proof. By the previous theorem (a_n) has a monotone subsequence which is also bounded. By Theorem 5.3 this subsequence converges. ■

Theorem 5.21 ***(Bolzano-Weierstrass Theorem in*** *$\mathbb{C}$) A bounded sequence in $\mathbb{C}$ has a convergent subsequence.*

Proof. Let (z_n) be a bounded sequence in $\mathbb{C}$. If we write $z_n = x_n + iy_n$ then we also have that (x_n) and (y_n) are bounded sequences. By the Bolzano-Weierstrass Theorem (x_n) has a convergent subsequence (x_{n_k}) which converges to L_1, say. As (y_{n_k}) is also bounded then it in turn has a convergent subsequence $\left(y_{n_{k_r}}\right)$ which converges to L_2, say.

As $\left(x_{n_{k_r}}\right)$ is a subsequence of (x_{n_k}) then it too converges to L_1 by Proposition 5.18. We then have that $\left(z_{n_{k_r}}\right)$ converges to $L_1 + iL_2$ as its real and imaginary parts converge (Theorem 3.25). ■

Here is alternative way of phrasing the Bolzano-Weierstrass Theorem.

Definition 5.22 *Let $S \subseteq \mathbb{R}$ We say that x is a* ***limit point*** *or* ***accumulation point*** *of S if for every $\varepsilon > 0$ there exists $y \in S$, such that*

$$0 < |y - x| < \varepsilon.$$

Note that x itself need not be in the set. The set of limit points of S is denoted S'.

Example 5.23 *The set of limit points of* $(0, 1)$ *is* $[0, 1]$

The set of limit points of $\mathbb{Q}$ is $\mathbb{R}$.

The set of limit points of $\mathbb{Z}$ is $\varnothing$.

The set of limit points of $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ *is* $\{0\}$.

Remark 5.24 *The Bolzano-Weierstrass Theorem can be rephrased as: 'An infinite bounded subset of $\mathbb{R}$ or $\mathbb{C}$ has a limit point'. Given such a set, S, then we can select a sequence (x_n) of points of S and by the Bolzano-Weierstrass Theorem this sequence has a subsequence (x_{n_r}) which converges to a limit L. It is not hard to show that L is then a limit point of the set $\{x_{n_1}, x_{n_2}, x_{n_2}, \ldots\} \subseteq S$ and so of the set S.*

5.3 The Cauchy Convergence Criterion

A first difficulty in proving that a sequence converges is in investigating the limit. Cauchy saw that a (real or complex) sequence would converge if and only if the sequence's terms got sufficiently close. This makes it possible to demonstrate convergence without knowing the limit. Further, Cauchy's insight can be used to *construct* the reals from the rationals so that we could show the existence of a complete ordered field rather than assuming that a field satisfying all our axioms exists (see Remark 5.34).

Definition 5.25 *Let (a_n) be a real or complex sequence. We say that (a_n) is a* ***Cauchy sequence****, or simply is* ***Cauchy,*** *if*

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall m, n \geqslant N \quad |a_m - a_n| < \varepsilon.$$

Note that the definition makes no mention of a limit, but we shall see that this criterion is in fact equivalent to convergence in $\mathbb{R}$ or $\mathbb{C}$ (but not in $\mathbb{Q}$!).

Proposition 5.26 *A convergent sequence is Cauchy.*

Proof. Let (a_n) be a convergent sequence with limit L, and let $\varepsilon > 0$. Then there exists a natural number N such that

$$|a_k - L| < \varepsilon/2 \text{ for all } k \geqslant N.$$

So for all $m, n \geqslant N$,

$$|a_m - a_n| \leqslant |a_m - L| + |L - a_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

by the triangle inequality and hence (a_n) is Cauchy. ∎

Proposition 5.27 *A (real or complex) Cauchy sequence is bounded.*

Proof. Let (a_n) be a real or complex Cauchy sequence. Taking $\varepsilon = 1$, we know there exists N such that

$$|a_n - a_N| < 1 \quad \text{whenever} \quad n \geqslant N.$$

Hence, by the triangle inequality

$$|a_n| < |a_N| + 1 \text{ for all } n \geqslant N.$$

The above inequality bounds all but finitely many terms. So for all m we have

$$|a_m| \leqslant \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a_N| + 1\}$$

and we see that the sequence is bounded. ∎

Lemma 5.28 *If (a_n) is a real or complex Cauchy sequence such that a subsequence (a_{n_k}) converges to L, then (a_n) converges to L.*

Proof. Let $\varepsilon > 0$. So there exists $K \in \mathbb{N}$ such that

$$|a_{n_k} - L| < \varepsilon/2 \quad \text{whenever} \quad k \geqslant K.$$

As the sequence (a_n) is Cauchy then there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \varepsilon/2 \text{ whenever } m, n \geqslant N.$$

If we select take $k \geqslant \max(K, N)$ so that $n_k \geqslant N$ then we have, by the triangle inequality

$$|a_n - L| \leqslant |a_n - a_{n_k}| + |a_{n_k} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for all } n \geqslant N$$

and the proof is complete. ∎

Theorem 5.29 *(Cauchy, 1821) A real or complex Cauchy sequence is convergent.*

Proof. Let (a_n) be a real or complex Cauchy sequence. By Proposition 5.27 (a_n) is bounded, and so by the Bolzano-Weierstrass Theorem (a_n) has a convergent subsequence (a_{n_k}). By the previous lemma (a_n) converges to the same limit. ■

We have then establishded the **Cauchy Convergence Criterion** for real and complex sequences:

$$(a_n) \text{ is convergent} \iff (a_n) \text{ is Cauchy.}$$

Remark 5.30 *The Cauchy convergence criterion, together with the Archimedean property, is equivalent to the completeness axiom.*

Example 5.31 *The terminating decimal expansions of $\sqrt{2}$, namely the sequence (q_n):*

$$1, \quad 1.4, \quad 1.41, \quad 1.414, \ldots$$

is a sequence of rational numbers which is Cauchy (for example, because it is a convergent real sequence) but it is not convergent in the rationals – that is, it does not satisfy

$$\exists L \in \mathbb{Q} \quad \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geqslant N \quad |q_n - L| < \varepsilon.$$

Example 5.32 *(**Mercator's series**) For $n \in \mathbb{N}$ let*

$$s_n = 1 - \frac{1}{2} + \frac{1}{3} + \cdots + (-1)^{n+1}\frac{1}{n}.$$

Then with $m \geqslant n > 0$, and $m - n$ even we have

$$\begin{aligned}
|s_m - s_n| &= \left| \overbrace{\frac{1}{n+1} - \frac{1}{n+2}}^{>0} + \overbrace{\frac{1}{n+3} - \frac{1}{n+4}}^{>0} + \cdots + \overbrace{\frac{1}{m-1} - \frac{1}{m}}^{>0} \right| \\
&= \underbrace{\frac{1}{n+1}} \underbrace{- \frac{1}{n+2} + \frac{1}{n+3}}_{<0} - \cdots \underbrace{- \frac{1}{m-2} + \frac{1}{m-1}}_{<0} \underbrace{- \frac{1}{m}}_{<0} \\
&\leqslant \frac{1}{n+1}.
\end{aligned}$$

If $m - n$ is odd, we write

$$\begin{aligned}
|s_m - s_n| &= \left| \overbrace{\frac{1}{n+1} - \frac{1}{n+2}}^{>0} + \overbrace{\frac{1}{n+3} - \frac{1}{n+4}}^{>0} + \cdots + \overbrace{\frac{1}{m-2} - \frac{1}{m-1}}^{>0} + \frac{1}{m} \right| \\
&= \underbrace{\frac{1}{n+1}} \underbrace{- \frac{1}{n+2} + \frac{1}{n+3}}_{<0} - \cdots \underbrace{- \frac{1}{m-1} + \frac{1}{m}}_{<0} \\
&\leqslant \frac{1}{n+1}.
\end{aligned}$$

Let $\varepsilon > 0$ and take $N > \frac{1}{\varepsilon}$. Then $|s_n - s_m| < \varepsilon$ whenever $m, n \geqslant N$ and we see that (s_n) is Cauchy. This shows that the sequence is convergent even though we currently have no idea of its limit. In due course we shall see that the limit is $\log 2$ *(Sheet 6, Exercise 6). The sum was first published by Mercator in 1668.*

Remark 5.33 ***(Double sequences)*** *A (real) double sequence is a map $x\colon \mathbb{N}^2 \to \mathbb{R}$ and we write $x_{m,n}$ for $x(m,n)$. We write that*

$$\lim_{m,n\to\infty} x_{m,n} = L$$

if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall m, n \geqslant N \quad |x_{m,n} - L| < \varepsilon.$$

So we may rewrite the Cauchy convergence criterion as

$$(a_n) \quad \textit{is Cauchy if} \quad |a_n - a_m| \to 0 \quad \textit{as } m, n \to \infty.$$

Given a double sequence $(x_{m,n})$ the limits

$$\lim_{m,n\to\infty} x_{m,n} \qquad \lim_{m\to\infty}\left(\lim_{n\to\infty} x_{m,n}\right) \qquad \lim_{n\to\infty}\left(\lim_{m\to\infty} x_{m,n}\right)$$

are different notions and may independently exist or not as seen in Sheet 5, Exercise 8.

Remark 5.34 ***(Construction of the real numbers)*** *(Off-syllabus)*

We mentioned in Remark 1.58 the matter of existence and uniqueness of the real numbers. These issues were posed in the sense of 'can the real numbers be constructed from more concrete sets such as $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Q}$?'

Construction of the natural numbers.

One approach to define the natural numbers is due to Peano from 1889. Peano's description essentially states:

$\mathbb{N}$ is the smallest set such that (i) $0 \in \mathbb{N}$, (ii) if $n \in \mathbb{N}$ then $n + 1 \in \mathbb{N}$.

A later model, in the style of the Zermelo-Fraenkel axioms for set theory (1908,1922), was Von Neumann's model from 1923 where he identified 0 with $\varnothing$, 1 with $\{\varnothing\}$, 2 with $\{\varnothing, \{\varnothing\}\}$ and in general n with $\{0, 1, \ldots, n-1\}$. as a collection of sets meeting Peano's axioms.

Construction of the integers.

From the set $\mathbb{N}$ we can define the set of integers $\mathbb{Z}$ from $\mathbb{N}^2$. We define the equivalence relation $\sim$ on $\mathbb{N}^2$ by $(m_1, m_2) \sim (n_1, n_2)$ iff $m_1 + n_2 = n_1 + m_2$. Then $\mathbb{Z} = \mathbb{N}^2/\sim$. Essentially we are identifying an integer with pairs of natural numbers that differ by that integer.

Construction of the rational numbers.

Having defined $\mathbb{Z}$ we can define $\mathbb{Q}$ as a set of equivalence classes of $\mathbb{Z} \times (\mathbb{N}\backslash\{0\})$. We set $(m_1, n_1) \sim (m_2, n_2)$ iff $m_1 n_2 = n_1 m_2$. Then $\mathbb{Q} = \mathbb{Z} \times (\mathbb{N}\backslash\{0\})/\sim$. Essentially we are identifying an rational with all fractions $\frac{m}{n}$ which represent that rational.

Construction of the real numbers. *Having defined $\mathbb{Q}$ we set*

$$S = \{(a_n) \mid (a_n) \ \textit{is a rational Cauchy sequence}\}.$$

At this point we have yet to define the real numbers, but we know that a rational Cauchy sequence converges to some real limit. These limits are what we want as our model of the real numbers but we can't refer to such limits, irrational ones in particular, whilst only being able to refer to the rational numbers. Also many sequences in S will converge to the same limit so at this stage each real is overrepresented.

We can deal at least with this last point within the context of real numbers: for (a_n), $(b_n) \in S$ we set

$$(a_n) \sim (b_n) \quad \sim \quad a_n - b_n \to 0.$$

As we see in Sheet 5, Exercise 2, for (a_n), $(b_n) \in S$ and $c \in \mathbb{R}$ then

$$(a_n \pm b_n), \quad (ca_n), \quad (a_n b_n)$$

are in S and if $a_n \nrightarrow 0$ then $1/a_n \in S$.

Further these operation are well-defined in $S/\sim$. So if $(a_n) \sim (\alpha_n)$ and $(a_n) \sim (\beta_n)$ then

$$(a_n \pm b_n) \sim (\alpha_n \pm \beta_n), \quad (c\alpha_n) \sim (c\alpha_n), \quad (a_n b_n) \sim (\alpha_n \beta_n),$$

and if $a_n \nrightarrow 0$ and $\alpha_n \nrightarrow 0$ then $(1/a_n) \sim (1/\alpha_n)$. All these results follow bt AOL.

Regarding order we define $(a_n) \leqslant (b_n)$ if $b_n - a_n \geqslant 0$ in some tail.

All this gives $\mathbb{R} = S/\sim$ the structure of an ordered field. It can further be shown that any non-empty bounded subset of $S/\sim$ has a least upper bound; this result is not particularly difficult but is non-trivial (Körner pp. 352-353).

Construction of the complex numbers.

We showed in Section 1.4 how $\mathbb{C}$ can be constructed from $\mathbb{R}$ by identifying a complex number with an ordered pair of real numbers and defining addition and multiplication as one would expect of complex numbers.