6. SERIES

6.1 Infinite Series

Looking back at the field axioms, given any pair of real numbers a, b we can form their sum $a+b$. By induction, we can form any finite sum $\sum_1^n a_k$. The associative law means we don't have to worry about the order is which the necessary additions are executed.

What our axioms *don't* do is licence us to start writing down **infinite sums**, and behaving as though the mere act of writing down similar looking signs ($\sum_1^\infty$, say) entitles us to assume that all the properties of finite sums still hold. In fact, we will see that there are *conditionally convergent* series that give different sums depending on the order in which the terms are added. (See Sheet 6, Exercise 8 for the *Cauchy Root Test* and *Dirichlet's Test*.)

Definition 6.1 *Let $(a_n)_1^\infty$ be a sequence of (real or complex) numbers. For $n \geqslant 1$, the n**th partial sum** of (a_n) is the finite sum*

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

*By the **series***

$$\sum_{k=1}^{\infty} a_k \quad \textit{or just} \quad \sum a_k,$$

we mean the sequence of partial sums (s_n).

Example 6.2 *(a) **The geometric series.** Let $x \in \mathbb{C}$, and let $a_n = x^n$ Then $\sum x^n$ is*

$$(1,\ 1+x,\ 1+x+x^2,\ \ldots\ ,\ 1+x+x^2+\cdots+x^n,\ \ldots\ ).$$

*(b) **The harmonic series.** Let $a_n = \frac{1}{n}$. Then $\sum \frac{1}{n}$ is*

$$\left(1,\ 1+\frac{1}{2},\ 1+\frac{1}{2}+\frac{1}{3},\ \ldots\ \right).$$

*(c) **The exponential series.** Let $x \in \mathbb{C}$ and let $a_n = x^n/n!$. Then $\sum x^n/n!$ is*

$$\left(1,\ 1+x,\ 1+x+\frac{x^2}{2!},\ \ldots\ \right).$$

*(d) **The cosine series.** Let $x \in \mathbb{C}$ and set*

$$a_n = \begin{cases} \frac{x^{2m}}{(2m)!}(-1)^m & \textit{if } \ n = 2m \\ 0 & \textit{otherwise.} \end{cases}$$

Then $\sum a_n$ is

$$\left(1,\ 1,\ 1-\frac{x^2}{2!},\ 1-\frac{x^2}{2!},\ 1-\frac{x^2}{2!}+\frac{x^4}{4!},\ \ldots\right).$$

Definition 6.3 *Let (a_n) be a (real or complex) sequence. We say that the series $\sum_1^\infty a_k$* ***converges*** *(resp.* ***diverges****) if the sequence (s_n) of partial sums converges (resp. diverges). If $s_n \to L$ as $n \to \infty$ then we write*

$$\sum_{k=1}^{\infty} a_k = L.$$

We refer to L as the ***sum*** *(or* ***infinite sum****) of the series.*

Remark 6.4 *Our earlier results regarding the tails of sequences still apply – it follows that $\sum_1^\infty a_k$ converges if and only $\sum_K^\infty a_k$ converges for some K (Proposition 3.13). Consequently it makes sense to discuss the convergence (or otherwise) of $\sum a_k$ without needing to identify the initial term. But to determine the sum of a convergent series exactly we do need to specify the initial term.*

Proposition 6.5 *Say that $\sum a_n$ is convergent. Then $a_n \to 0$* ***but the converse is not true.***

Proof. Let s_n denote the nth partial sum; then $s_n \to L$ for some sum L. By AOL

$$a_n = s_n - s_{n-1} \to L - L = 0.$$

But recall from Example 3.38 that $\sum \frac{1}{n}$ is divergent, yet $a_n = \frac{1}{n} \to 0$. ■

Example 6.6 *Let $a_n = x^n$ for $n \geqslant 0$ where $x \in \mathbb{C}$.*

(a) If $x \neq 1$ then

$$s_n = 1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}.$$

(b) If $|x| < 1$ then $\sum x^n$ is convergent noting $x^n \to 0$ and using the algebra of limits.

(c) If $|x| \geqslant 1$ then $\sum x^n$ is divergent as $a_n = x^n \nrightarrow 0$.

Example 6.7 *Let $a_n = \frac{1}{n^2}$. Then $\sum \frac{1}{n^2}$ is convergent.*

Proof. Clearly the partial sums form an increasing sequence. By comparison with a telescoping sum we note

$$s_n = \sum_{k=1}^{n} \frac{1}{k^2} \leqslant 1 + \sum_{k=2}^{n} \frac{1}{k(k-1)} = 1 + \sum_{k=2}^{n} \left\{\frac{1}{k-1} - \frac{1}{k}\right\} = 1 + \sum_{k=1}^{n-1} \frac{1}{k} - \sum_{k=2}^{n} \frac{1}{k} = 1 + 1 - \frac{1}{n} \leqslant 2.$$

Hence (s_n) is a bounded increasing sequence and so convergent. [In due course we will meet, with the Integral Test, a systematic way of dealing with such series and won't have to resort to such algebraic tricks.] ■

Remark 6.8 *The exact sum $\sum_1^\infty \frac{1}{n^2}$ is known to be $\pi^2/6$. This sum was first found by Euler in 1734 and is known as the* ***Basel problem****, Basel being Euler's hometown.*

Applying Cauchy's criterion for convergence for sequences to series (which, recall, is just a sequence of partial sums) we have:

Theorem 6.9 ***(Cauchy's Criterion for Series)*** *The series $\sum_0^\infty a_k$ converges if and only if for all $\varepsilon > 0$ there exists N such that for all $m, n \geqslant N$ we have*

$$|s_n - s_m| = \left|\sum_{m+1}^{n} a_k\right| < \varepsilon.$$

Definition 6.10 *Let (a_n) be a real or complex sequence. Then we say that $\sum a_n$ is* ***absolutely convergent*** *or* ***AC*** *if the series $\sum |a_n|$ converges. A series which is convergent, but not absolutely convergent, is called* ***conditionally convergent.***

Theorem 6.11 *An AC (real or complex) series is convergent*

Proof. Suppose that $\sum a_n$ is AC and let $\varepsilon > 0$. By Cauchy's criterion there exists N such that

$$l > k \geqslant N \Longrightarrow \left|\sum_{k+1}^{l} |a_n|\right| < \varepsilon.$$

By the triangle inequality

$$l > k \geqslant N \Longrightarrow \left|\sum_{k+1}^{l} a_n\right| \leqslant \sum_{k+1}^{l} |a_n| = \left|\sum_{k+1}^{l} |a_n|\right| < \varepsilon,$$

and hence $\sum a_n$ is Cauchy and so converges. ■

Example 6.12 *(a) $\sum_0^\infty x^n$ is AC if $|x| < 1$ and diverges for $|x| \geqslant 1$.*
(b) $\sum_1^\infty \frac{(-1)^n}{n^2}$ is AC.
(c) $\sum_1^\infty \frac{\sin n}{n^3}$ is AC.
(d) $\sum_1^\infty \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

Solution. (a) See Example 6.6.
(b) See Example 6.7.
(c) Note that the partial sums

$$s_n = \sum_1^n \frac{|\sin k|}{k^3} \leqslant \sum_1^n \frac{1}{k^3} \leqslant \sum_1^n \frac{1}{k^2} \leqslant \sum_1^\infty \frac{1}{k^2}$$

form an increasing bounded sequence. Hence they converge.
(d) See Examples 5.32 and 3.38 ■

Definition 6.13 *Let $p\colon \mathbb{N} \longrightarrow \mathbb{N}$ be a bijection and set $b_n = a_{p(n)}$. Then $\sum b_n$ is called a* ***rearrangement*** *of the series $\sum a_n$.*

Example 6.14 *(See also Sheet 6, Exercise 6) If we rearrange the* $\log 2$ *series from Example 5.32 then we can change the sum:–*

$$\begin{aligned} 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots &= \log 2 \\ 1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots &= \frac{3}{2}\log 2. \end{aligned}$$

Theorem 6.15 *(**Dirichlet, 1837**) (Off-syllabus) If* $\sum a_n$ *is AC then* $\sum a_{p(n)}$ *is AC for any rearrangement* p *and*

$$\sum_1^\infty a_n = \sum_1^\infty a_{p(n)}$$

Theorem 6.16 *(**Riemann Rearrangment Theorem, 1853**) (Off-syllabus) If* $\sum_1^\infty a_n$ *is a real conditionally convergent series and* $-\infty \leqslant L \leqslant \infty$ *then there exists a bijection* $p\colon \mathbb{N}\to\mathbb{N}$ *such that*

$$\sum_1^\infty a_{p(n)} = L.$$

Hence a real series is AC if and only if it unconditionally convergent.

Theorem 6.17 *(**Cauchy Multiplication of Series, 1821**) (Off-syllabus) Suppose* $\sum_0^\infty a_n$ *and* $\sum_0^\infty b_n$ *are AC. For each* $n\in\mathbb{N}$ *we set*

$$c_n = \sum_{k=0}^n a_k b_{n-k}.$$

Then $\sum_1^\infty c_n$ *is AC and*

$$\sum_0^\infty c_n = \left(\sum_0^\infty a_n\right)\left(\sum_0^\infty b_n\right)$$

Proof. See Sheet 6, Exercise 7. ■

Remark 6.18 *Mertens, in 1875, showed that if just one of* $\sum_0^\infty a_n$ *and* $\sum_0^\infty b_n$ *is AC and the other convergent, then* $\sum_0^\infty c_n$ *converges. (See Apostol, Theorem 12-46.)*

Example 6.19 *For* $x,y\in\mathbb{C}$

$$\left(\sum_0^\infty \frac{x^n}{n!}\right)\left(\sum_0^\infty \frac{y^n}{n!}\right) = \sum \frac{(x+y)^n}{n!}$$

Proof. Let $a_n = \frac{x^n}{n!}$, $b_n = \frac{y^n}{n!}$. Then the series $\sum a_n$ and $\sum b_n$ are absolutely convergent (see Example 6.26). Then

$$c_n = \sum_{r+s=n} \frac{x^r}{r!}\frac{y^s}{s!} = \frac{1}{n!}\sum_{r=0}^n \binom{n}{r} x^r y^{n-r} = \frac{(x+y)^n}{n!}$$

by the binomial theorem. ■

6.2 Some Tests for Convergence

Here we discuss some classic tests for convergence and divergence. The idea that there are 'tests' is very attractive, but in practice (for problems arising from real-word situations) these tests may not apply. However the tests *do* give us clues, suggest ways of thinking about series, what sort of estimates need to be made, and a sense of the relative magnitude of terms.

Proposition 6.20 ***(A Simple Test for Divergence)*** *If $\sum a_n$ converges then $a_n \to 0$. The converse is not true.*

As the converse is not true then, in practice, the contrapositive is used more: if a_n does not tend to 0 then $\sum a_n$ diverges.

Proof. We already noted this in Proposition 6.5. ■

Theorem 6.21 ***(The Comparison Test)*** *Let (a_n), (b_n) be real sequences with $0 \leqslant a_n \leqslant b_n$. Then*

- *$\sum b_n$ is convergent $\Longrightarrow \sum a_n$ is convergent;*
- *$\sum a_n$ is divergent $\Longrightarrow \sum b_n$ is divergent.*

Proof. Note that the second statement is just the contrapositive of the first, and so it is enough to just prove the first. Suppose that $\sum b_k$ converges. Then the partial sums $\sum_1^n a_k$ satisfy

$$\sum_1^n a_k \leqslant \sum_1^n b_k \leqslant \sum_1^\infty b_k$$

and hence form an increasing bounded sequence which converges. ■

Remark 6.22 *At first glance, the comparison test seems limited as it only applies to non-negative terms. In practice, however, it is often used to show a series is AC and hence convergent. (See Example 6.23 (d).)*

And as with the sandwich test for sequences, the comparison test can be used to take care of expresssions that are awkward without being impactful. For example, the term $(2+\cos n)^{-1}$ in Example 6.23 (b) lies between $1/3$ and 1.

Example 6.23 *The following sequences*

$$\text{(a)} \quad \sum_1^\infty n^{-5/2}, \quad \text{(b)} \quad \sum_1^\infty \frac{1}{n\,(n+1)\,(2+\cos n)},$$
$$\text{(c)} \quad \sum_1^\infty \frac{x^n}{n} \quad \text{where } |x|<1, \quad \text{(d)} \quad \sum_1^\infty \frac{\sin n}{n^2+1},$$

all converge.

Solution. (a) This converges by comparison with $\sum n^{-2}$.

(b) This converges by comparison with $\sum n^{-2}$.

(c) Even though the terms are not non-negative, this is AC by comparison with $\sum |x|^n$ and hence is convergent.

(d) Even though the terms are not non-negative, this is AC by comparison with $\sum n^{-2}$ and hence is convergent. ■

Theorem 6.24 ***(The Ratio Test)*** *Let (a_n) be a real or complex sequence with $a_n \neq 0$ for all n. Suppose that*

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

exists.

- *If $L < 1$ then $\sum a_n$ converges absolutely;*
- *If $L > 1$ then $\sum a_n$ diverges;*
- *If $L = 1$ then $\sum a_n$ may converge or diverge (that is, the test is inconclusive).*

Proof. (a) Choose K such that $|L| < K < 1$. As $\varepsilon = K - |L| > 0$ there exists N such that

$$n \geqslant N \quad \Longrightarrow \quad \left| \left| \frac{a_{n+1}}{a_n} \right| - L \right| < \varepsilon,$$

so that for $n \geqslant N$

$$\left| \frac{a_{n+1}}{a_n} \right| \leqslant \varepsilon + |L| = K.$$

So for $k \geqslant 0$

$$|a_{N+k}| = \left| \frac{a_{N+k}}{a_{N+k-1}} \right| \times \left| \frac{a_{N+k-1}}{a_{N+k-2}} \right| \times \cdots \times \left| \frac{a_{N+1}}{a_N} \right| \times |a_N| \leqslant |a_N| K^k.$$

Now $\sum K^k$ is a convergent geometric series, and so the tail $\sum_N^\infty a_n$ is AC by the comparison test. Hence $\sum a_n$ is AC as it has an AC tail.

(b) Choose K such that $1 < K < |L|$. Then there exists N such that

$$n \geqslant N \quad \Longrightarrow \quad \left| \frac{a_{n+1}}{a_n} \right| > K.$$

Arguing as in (a), $|a_{N+k}| \geqslant K^k |a_N|$ and hence we see a_n does not tend to 0. So $\sum a_n$ is divergent by Proposition 6.20.

(c) For each of the series $\sum n^{-1}$ and $\sum n^{-2}$ we have $L = 1$ yet the former diverges and the latter converges. ■

Remark 6.25 *If $a_n > 0$ for all n and $\sum a_n$ converges, this does not mean that* $\lim |a_{n+1}/a_n|$ *exists; for example*

$$1 + \frac{1}{3} + \frac{1}{2} + \frac{1}{9} + \frac{1}{4} + \frac{1}{27} + \frac{1}{8} + \frac{1}{81} + \cdots$$

converges absolutely whilst $|a_{n+1}/a_n|$ does not have a limit.

Example 6.26 ***(Exponential Series)*** *For all $x \in \mathbb{C}$, the exponential series*

$$\sum_{0}^{\infty} \frac{x^n}{n!}$$

converges absolutely.

Solution. The case $x = 0$ is trivial. If $x \neq 0$ then

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{|x|}{n+1} \to 0 < 1 \quad \text{as } n \to \infty$$

and apply the ratio test. ■

Example 6.27 *The series*

$$\sum_{1}^{\infty} (\sinh n)\, x^n$$

converges absolutely for $|x| < e^{-1}$ and diverges for $|x| \geqslant e^{-1}$.

Solution. By definition $\sinh n = (e^n - e^{-n})/2$ and so

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \frac{\sinh (n+1)}{\sinh n} |x| \\ &= \frac{e^{n+1} - e^{-n-1}}{e^n - e^{-n}} |x| \\ &= \frac{e - e^{-2n-1}}{1 - e^{-2n}} |x| \\ &\to e\,|x| \end{aligned}$$

as $n \to \infty$. If $x = e^{-1}$ then the ratio test is inconclusive but

$$a_n = \sinh n \times e^{-n} \to \frac{1}{2} \neq 0$$

and so the series does not converge. ■

Theorem 6.28 ***(Leibniz Alternating Series Test, 1676)*** *Let (a_n) be a non-negative decreasing series which tends to 0. Then*

$$\sum_{n=0}^{\infty} (-1)^n a_n$$

converges.

Proof. If we consider the partial sums $s_n = \sum_{k=0}^{n} (-1)^k a_k$ we see that

$$\begin{aligned} s_{2k} &= \underbrace{(a_0 - a_1)}_{\geqslant 0} + \underbrace{(a_2 - a_3)}_{\geqslant 0} + \cdots + \underbrace{(a_{2k-2} - a_{2k-1})}_{\geqslant 0} + a_{2k} \\ &= a_0 + \underbrace{(-a_1 + a_2)}_{\leqslant 0} + \underbrace{(-a_3 + a_4)}_{\leqslant 0} + \cdots + \underbrace{(-a_3 + a_4)}_{\leqslant 0} \\ &\leqslant a_0. \end{aligned}$$

Hence s_{2k} is an increasing sequence bounded above by a_0 and so s_{2k} converges to a limit L. We also have

$$s_{2k+1} = s_{2k} - a_{2k+1} \to L - 0 = L$$

by AOL. Hence s_k converges to L by Sheet 5, Exercise 4(i). ∎

Remark 6.29 *Nothing we have done so far lets us tackle series like* $\sum_2^\infty \frac{1}{n(\log n)^2}$*, to evaluate* $\sum_1^\infty \frac{(-1)^{n+1}}{n}$ *or define general exponents. In the remainder of this section we deal with these: but in order to do so we need to make use of the properties of integration and logarithms. We will define logarithms and general powers in the next chapter but we will not meet integration rigorously until Analysis III in Trinity. At the end of the year you will be able to persuade yourself that these properties which we now use do not depend on any of the results of this section, and that no circular arguments have been made. Basically, it is just impatience that forces us to deal with this test now and not wait until Trinity Term.*

Theorem 6.30 *Let* $K \in \mathbb{N}$ *and let* $f\colon [K, \infty) \to [0, \infty)$ *be continuous and decreasing. For* $n > K$ *we define*

$$\delta_n = \sum_{K}^{n-1} f(k) - \int_K^n f(x)\, \mathrm{d}x.$$

Then for $n > K$

$$0 \leqslant \delta_n \leqslant \delta_{n+1} \leqslant f(K)$$

and hence δ_n *converges.*

Corollary 6.31 *(**The Integral Test**) With* f *as above, the series* $\sum_K^\infty f(k)$ *is convergent if and only if* $\int_K^n f(x)\, \mathrm{d}x$ *is convergent.*

We postpone the proof for now and instead apply the integral test to a few series.

Example 6.32 $a_n = 1/n^\alpha$ *where* $\alpha \in \mathbb{R}$. *(We will not properly define general exponents until the next chapter.) If* $\alpha \leqslant 0$ *then* a_n *does not tend to* 0 *and so* $\sum a_n$ *diverges. Let* $\alpha > 0$. *Consider the function* $f(x) = x^{-\alpha} \geqslant 0$ *which is continuous and decreasing on* $(0, \infty)$. *We take* $K = 1$ *and note if* $\alpha \neq 1$ *that*

$$\int_1^n f(t)\, \mathrm{d}t = \left[\frac{t^{1-\alpha}}{1-\alpha}\right]_1^n = \frac{n^{1-\alpha} - 1}{1-\alpha}$$

which converges as $n \to \infty$ if $\alpha > 1$ and diverges if $\alpha < 1$. If $\alpha = 1$ then

$$\int_1^n f(t)\,\mathrm{d}t = [\log t]_1^n = \log n$$

which diverges. Hence

$$\sum_1^\infty \frac{1}{n^\alpha}$$

converges when $\alpha > 1$ and diverges for $\alpha \leqslant 1$.

Example 6.33 *$a_n = (n \log n)^{-1}$ for $n \geqslant 2$. Hence we define $f(x) = \frac{1}{x \log x}$ on $(2, \infty)$, and note $f(x)$ is decreasing as x and $\log x$ are increasing. Then*

$$\int_2^n \frac{1}{x \log x}\,\mathrm{d}x = \log\log n - \log\log 2 \to \infty \quad as \quad n \to \infty.$$

Therefore $\sum \frac{1}{n \log n}$ is divergent.

Proof. (Of Theorem 6.30) We set

$$\delta_n = \sum_K^{n-1} f(k) - \int_K^n f(x)\,\mathrm{d}x.$$

In the diagram below, which includes a graph of $y = 1/x$ for $x \geqslant K = 1$ we can see δ_4 as the "excess area" above the graph between $1 \leqslant x \leqslant 4$.

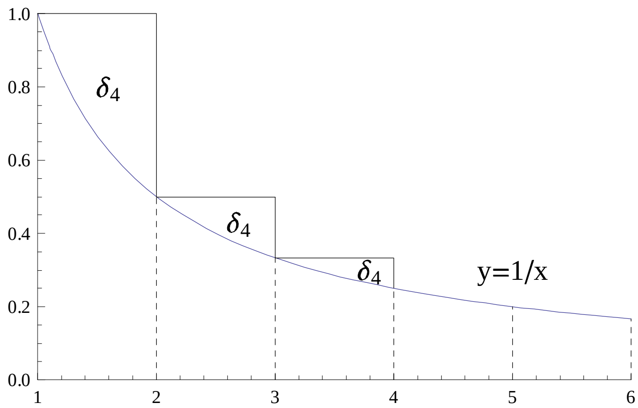

Fig. 6.1 – Proving the Integral Test

As f is decreasing,

$$f(k+1) \leqslant f(x) \leqslant f(k), \text{ if } \ k \leqslant x \leqslant k+1$$

We use the following properties of integration:

- $\int$ preserves weak inequalities;

- $\int_n^{n+1} 1 \, \mathrm{d}x = 1$;
- $\int$ is additive: $\int_a^b = \int_a^c + \int_c^b$;
- $\int$ is a linear map on the space of integrable functions.

So we get:

$$f(k+1) \leqslant \int_k^{k+1} f(x) \, \mathrm{d}x \leqslant f(k)$$

and we can add such equations to get

$$f(K+1) + f(K+2) + \cdots + f(n) \leqslant \int_K^n f(t) \, \mathrm{d}t \leqslant f(K) + f(K+1) + \cdots + f(n-1).$$

So using the second inequality above we have

$$0 \leqslant \sum_{r=K}^{n-1} f(r) - \int_K^n f(t) \, \mathrm{d}t$$

which shows $0 \leqslant \delta_n$. Using the first inequality we have

$$\delta_n = \sum_{r=K}^{n-1} f(r) - \int_K^n f(t) \, \mathrm{d}t \leqslant \sum_{r=K}^{n-1} f(r) - \sum_{r=K+1}^{n} f(r) = f(K) - f(n) \leqslant f(K).$$

We also have

$$\delta_{n+1} - \delta_n = f(n) - \int_n^{n+1} f(t) \, \mathrm{d}t \geqslant 0.$$

Hence (δ_n) is bounded above, increasing and so convergent.

Finally

$$\sum_K^{n-1} f(k) \quad \text{and} \quad \int_K^n f(x) \, \mathrm{d}x$$

differ by a convergent sequence. Therefore they both converge or both diverge by AOL. ■

Example 6.34 *(**Euler's Constant** γ, **1734**) If we apply Theorem 6.30 to $f(x) = 1/x$ we get*

$$\begin{aligned} \gamma_n &= 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \\ &= 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \int_1^n \frac{\mathrm{d}x}{x} \\ &= \delta_n + \frac{1}{n} \end{aligned}$$

is convergent. This limit is called Euler's constant, and often denoted as γ:

$$\gamma = \lim_{n \to \infty} \left(\sum_1^n \frac{1}{k} - \log n \right).$$

The approximate numerical value of γ is

$$0.57721566490153286060\ldots$$

Relatively little is known about γ – for example, it is an open problem still as to whether γ is irrational.

Example 6.35 *We make use of γ in Sheet 6, Exercise 6 to show that*

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots=\log 2.$$

Example 6.36 *(**Euler's Number** e) In Sheet 4, Exercise 5, we showed that*

$$e=1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots=\sum_{r=0}^{\infty}\frac{1}{r!}$$

converges to an irrational number e. Its approximate numerical value is

$$2.7182818284590452353\ldots$$

In fact, the constant e had been studied well before Euler, with some interest in the constant shown by Napier, Harriot and Huygens. The constant was explicitly defined by Jacob Bernoulli in 1683 as $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ while investigating 'continuous compounding' but it was Euler who recognized the importance of the constant and its connection with the 'antilogarithm' function.

Proposition 6.37 *(Euler, 1748)*

$$e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Proof. Let

$$\alpha_n=\left(1+\frac{1}{n}\right)^n \quad \text{and} \quad \beta_n=\sum_{k=0}^{n}\frac{1}{k!}.$$

It was shown in Sheet 4, Exercise 5 that $\lim\beta_n$ exists and we defined e as this limit. It was also shown in Sheet 1, Exercise 6, that α_n is an increasing sequence bounded above and so also converges. By the binomial theorem

$$\begin{aligned}\alpha_n &= 1+n\left(\frac{1}{n}\right)+\frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3+\cdots+\frac{1}{n^n}\\ &= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\frac{1}{n}\\ &\leqslant 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}=\beta_n.\end{aligned}$$

From this we have $\lim\alpha_n\leqslant e$. On the other hand for $1\leqslant m<n$ and focusing on the first $m+1$ terms in the binomial expansion of α_n we see

$$1+1+\left(1-\frac{1}{n}\right)\frac{1}{2!}+\cdots+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right)\frac{1}{m!}\leqslant\alpha_n.$$

Fixing m and letting $n \to \infty$ we have, using AOL and recalling that limits respect weak inequalities,

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{m!} \leqslant \lim \alpha_n.$$

Finally letting $m \to \infty$ we have $e \leqslant \lim \alpha_n$ and the result follows. ■

Remark 6.38 *It's important to note why we took the first $m+1$ terms in the binomial expansion of α_n earlier. In that expansion there are $n+1$ terms and so, as n variees, the number of terms varies. AOL applies to a fixed finite number of terms – fixed in the sense of not depending on the variable that's tending. For exampl it's clear*

$$1 = \underbrace{\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}}_{n \text{ times}}.$$

If AOL could be applied to a varying number of terms, letting $n \to \infty$ we would find

$$1 = 0 + 0 + 0 + \cdots = 0,$$

which is false.

Whilst the tests are useful series are not usually met in such a straightforward way that a single convergence test can be employed. If they can be employed at all, some combination of the tests may be needed.

Example 6.39 *Discuss the convergence or divergence of the following series.*

- $$\sum \frac{\cos (n^2 + 1)}{n^2 + \log n}.$$

  We note that

  $$0 \leqslant \left| \frac{\cos (n^2 + 1)}{n^2 + \log n} \right| \leqslant \frac{1}{n^2 + \log n} \leqslant \frac{1}{n^2}$$

  and so the series is AC by comparison with $\sum n^{-2}$.

- $$\sum (-1)^n \frac{\log (n^2 + 1)}{\sqrt{n+2}}$$

  If $y(x) = \log (x^2 + 1) (x+2)^{-1/2}$ then

  $$\begin{aligned} y'(x) &= \frac{1}{\sqrt{x+2}} \frac{2x}{(x^2+1)} - \frac{1}{2(x+2)^{3/2}} \log \left(x^2 + 1\right) \\ &= \frac{1}{(x+2)^{3/2}} \left[\frac{2x(x+2)}{x^2+1} - \frac{1}{2} \log \left(x^2+1\right) \right] \\ &< \frac{1}{(x+2)^{3/2}} \left[4 - \frac{1}{2} \log \left(x^2+1\right) \right] \\ &< 0 \text{ for } x > e^4. \end{aligned}$$

So $y(n)$ is eventually decreasing – using a result from Analysis II and by Leibniz's Test a tail of the series converges. Hence the whole series converges.

- $$\sum \frac{1}{\sqrt{n^2+n}}$$

  We see
  $$\frac{1}{\sqrt{n^2+n}} > \frac{1}{\sqrt{2n^2}} = \frac{1}{n\sqrt{2}}$$
  and so the series diverges by comparison with the harmonic series.

- $$\sum \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}}$$

  Note
  $$\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$$
  and hence
  $$\sum_1^N \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} = 1 - \frac{1}{\sqrt{N+1}} \to 1 \quad \text{as} \quad N \to \infty.$$

Proposition 6.40 ***(Stirling's Approximation, 1730)*** *(Proof off-syllabus) As $n \to \infty$ then*

$$\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} \to 1.$$

This same result is often written as

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Proof. Firstly we note
$$\log n! = \log 2 + \log 3 + \cdots + \log n.$$

We can find a good approximation to the sum on the RHS by applying the trapezium rule to $\log x$ on the interval $[1, n]$. Let $f(x)$ denote the approximating function to $\log x$ whose integral the trapezium rule determines using $n-1$ steps – that is $f(x)$ satisfies $f(k) = \log k$ for each integer $k = 1, 2, \ldots, n$ and is piecewise linear between those values.

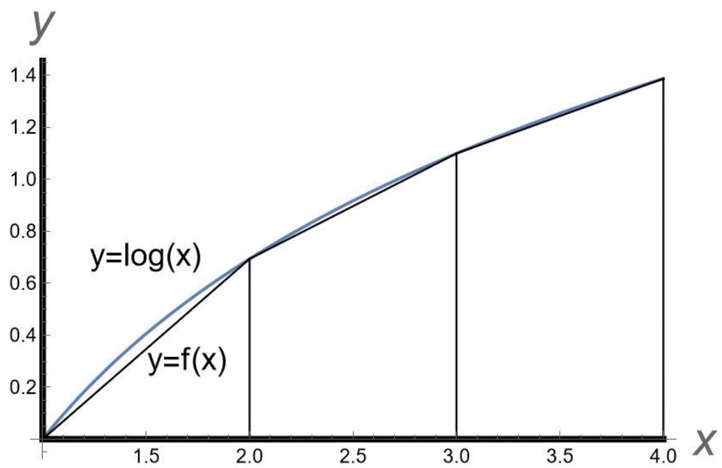

Fig. 6.2 – Trapezium Rule for $\log x$

Note for x in the range $k \leqslant x \leqslant k+1$ we have

$$\frac{1}{k+1} \leqslant \frac{1}{x} \leqslant \frac{1}{k}$$

and so integrating we have

$$\int_k^x \frac{\mathrm{d}t}{k+1} \leqslant \int_k^x \frac{\mathrm{d}t}{t} \leqslant \int_k^x \frac{\mathrm{d}t}{k}$$

or equivalently

$$\log k + \left(\frac{x-k}{k+1}\right) \leqslant \log x \leqslant \log k + \left(\frac{x-k}{k}\right).$$

Now $\log x$ is concave (that is, a chord connecting two points of the graph lies under the graph), and so $f(x) \leqslant \log x$ on the interval $[k, k+1]$. Further as $f'(x) \geqslant (k+1)^{-1}$ on the interval (that being the minimum gradient of $\log x$ whilst $f'(x)$ has the average gradient) we have

$$\log k + \left(\frac{x-k}{k+1}\right) \leqslant f(x) \leqslant \log x \qquad \text{for } k \leqslant x \leqslant k+1.$$

So we have the inequalities

$$0 \leqslant \log x - f(x) \leqslant \left(\frac{1}{k} - \frac{1}{k+1}\right)(x-k) \qquad \text{for } k \leqslant x \leqslant k+1,$$

and integrating on the interval $[k, k+1]$ we find

$$0 \leqslant \int_k^{k+1} (\log x - f(x))\ \mathrm{d}x \leqslant \left(\frac{1}{k} - \frac{1}{k+1}\right) \int_k^{k+1} (x-k)\ \mathrm{d}x = \frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+1}\right).$$

Summing up the contributions from the intervals $[1,2],[2,3],\ldots,[n-1,n]$ we find

$$0 \leqslant \int_1^n (\log x - f(x))\ \mathrm{d}x \leqslant \frac{1}{2}\sum_{k=1}^{n-1}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{1}{2}\left(1 - \frac{1}{n}\right),$$

as most of the terms in the above sum cancel consecutively.

Recalling an antiderivative of $\log x$ to be $x\log x - x$ and using the formula for the trapezium rule we then have

$$\begin{aligned}
I_n &= \int_1^n (\log x - f(x))\ \mathrm{d}x \\
&= [x\log x - x]_1^n - 1\left(\frac{\log 1}{2} + \log 2 + \log 3 + \cdots + \log(n-1) + \frac{\log n}{2}\right) \\
&= n\log n - n + 1 - \left(\log n! - \frac{1}{2}\log n\right) \\
&= \left(n + \frac{1}{2}\right)\log n - n + 1 - \log n!.
\end{aligned}$$

So we have

$$0 \leqslant \int_1^n (\log x - f(x))\ \mathrm{d}x = \left(n + \frac{1}{2}\right)\log n - n + 1 - \log n! \leqslant \frac{1}{2}\left(1 - \frac{1}{n}\right).$$

(I_n) is an increasing sequence of numbers which we see are bounded above by $1/2$ and hence they converge to some L.

Applying the exponential function we find

$$e^{L-1} = \lim \frac{(n/e)^n\sqrt{n}}{n!}.$$

Whilst in Sheet 6, Exercise 11(ii) , we proved

$$\binom{2n}{n}\frac{\sqrt{n}}{2^{2n}} \to \frac{1}{\sqrt{\pi}}.$$

We can combine these facts to note

$$\begin{aligned}
\frac{1}{\sqrt{\pi}} &= \lim \binom{2n}{n}\frac{\sqrt{n}}{2^{2n}} \\
&= \lim \frac{(2n)!}{n!n!}\frac{\sqrt{n}}{2^{2n}} \\
&= \sqrt{2} \times \lim\left(\frac{(2n!)}{\sqrt{2n}\left(\frac{2n}{e}\right)^{2n}}\right) \times \lim\left(\frac{\sqrt{n}\left(\frac{n}{e}\right)^n}{n!}\right)^2 \\
&= \sqrt{2} \times \left(e^{L-1}\right)^{-1} \times \left(e^{L-1}\right)^2 \\
&= \sqrt{2}e^{L-1}
\end{aligned}$$

Hence $e^{1-L} = \sqrt{2\pi}$ and

$$\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} \to \frac{e^{1-L}}{\sqrt{2\pi}} = 1.$$

■

Remark 6.41 *In terms of relative error, Stirling's formula is a very accurate underestimate. For $n = 10$ the relative error is under 1%.*

An improvement on the above approximation is

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right)$$

and there are yet more accurate approximations

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\left\{\sum_{k=2}^{m} \frac{(-1)^k B_k}{k(k-1)n^{k-1}} + O\left(\frac{1}{n^m}\right)\right\},$$

where B_k is the kth Bernoulli number (see Sheet 7, Exercise 9).