7. POWER SERIES

7.1 The Disc and Radius of Convergence

Definition 7.1 *By a **power series** we will mean a series of the form*

$$\sum_{n=0}^{\infty} a_n z^n$$

where $(a_n)_0^\infty$ is a complex sequence and $z \in \mathbb{C}$. We consider (a_n) as fixed for this series, and z as a variable. Clearly the series might converge for some values of z and not for others.

Remark 7.2 *The above power series is a power series centred at the origin. Given $z_0 \in \mathbb{C}$ then we can also consider power series centred at z_0, which take the form*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Though we will not consider such power series in this chapter, the theory we will develop literally translates to an identical theory for power series centred at $z_0 \neq 0$.

Example 7.3

- *$a_n = 1$: $\sum_0^\infty z^n$: **Geometric series** : as we have already seen (Example 6.6), this series is convergent when $|z| < 1$ and divergent when $|z| \geqslant 1$.*
- *$a_n = 1/n!$: $\sum_0^\infty z^n/n!$: **Exponential series** : we have shown (Example 6.26) that this series is convergent for all $z \in \mathbb{C}$.*
- *$a_n = 1/n$: $\sum_{n=1}^\infty z^n/n$: **Logarithmic series** : convergent for $|z| < 1$. This follows from the ratio test as*

  $$\left| \frac{z^{n+1}/(n+1)}{z^n/n} \right| = \frac{n\,|z|}{n+1} \to |z|\,.$$

  The series converges at $z = -1$ (Leibniz test and Sheet 6, Exercise 6) and diverges at $z = 1$ (as it's the harmonic series). What about for other values where $|z| = 1$? Well at $z = i$ we have

  $$\sum_1^{2N} \frac{i^n}{n} = \left(-\frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \cdots + \frac{(-1)^N}{2N} \right) + i \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{N-1}}{2N-1} \right)$$

  and we see that both real and imaginary parts converge by the Leibniz test. In fact we know the above partial sums to converge to $-\frac{1}{2}\log 2 + \frac{i\pi}{4}$ (Sheet 6, Exercise 6 and Sheet 5, Exercise 6). More generally it can be shown that the logarithmic series converges on the circle $|z| = 1$ except at $z = -1$.

- $\left.\begin{array}{rcl} a_{2n} & = & (-1)^n/(2n)! \\ a_{2n+1} & = & 0 \end{array}\right\} : \sum \frac{(-1)^n}{(2n)!} z^{2n}$: ***Cosine series*** : *convergent for all z by the ratio test.*
- $\left.\begin{array}{rcl} a_{2n} & = & 0 \\ a_{2n+1} & = & (-1)^n/(2n+1)! \end{array}\right\} : \sum \frac{(-1)^n}{(2n+1)!} z^{2n+1}$: ***Sine series*** : *convergent for all z by the ratio test again.*

Definition 7.4 *Given a power series $\sum a_n z^n$ the set*

$$S = \left\{ z \in \mathbb{C} \mid \sum a_n z^n \text{ converges} \right\} \subseteq \mathbb{C}$$

*is either bounded or unbounded. Note also, that S is non-empty as $0 \in S$. We define the power series' **radius of convergence** R as*

$$R = \begin{cases} \sup \{|z| \mid z \in S\} & \text{when } S \text{ is bounded,} \\ \infty & \text{when } S \text{ is unbounded.} \end{cases}$$

Lemma 7.5 *Suppose that the power series $\sum a_n (z_0)^n$ converges. Then $\sum a_n z^n$ converges absolutely when $|z| < |z_0|$.*

Proof. As $\sum a_n (z_0)^n$ converges then $a_n (z_0)^n \to 0$ and in particular the sequence $a_n (z_0)^n$ is bounded; say $|a_n (z_0)^n| < M$ for all n. Then, for $|z| < |z_0|$,

$$|a_n z^n| = |a_n (z_0)^n| \left| \frac{z}{z_0} \right|^n < M \left| \frac{z}{z_0} \right|^n$$

and so $\sum |a_n z^n|$ converges by comparison with the convergent geometric series $\sum M |z/z_0|^n$. ■

Theorem 7.6 *Given a power series $\sum a_n z^n$ with radius of convergence R,*

- *$\sum a_n z^n$ is AC when $|z| < R$,*
- *$\sum a_n z^n$ diverges when $|z| > R$.*

Note that when $R = \infty$ then $\sum a_n z^n$ converges absolutely for all $z \in \mathbb{C}$.

Proof. If $|z| < R$ then, by the approximation property, $|z| < |z_0| < R$ for some $z_0 \in S$ and hence $\sum a_n z^n$ is AC by the previous lemma. On the other hand if $|z| > R$ then $z \notin S$ and hence $\sum a_n z^n$ diverges. ■

Definition 7.7 *The set S is called the **disc of convergence**.*

Remark 7.8 *So a power series is AC strictly within its radus of convergence and diverges strictly beyond the disc of convergence. For z on the boundary $|z| = R$ the series may converge or diverge. It's quite easy to construct power series that converge at only finitely many points of the boundary, or power series that converge everywhere on the boundary except finitely many points. The general question – for which subsets of $|z| = R$ is there a power series which converges exactly on that subset? – remains an open problem.*

Remark 7.9 *Commonly we will use the ratio test to determine the radius of convergence, but it is not hard to produce examples where the ratio test can not be employed, or at least has to be used more subtly. See the third example below.*

Remark 7.10 *(Off-syllabus) As a consequence of Cauchy's root test (Sheet 6, Exercise 8(i)) an exact formula for the radius of convergence is*

$$R = \left(\limsup \sqrt[n]{|a_n|}\right)^{-1}. \tag{7.1}$$

Example 7.11 *Find the radius of convergence of the following examples, and consider the series' convergence on the disc's boundary.*

- $\sum_1^\infty z^n/n^2$. *If we set* $a_n = z^n/n^2$ *then*

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|z|^{n+1}/(n+1)^2}{|z|^n/n^2} = \left(1+\frac{1}{n}\right)^2 |z| \to |z|.$$

  Hence, by the ratio test the series converges absolutely when $|z| < 1$ *but diverges when* $|z| > 1$. *In fact, by comparison with* $\sum n^{-2}$ *we see that the series is AC when* $|z| = 1$.

- $\sum z^n/n$. *If we set* $a_n = z^n/n$ *we can argue as above to see* $R = 1$. *This is the logarithmic series and we have commented that it diverges at* $z = 1$ *and otherwise converges on* $|z| = 1$.

- $\sum z^p$ *where the sum is taken over all primes* p. *Then* $R = 1$. *To see this we can note* z^p *does not tend to* 0 *when* $|z| \geqslant 1$. *On the other hand* $\sum z^p$ *is AC when* $|z| < 1$ *by comparison with the geometric series* $\sum z^n$.

- *Cosine series:* $\sum_0^\infty (-1)^n z^{2n}/(2n)!$ *If we set* $a_n = (-1)^n z^{2n}/(2n)!$ *then, for all* $z \in \mathbb{C}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|z|^{2n+2}/(2n+2)!}{|z|^{2n}/(2n)!} = \frac{|z|^2}{(2n+2)(2n+1)} \to 0 \quad \text{as } n \to \infty.$$

  Hence by the ratio test the cosine series is AC for all z.

- *Sine series:* $\sum_0^\infty (-1)^n z^{2n+1}/(2n+1)!$ *If we set* $a_n = (-1)^n z^{2n+1}/(2n+1)!$ *then, for all* $z \in \mathbb{C}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|z|^{2n+3}/(2n+3)!}{|z|^{2n+1}/(2n+1)!} = \frac{|z|^2}{(2n+2)(2n+3)} \to 0 \quad \text{as } n \to \infty.$$

  Hence by the ratio test the sine series is AC for all z.

Example 7.12 *Use (7.1) to determine the radii of convergence of the series*

$$\sum_1^\infty \frac{z^n}{n}, \qquad \sum_{prime\ p} z^p, \qquad \sum_0^\infty \frac{z^n}{n!}.$$

Solution.

- $a_n = \frac{1}{n}$. Now $n^{1/n} \to 1$ (Sheet 3, Exercise 3) and so

$$\limsup \sqrt[n]{|a_n|} = \lim \left(\sqrt[n]{n}\right)^{-1} = 1$$

  so that $R = 1$.

- $a_p = 1$. As there are infinitely many primes then $\limsup \sqrt[n]{|a_n|} = \limsup 1 = 1$ and so $R = 1$.

- $a_n = 1/n!$. By Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

  and hence

$$\limsup \sqrt[n]{|a_n|} = \limsup \frac{1}{\sqrt[n]{n!}} = \lim \frac{1}{\sqrt[2n]{2\pi n}}\left(\frac{e}{n}\right) = 0,$$

  giving $R = \infty$.

■

The following theorem is beyond the scope of this course, but will be proved in Hilary term. This theorem will prove very useful when proving various properties of the elementary functions in the next section.

Theorem 7.13 ***(Term-by-term differentiation)*** *Suppose the (real or complex) power series $\sum_0^\infty a_n z^n$ has radius of convergence R. Then the power series defines a differentiable function on $|z| < R$.*

Term-by-term differentiation is valid within $|z| < R$ so that

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\sum_0^\infty a_n z^n\right) = \sum_1^\infty n a_n z^{n-1} = \sum_0^\infty (n+1)\, a_{n+1} z^n.$$

*The power series $\sum_0^\infty (n+1)\, a_{n+1} z^n$ is called the **derived series** and also has radius of convergence R.*

Remark 7.14 ***(Uniqueness of Coefficients)*** *Say that a function $f(x) = \sum_0^\infty a_n x^n$ is defined on the interval $|x| < R$. By repeated differentiation we see that*

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

*So, if a function is locally defined by a power series, that is $f(x)$ is **analytic**, then the coefficients a_n are unique.*

As a corollary to this, if an analytic function satisfies $f'(x) = 0$ for all x then $a_n = 0$ for all $n \geqslant 1$, and $f(x) = a_0$ is constant.

Remark 7.15 ***(Existence of Coefficients)*** *A real function is said to be analytic (at* 0*) if it can be locally defined by a power series on some* $(-R,R)$ *. As we may differentiate term-by-term, then* $f(x)$ *is necessarily* ***smooth*** *– that is,* $f(x)$ *has derivatives of all orders. Unfortunately smoothness is not a sufficient condition though. For example, the function*

$$f(x) = \begin{cases} \exp\left(-1/x^2\right) & x \neq 0, \\ 0 & x = 0, \end{cases}$$

can be shown to have derivatives of all orders at $x = 0$ *with* $f^{(n)}(0) = 0$ *for all* $n \geqslant 0$*. So if* $f(x)$ *could be defined by a power series on some* $(-R,R)$ *then we'd have*

$$f(x) = \sum_0^\infty \frac{f^{(n)}(0)}{n!} x^n = 0,$$

but $f(x) \neq 0$ *except at* $x = 0$*. So* $f(x)$ *is smooth, but isn't analytic.*

In the Part A Complex Analysis course, you will see that the situation is very different for complex functions. A complex function which is differentiable (just once!) on an open disc about the origin will be analytic.

Proposition 7.16 *Let* $f(x) = \sum_0^\infty a_n x^n$ *converge on* $(-R,R)$ *.*

(a) $f(x)$ *is an even function if and only if* $a_{2n+1} = 0$ *for each* $n \geqslant 0$*.*

(b) $f(x)$ *is an odd function if and only if* $a_{2n} = 0$ *for each* $n \geqslant 0$*.*

Proof. (a) If $a_{2n+1} = 0$ for each $n \geqslant 0$, then

$$f(x) = \sum_{n=0}^\infty a_{2n} x^{2n} = \sum_{n=0}^\infty a_{2n} (-x)^{2n} = f(-x)$$

is even. Conversely say that $f(x)$ is even. Then $f^{(n)}(x)$ is even when n is even and odd when n is odd – these facts follow from the chain rule. So $f^{(2n+1)}(x)$ is odd and in particular $f^{(2n+1)}(0) = 0$. Hence

$$a_{2n+1} = \frac{f^{(2n+1)}(0)}{(2n+1)!} = 0$$

as required. The proof of (b) is almost identical. ■

7.2 The Elementary Functions

The *elementary functions* include polynomials, rational functions, exponentials, logarithms and trigonometric functions. In contrast there are *special functions* such as Bessel functions (Sheet 7, Exercise 4), Gauss's *error function,* the *gamma function,* etc. and there are deep theorems showing the special functions cannot be expressed in terms of the elementary functions.

In this section we give rigorous definitions for exponentials, logarithms, general exponents and the trigonometric and hyperbolic functions.

Definition 7.17 *(**Exponential Function**) The exponential function* $\exp\colon \mathbb{C} \to \mathbb{C}$ *is defined by the infinite series*

$$\exp(z) = \sum_{0}^{\infty} \frac{z^n}{n!}.$$

1. For all $z \in \mathbb{C}$, $\sum z^n/n!$ is convergent by the ratio test (Example 6.26): so $R = \infty$.
2. $\exp(0) = 1$
3. $\exp(1) = e$
4. $\exp'(z) = \exp(z)$.

   Proof. We use Theorem 7.13 for this. Note

   $$\frac{\mathrm{d}}{\mathrm{d}z}\exp z = \frac{\mathrm{d}}{\mathrm{d}z}\left(\sum_{0}^{\infty}\frac{z^n}{n!}\right) = \sum_{0}^{\infty}\frac{(n+1)\,z^n}{(n+1)!} = \sum_{0}^{\infty}\frac{z^n}{n!} = \exp z.$$

   ■
5. $\exp(x+y) = \exp(x)\exp(y)$.

   Proof. We proved this in Example 6.19. We can also use Theorem 7.13 to show this: for fixed $c \in \mathbb{C}$ we define

   $$F(z) = \exp(z+c)\exp(-z).$$

   By the product rule

   $$F'(z) = \exp(z+c)\exp(-z) - \exp(z+c)\exp(-z) = 0.$$

   So $F(z)$ is constant by Remark 7.14 and, as $F(0) = \exp(c)$, then

   $$\exp(z+c)\exp(-z) = \exp(c) \qquad \text{for all } z \in \mathbb{C}.$$

   Set $c = x+y$ and $z = -y$ for the required result. ■
6. $\exp(z) \neq 0$. (In fact we will see below that the image of exp is $\mathbb{C}\backslash\{0\}$.)

   Proof. For any $z \in \mathbb{C}$ we have

   $$\exp(z)\exp(-z) = \exp(0) = 1.$$

   ■
7. $\exp(q) = e^q$ for rational q.

   Proof. Say $q = m/n$ then

   $$\left(\exp\left(\frac{m}{n}\right)\right)^n = \exp\left(n\frac{m}{n}\right) = \exp(m \times 1) = (\exp 1)^m = e^m.$$

   By the uniqueness of positive nth roots, we have $\exp(q) = \exp(m/n) = \sqrt[n]{e^m} = e^q$. ■

It seems appropriate to make the following definitions here, though some of what follows requires theory from Hilary Term. We now restrict our attention to the real exponential $\exp\colon \mathbb{R} \to \mathbb{R}$. It is clear from the power series definition of exp that $\exp x > 0$ if $x \geqslant 0$. Further if $x < 0$ then

$$\exp(x) = \frac{1}{\exp(-x)} > 0$$

also. So $\exp(\mathbb{R}) \subseteq (0, \infty)$.

Now $\exp' x = \exp x > 0$ and so exp is an increasing function; in particular this means that $\exp\colon \mathbb{R} \to (0,\infty)$ is injective. Also for $x > 0$, $\exp x > x$ and so exp takes arbitrarily large values of x and similarly $\exp(-x) = 1/\exp(x)$ takes arbitrarily small positive values. So, by the Intermediate Value Theorem (proved in HT), we have

- $\exp\colon \mathbb{R} \to (0,\infty)$ is a bijection and hence invertible.
- The inverse is denoted as $\log\colon\ (0,\infty) \to \mathbb{R}$, and by a HT result log is differentiable.

Definition 7.18 *The* ***natural logarithm*** $\log x$*, or* $\ln x$*, is the inverse of the real exponential function* $\exp\colon \mathbb{R} \to (0,\infty)$.

Proposition 7.19 *For* $x > 0$,

$$\log' x = \frac{1}{x}.$$

Proof. As $\exp(\log x) = x$ on $(0.\infty)$ then, by the chain rule,

$$\log'(x) \times \exp(\log x) = 1$$

and the result follows. ■

Example 7.20 *The image of* $\exp\colon \mathbb{C} \to \mathbb{C}$ *is* $\mathbb{C}\backslash\{0\}$.

Solution. We previously showed 0 is not in the image. Take $z = re^{i\theta} \neq 0$. We need to find $w = x + iy \in \mathbb{C}$ such that $\exp(w) = z$. This means $e^x e^{iy} = re^{i\theta}$. Setting

$$x = \log r \qquad \text{and} \qquad y = \theta,$$

gives one solution to $\exp(w) = z$. ■

Definition 7.21 *(**General Exponents**) Given* $a > 0$ *and* $x \in \mathbb{R}$*, we define*

$$a^x = \exp(x \log a).$$

Note, with this definition,

$$e^x = \exp x \textit{ for } x \in \mathbb{R}.$$

Proposition 7.22 *Let* $a, b > 0$ *and* $x \in \mathbb{R}$*. Then*

$$\log(ab) = \log a + \log b, \qquad \log(a^x) = x \log a.$$

Proof. Note

$$\exp(\log a + \log b) = \exp(\log a)\exp(\log b) = ab = \exp(\log(ab))$$

and then take the log of both sides. Also

$$\log(a^x) = \log(\exp(x \log a)) = x \log a.$$

■

Proposition 7.23 *Let $a > 0$ and $x, y \in \mathbb{R}$. Then*

$$a^{x+y} = a^x a^y, \qquad (a^x)^y = a^{(xy)}.$$

Proof. Note

$$\begin{aligned} a^{x+y} &= \exp((x+y)\log a) \\ &= \exp((x \log a) + (y \log a)) \\ &= \exp(x \log a)\exp(y \log a) \\ &= a^x a^y. \end{aligned}$$

Also

$$\log(a^x)^y = y \log(a^x) = y(x \log a) = (xy)\log a = \log\left(a^{(xy)}\right)$$

and apply the expoential to both sides. ■

Proposition 7.24 *For $x > 0$ and real a,*

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^a) = a x^{a-1}.$$

Proof. By the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^a) = \frac{\mathrm{d}}{\mathrm{d}x}(\exp(a \log x)) = \frac{a}{x}\exp(a \log x) = a x^{-1} x^a = a x^{a-1}.$$

■

Definition 7.25 ***(The Trigonometric and Hyperbolic Functions)***

1. For all $z \in \mathbb{C}$ we define *cosine* and *sine* by

$$\cos z = \frac{\exp(iz) + \exp(-iz)}{2}, \qquad \sin z = \frac{\exp(iz) - \exp(-iz)}{2i}$$

2. Then
$$\cos z = \sum_0^\infty \frac{(-1)^n z^{2n}}{(2n)!} \qquad \text{and} \qquad \sin z = \sum_0^\infty \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$
with these series converging for all $z \in \mathbb{C}$.

Proof.
$$\begin{aligned}
\frac{\exp(iz) + \exp(-iz)}{2} &= \frac{1}{2}\sum_0^\infty \frac{(i^n + (-i)^n)}{n!} z^n \\
&= \frac{1}{2}\sum_{k=0}^\infty \frac{(-1)^k + (-1)^k}{(2k)!} z^{2k} = \sum_0^\infty \frac{(-1)^k z^{2k}}{(2k)!}; \\
\frac{\exp(iz) - \exp(-iz)}{2i} &= \frac{1}{2i}\sum_0^\infty \frac{(i^n - (-i)^n)}{n!} z^n \\
&= \frac{1}{2i}\sum_{k=0}^\infty \frac{(-1)^k i + (-1)^k i}{(2k+1)!} z^{2k+1} = \sum_0^\infty \frac{(-1)^k z^{2k+1}}{(2k+1)!}.
\end{aligned}$$

■

3. $\cos 0 = 1$, $\sin 0 = 0$.
4.
$$\exp(iz) = \cos z + i \sin z.$$

Proof.
$$\cos z + i\sin z = \left(\frac{\exp(iz) + \exp(-iz)}{2}\right) + i\left(\frac{\exp(iz) - \exp(-iz)}{2i}\right) = \exp(iz).$$

■

5. $\cos z = \cos(-z)$ and $\sin z = -\sin(-z)$.

Proof. The powers series of cos (resp. sin) involves only even (resp. odd) powers. ■

6.
$$\cos'(z) = -\sin z \qquad \text{and} \qquad \sin'(z) = \cos z.$$

Proof. Using $\exp' z = \exp z$ then
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}(\cos z) &= \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\exp(iz) + \exp(-iz)}{2}\right) = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{i\exp(iz) - i\exp(-iz)}{2}\right) = -\sin z, \\
\frac{\mathrm{d}}{\mathrm{d}z}(\sin z) &= \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\exp(iz) - \exp(-iz)}{2i}\right) = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\exp(iz) + \exp(-iz)}{2}\right) = \cos z.
\end{aligned}$$
or we can just calculate the derived series of $\cos z$ and $\sin z$. ■

7.

$$\begin{aligned}\sin(z+w) &= \sin z\cos w+\cos z\sin w,\\ \cos(z+w) &= \cos z\cos w-\sin z\sin w.\end{aligned}$$

Proof. By definition $\sin z\cos w+\cos z\sin w$ equals

$$\begin{aligned}&\left(\frac{\exp(iz)-\exp(-iz)}{2i}\right)\left(\frac{\exp(iw)+\exp(-iw)}{2}\right)\\ &+\left(\frac{\exp(iz)+\exp(-iz)}{2}\right)\left(\frac{\exp(iw)-\exp(-iw)}{2i}\right)\end{aligned}$$

which rearranges to

$$\begin{aligned}&\frac{1}{4i}\left[2\exp(iz)\exp(iw)-2\exp(-iz)\exp(-iw)\right]\\ =\ &\frac{\exp(iz+iw)-\exp(-iz-iw)}{2i}\\ =\ &\sin(z+w).\end{aligned}$$

The second identity can be proved in a similar manner or by differentiating the first identity with respect to z. ■

8.

$$\cos^2 z+\sin^2 z=1$$

Proof. Set $z=-w$ in the previous identity for $\cos(z+w)$. Alternatively, differentiating gives

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\cos^2 z+\sin^2 z\right)=-2\cos z\sin z+2\sin z\cos z=0.$$

So $\cos^2 z+\sin^2 z$ is constant by Remark 7.14 and takes value $1^2+0^2=1$ at $z=0$. Or we can argue

$$\cos^2 z+\sin^2 z=(\cos z+i\sin z)(\cos z-i\sin z)=\exp(iz)\exp(-iz)=\exp(iz-iz)=1.$$

■

9. It is easy to note that $\cos 0=1$ and that

$$\begin{aligned}\cos 2 &= \sum_0^\infty\frac{(-1)^n\,2^{2n}}{(2n)!}\\ &= 1-\frac{2^2}{2!}+\frac{2^4}{4!}-\frac{2^6}{6!}+\frac{2^8}{8!}-\cdots\\ &= \underbrace{1-2+\frac{2}{3}}_{<0}-\underbrace{\frac{2^6}{6!}\left(1-\frac{2^2}{7\times 8}\right)}_{>0}-\underbrace{\frac{2^{10}}{10!}\left(1-\frac{2^2}{11\times 12}\right)}_{>0}-\cdots\\ &< 0.\end{aligned}$$

It follows from theorems we will meet in Hilary Term that there exists a smallest positive root to the equation $\cos x=0$. We will define $\pi/2$ as the smallest root of cosine.

10. As $\cos^2 z + \sin^2 z = 1$ then $\sin(\pi/2) = \pm 1$ (in fact it equals 1 as we know) and

$$\exp(\pi i/2) = \cos(\pi/2) + i\sin(\pi/2) = \pm i.$$

Then

$$\exp(z + 2\pi i) = \exp z\left(\exp\frac{\pi i}{2}\right)^4 = (\exp z)(\pm i)^4 = \exp z.$$

Hence exp has period $2\pi i$ and cosine and sine have period 2π – i.e.

$$\cos(z + 2\pi) = \cos(z) \quad \text{and} \quad \sin(z + 2\pi) = \sin(z).$$

11. For the other trigonometric functions we define

$$\begin{aligned} \sec x &= \frac{1}{\cos x}, \qquad \tan x = \frac{\sin x}{\cos x}, \\ \csc x &= \frac{1}{\sin x}, \qquad \cot x = \frac{\cos x}{\sin x}. \end{aligned}$$

12. We define *hyperbolic cosine* and and *hyperbolic sine* by

$$\begin{aligned} \cosh z &= \frac{\exp(z) + \exp(-z)}{2} = \sum_0^\infty \frac{z^{2n}}{(2n)!}, \\ \sinh z &= \frac{\exp(z) - \exp(-z)}{2} = \sum_0^\infty \frac{z^{2n+1}}{(2n+1)!}, \end{aligned}$$

where these series converge for all $z \in \mathbb{C}$. Note that

$$\begin{aligned} \cos iz &= \cosh z, \qquad \cosh iz = \cos z, \\ \sin iz &= i\sinh z, \qquad \sinh iz = i\sin z, \\ \cosh(-z) &= \cosh z, \qquad \sinh(-z) = -\sinh z, \\ \cosh' z &= \sinh z, \qquad \sinh' z = \cosh z, \\ \sin(x + iy) &= \sin x\cosh y + i\cos x\sinh y, \\ \cos(x + iy) &= \cos x\cosh y - i\sin x\sinh y, \\ \cosh^2 z - \sinh^2 z &= 1. \end{aligned}$$

13. (*Inverse hyperbolic functions*) (a) Let $x \in \mathbb{R}$. Then

$$\sinh^{-1} x = \log\left(x + \sqrt{x^2 + 1}\right).$$

(b) Let $x \geqslant 1$

$$\cosh^{-1} x = \log\left(x + \sqrt{x^2 - 1}\right)$$

(c) Let $-1 < x < 1$. Then

$$\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right).$$

Proof. (a) We need to solve

$$\frac{e^y - e^{-y}}{2} = x \quad \Longleftrightarrow \quad e^{2y} - 2xe^y - 1 = 0 \quad \Longleftrightarrow \quad e^y = x \pm \sqrt{x^2+1}.$$

Only one of the options on the RHS (the plus option) is positive, so

$$y = \sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right).$$

Both (b) and (c) can be solved similarly by creating a quadratic in e^y. ■

Example 7.26 *(See also Sheet 7, Exercise 9.) Find the power series for* $\tan z$ *up to the* z^5 *term.*

Solution. As $\tan z$ is odd then we only have to calculate coefficients for z, z^3 and z^5 (Proposition 7.16). One approach would involve differentiating repeatedly, but we would then need to calculate the fifth derivative of $\tan z$. Instead we will use the binomial theorem.

Recall that

$$\sin z = z - \frac{z^3}{6} + \frac{z^5}{120} + O(z^7), \qquad \cos z = 1 - \frac{z^2}{2} + \frac{z^4}{24} + O(z^6),$$

so that

$$\tan z = \frac{z - \frac{z^3}{6} + \frac{z^5}{120} + O(z^7)}{1 - \frac{z^2}{2} + \frac{z^4}{24} + O(z^6)}.$$

By the binomial theorem, $(1-y)^{-1} = \sum_0^\infty y^n$ for $|y| < 1$, for suitably small z,

$$\begin{aligned}
&\left(1 - \left(\frac{z^2}{2} - \frac{z^4}{24} + O(z^6)\right)\right)^{-1} \\
= \ & 1 + \left(\frac{z^2}{2} - \frac{z^4}{24} + O(z^6)\right) + \left(\frac{z^2}{2} - \frac{z^4}{24} + O(z^6)\right)^2 + O(z^6) \\
= \ & 1 + \left(\frac{z^2}{2} - \frac{z^4}{24}\right) + \left(\frac{z^4}{4}\right) + O(z^6) \\
= \ & 1 + \frac{z^2}{2} + \frac{5z^4}{24} + O(z^6).
\end{aligned}$$

So

$$\begin{aligned}
\tan z &= \left(z - \frac{z^3}{6} + \frac{z^5}{120} + O(z^7)\right)\left(1 + \frac{z^2}{2} + \frac{5z^4}{24} + O(z^6)\right) \\
&= z + \left(\frac{1}{2} - \frac{1}{6}\right) z^3 + \left(\frac{1}{120} - \frac{1}{12} + \frac{5}{24}\right) z^5 + O(z^7) \\
&= z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + O(z^7).
\end{aligned}$$

■

Remark 7.27 *Note that many of the properties of complex sine and cosine differ significantly from their real counterparts. Whilst*

$$\cos^2 z + \sin^2 z = 1$$

is true for all z, this does not mean that $|\cos z| \leqslant 1$ or $|\sin z| \leqslant 1$. In fact, $\cos\colon \mathbb{C} \to \mathbb{C}$ *and* $\sin\colon \mathbb{C} \to \mathbb{C}$ *are onto, and so in, particular, are unbounded.*

Given $w \in \mathbb{C}$ then

$$\begin{aligned} \sin z = w \qquad &\iff \qquad \frac{\exp(iz) - \exp(-iz)}{2i} = w \\ &\iff \qquad \exp(iz)^2 - 2iw\exp(iz) - 1 = 0 \\ &\iff \qquad \exp(iz) = i \pm \sqrt{1 - w^2}. \end{aligned}$$

Recall $\exp$ *takes all values except* 0. *So unless $w^2 = 1$, the RHS represents two distinct complex numbers, so there is a solution z to at least one of the two equations. And if $w^2 = 1$ as $i \neq 0$ then* $\exp(iz) = i$ *has a solution. (See also Example 7.28.)*

Also, whilst $\exp(iz) = \cos z + i\sin z$ *is true for all complex z, it's not generally true that* $\cos z = \operatorname{Re}(\exp(iz))$.

Example 7.28 *Find all the solutions to* $\sin z = 2$.

Solution. Set $z = x + iy$ so that

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y = 2.$$

Comparing real and imaginary parts, we have

$$\sin x \cosh y = 2, \qquad \cos x \sinh y = 0.$$

If $y = 0$ then $\sin x = 2$ which has no solutions. Hence $\cos x = 0$ and $x = (2n+1)\frac{\pi}{2}$. Then

$$2 = \sin\left((2n+1)\frac{\pi}{2}\right)\cosh y = (-1)^n \cosh y.$$

So n must be even and we have $y = \pm\cosh^{-1} 2$. So the solutions to $\sin z = 2$ are

$$z = (2n+1)\frac{\pi}{2} \pm i\cosh^{-1} 2.$$

■

Example 7.29 *Show that* $\cos z = \operatorname{Re}(\exp(iz))$ *holds if and only if z is real.*

Solution. We know this is true for real z. To prove the converse, say $z = x + iy$. Then

$$\begin{aligned} \cos(x + iy) &= \cos x \cosh y - i \sin x \sinh y, \\ \operatorname{Re}(\exp(iz)) &= \operatorname{Re}(\exp(-y + ix)) = e^{-y}\cos x. \end{aligned}$$

Comparing real and imaginary parts we have

$$e^{-y}\cos x=\cos x\cosh y,\qquad \sin x\sinh y=0.$$

If $y=0$ then both equations are satisfied and z is real. If $y\neq 0$ then $\sin x=0$ and so $x=n\pi$ for some integer n, so that $\cos x=(-1)^n\neq 0$. Finally

$$\cosh y=e^{-y}\quad\Longleftrightarrow\quad\frac{e^{y}+e^{-y}}{2}=e^{-y}\quad\Longleftrightarrow\quad e^{2y}=1\quad\Longleftrightarrow\quad y=0,$$

and hence z is real. ■

Example 7.30 *Show that* $|\exp(iz)|=1$ *if and only if* z *is real.*

Solution. Let $z=x+iy$. Note

$$1=|\exp(iz)|=|\exp(-y+ix)|=e^{-y}$$

if and only if $y=0$ and so z is real. ■

Remark 7.31 *(**Complex Logarithm and Powers**) (Off-syllabus) We saw that* $\exp\colon\mathbb{C}\to\mathbb{C}$ *has image* $\mathbb{C}\setminus\{0\}$ *and has period* $2\pi i$*. So for any* $z\neq 0$ *there is a solution* w_0 *to* $\exp(w_0)=z$ *and for any integer* n *then*

$$w=w_0+2n\pi i$$

will also be a solution – in fact, these will be all the solutions. This can be argued as follows:

$$\begin{aligned}1\quad &=\quad e^{x+iy}=e^{x}(\cos y+i\sin y)\\ &\Longleftrightarrow\quad e^{x}=1,\ \cos y=1,\ \sin y=0\\ &\Longleftrightarrow\quad x=0\ \text{ and }\ y=2n\pi\quad\text{ for some } n\in\mathbb{Z},\end{aligned}$$

so that if $\exp w=\exp w_0=z$ *then*

$$\exp(w-w_0)=1\qquad\Longrightarrow\qquad w-w_0=2n\pi i.$$

These w *are the possible values of* $\log z$*. So complex logarithm is an example of a **multifunction**. Other examples of multifunctions include square root and the inverse trigonometric functions. We can make a genuine function from a multifunction by specifying certain principal values on the domain, for example by taking the positive square root or insisting* $\sin^{-1}\colon[-1.1]\to[-\pi/2,\pi/2]$.

Given $z=r\exp(i\theta)\neq 0$ *then the possible values of* $\log z$ *are*

$$\log z=\log r+i\theta.$$

θ *here is a choice of argument which needs specifying to define a single-valued function for* $\log$*. For* $z\in\mathbb{C}\setminus(-\infty,0]$ *we can uniquely write* $z=r\exp(i\theta)$ *where* $-\pi<\theta<\pi$*. We will denote this particular choice of* $\log z$ *as* $L(z)$ *which agrees with the real logarithm on the positive real axis (see Figure 6.3).*

If we were to take points z_+ and z_-, respectively just above and below the cut $(-\infty, 0]$, then we would have

$$z_+ = r\exp(i\theta_+) \quad \textit{where } \theta_+ \approx \pi; \qquad z_- = r\exp(i\theta_-) \quad \textit{where } \theta_- \approx -\pi.$$

So

$$L(z_+) \approx \log r + i\pi; \qquad L(z_-) \approx \log r - i\pi.$$

So across the negative real axis there is a jump of $\pm 2\pi i$ depending on which was the axis is crossed. The function L satisfies $\exp(L(z)) = z$ for all $w \in \mathbb{C}\backslash(-\infty, 0]$ and L is differentiable with $L'(z) = z^{-1}$ on $\mathbb{C}\backslash(-\infty, 0]$.

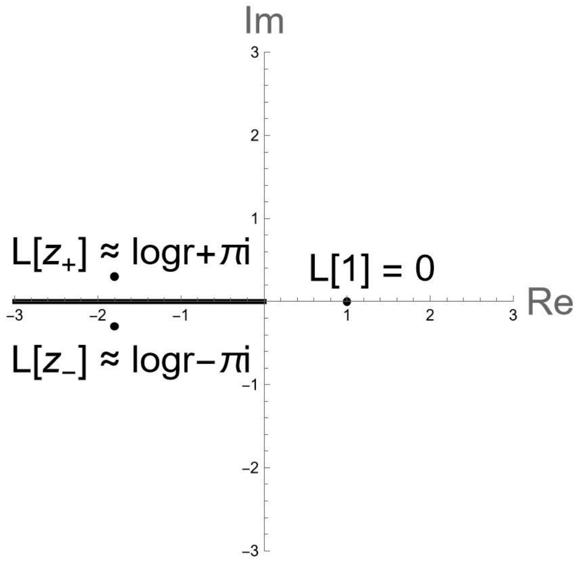

Fig. 6.3 – a branch L of log

We refer to $\mathbb{C}\backslash(-\infty, 0]$ as a ***cut-plane*** *and to L as a* ***branch*** *of* log*. It can be shown that there is no differentiable branch of* log *on $\mathbb{C}\backslash\{0\}$, so some cut to the origin is necessary. The only other differentiable branches of* log *on this cut-plane are*

$$L(z) + 2n\pi i$$

for integers n.

In the same way we defined general real exponents, for $z \in \mathbb{C}\backslash(-\infty, 0]$ and $\alpha \in \mathbb{C}$, we can define

$$z^\alpha = \exp(\alpha L(z)).$$

This defines a differentiable function on $\mathbb{C}\backslash(-\infty, 0]$ which has derivative $\alpha z^{\alpha-1}$. Considering the other possible branches of log*, we note that z^α takes a unique value if α is an integer, that z^α takes finitely many values if α is rational and otherwise z^α takes infinitely many values. When $\alpha = \frac{1}{2}$ then*

$$(z_+)^{1/2} \approx \sqrt{r}\exp(i\pi/2) = i\sqrt{r}; \qquad (z_-)^{1/2} \approx \sqrt{r}\exp(-i\pi/2) = -i\sqrt{r}.$$

We see this time that there is a sign change as we cross the cut.

The function in Figure 6.4a is $z^{1/2} = \exp(L(z)/2)$ *and the only other differentiable function on* $\mathbb{C}\backslash(-\infty, 0]$ *which satisfies* $w^2 = z$ *is* $-z^{1/2}$*, depicted in Figure 6.4b. This is because*

$$\exp\left\{\frac{L(z) + 2n\pi i}{2}\right\} = \begin{cases} z^{1/2} & \textit{if } n \textit{ is even,} \\ -z^{1/2} & \textit{if } n \textit{ is odd.} \end{cases}$$

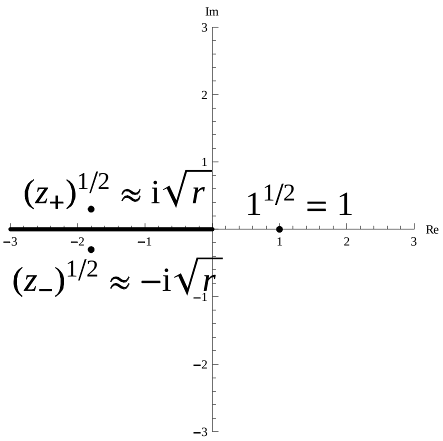

Fig. 6.4a – a branch of $\sqrt{z}$

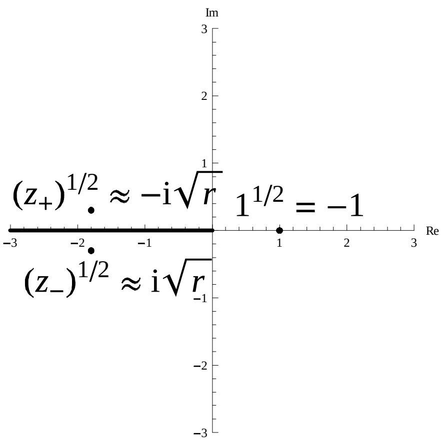

Fig. 6.4b – another branch

Definition 7.32 ***(Logarithmic Series)*** *Consider the power series*

$$\lambda(z) = \sum_1^\infty \frac{z^n}{n}.$$

The radius of convergence is 1 *(by the ratio test) and so converges for* $|z| < 1$*.*

For $-1 < x < 1$, by Theorem 7.13,

$$\lambda'(x) = \sum_1^\infty \frac{nx^{n-1}}{n} = \sum_1^\infty x^{n-1} = \frac{1}{1-x}.$$

Set

$$\mu(x) = (1-x)\exp(\lambda(x)).$$

By the chain and product rules,

$$\mu'(x) = -1 \times \exp\lambda(x) + (1-x)\lambda'(x)\exp\lambda(x) = 0.$$

It follows that $\mu(x)$ is constant and equals $\mu(0) = \exp\lambda(0) = \exp 0 = 1$. Hence

$$\exp(\lambda(x)) = \frac{1}{1-x} \qquad \text{for } -1 < x < 1$$

and, with using the definition of real logarithm (Definition 7.18)

$$\lambda(x) = \log\left(\frac{1}{1-x}\right) = -\log(1-x).$$

In terms of the branch L for complex logarithm defined earlier we have

$$\lambda(z) = -L(1-z) \qquad \text{for} \quad |z| < 1.$$

Example 7.33 *Let $\alpha, z \in \mathbb{C}$ with $|z| < 1$. Find the power series of*

$$B(z,\alpha) = (1+z)^{\alpha} = \exp(\alpha L(1+z)).$$

Solution. The composition of two analytic functions is itself analytic (which I do not prove here), so we may set

$$B(z,\alpha) = \sum_{0}^{\infty} a_n z^n.$$

By the chain rule

$$B'(z,\alpha) = \alpha L'(1+z)\exp(\alpha L(1+z)) = \frac{\alpha B(z,\alpha)}{1+z}$$

and so

$$(1+z)B'(z,\alpha) = \alpha B(z,\alpha).$$

We note $a_0 = 1$ and, focusing on the z^n term on each side, we obtain the recurrence relation

$$(n+1)a_{n+1} + na_n = \alpha a_n$$

so that

$$a_0 = 1, \qquad a_{n+1} = \frac{\alpha - n}{n+1} \quad \text{for } n \geqslant 0.$$

Hence

$$\begin{aligned}
a_n &= \left(\frac{\alpha-n+1}{n}\right)a_{n-1} \\
&= \left(\frac{\alpha-n+1}{n}\right)\left(\frac{\alpha-n+2}{n-1}\right)a_{n-2} \\
&= \cdots \\
&= \frac{(\alpha-n+1)(\alpha-n+2)\cdots(\alpha-1)\alpha}{n(n-1)\times\cdots\times 2\times 1}a_0 \\
&= \frac{(\alpha-n+1)(\alpha-n+2)\cdots(\alpha-1)\alpha}{n(n-1)\times\cdots\times 2\times 1}.
\end{aligned}$$

If we denote this last expression as $\binom{\alpha}{n}$ then we have determined the **binomial series for a general exponent:**

$$B(z,\alpha) = (1+z)^{\alpha} = \sum_{n=0}^{\infty}\binom{\alpha}{n}z^n.$$

Note that if α is a natural number then this is a finite sum and otherwise the above series is an infinite sum which converges for $|z| < 1$. ■