C3.8 Analytic Number Theory

Sheet 0 — MT25

Practice with estimates and analysis

The purpose of this sheet is to remind you of some basic techniques from analysis that we will use in the course, and to get a bit of practice in the big-O notation, which is the most important piece of asymptotic notation (see the introduction of the notes for more).

Recall that $f(x) = O(g(x))$ means that $|f(x)| \leqslant Cg(x)$ for some constant $C > 0$. Often, we do not need to specify any explicit value of C, and it could be quite hard to do so; this is why the notation is so useful.

1. Show that $(1+x)^3 = 1 + O(x)$ for $0 \leqslant x \leqslant 1$.
2. Show that $\log(1-x) = -x + O(x^2)$ for $0 \leqslant x \leqslant \frac{1}{2}$.

   The next two questions are about comparing sums to integrals.
3. Show that $\sum_{n=1}^{X} \frac{1}{n} = \log X + O(1)$ for $X \geqslant 1$.
4. Show that $\sum_{n>X} \frac{1}{n^2} = O(\frac{1}{X})$ for $X \geqslant 1$.

   The next two questions are about infinite sums and products.
5. What does it mean for a sum $\sum_{n=1}^{\infty} a_n$, with $a_n \in \mathbb{C}$, to converge absolutely? Let $t \in \mathbb{R}$. Show that $\sum_{n=1}^{\infty} \frac{\log n}{n^{2+it}}$ converges absolutely.
6. Understand the following statement and its proof: $\prod_{n=1}^{\infty}(1 + \frac{1}{n^2})$ converges.

   Finally, two more questions on estimating sums and integrals.
7. Show that $\int_2^X \frac{dt}{t \log t} = O(\log\log X)$ for $X \geqslant 3$.
8. Show that $\sum_{n=2}^{X} \frac{1}{\log n} = O(\frac{X}{\log X})$ for $X \geqslant 3$.

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