

BO1.1. History of Mathematics

Lecture III

Analytic geometry and the beginnings of calculus, part 1

MT25 Week 2

Summary

- ▶ Brief overview of the 17th century
- ▶ A cautionary tale
- ▶ Development of notation
- ▶ Use of algebra in geometry
- ▶ The beginnings of calculus

The 17th century

The main mathematical innovations of the 17th century:

- ▶ symbolic notation
- ▶ analytic (algebraic) geometry
- ▶ calculus
- ▶ infinite series [to be treated in later lectures]
- ▶ mathematics of the physical world [to be treated in later lectures]

Symbolic notation

Symbolic notation makes mathematics easier

- ▶ to read
- ▶ to write
- ▶ to communicate (though perhaps not orally)
- ▶ to think about — and thus stimulates mathematical advances?
- ▶ BUT it took a long time to develop
- ▶ why did it develop when it did?

Early European notation (abbreviation)

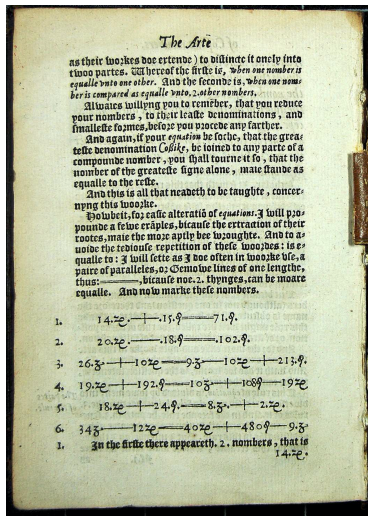
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4 + 5 Wile du das wys
 4 — 17 sen oder desgleys
 3 + 30 chen/So sumier
 4 — 19 die zentner vnd
 3 + 44 lb vnnnd was auß
 3 + 22 — ist/das ist mi
 Zentner 3 — 11 lb nus dz sen beson
 3 + 50 der vnnnd werden
 4 — 16 45 39 lb (So
 3 + 44 du die zentner
 3 + 29 zu lb gemacht
 3 — 12 hast vnnnd das /
 3 + 9 + das ist meer
 darzu Addierest) vnd > 5 minus. Nun
 solc du für Holz abschlahen allweg für
 ain legel 24 lb. Vnd das ist 13 mal 24.
 vnd mache 3 12 lb darzu addier das —
 das ist > 5 lb vnd werden 387. Dye suß
 erahier von 45 39. Vnd Bleyben 415 2
 lb. Nun sprich 100 lb das ist ein zentner
 pro 4 ft $\frac{1}{2}$ wie kummen 415 2 lb vnd kumē
 171 ft 5 4 heller? Vñ ist rechte gemacht

Wesser

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Johannes Widman, *Behende und hüpsche Rechenung auff allen Kauffmanschafft* (1489)



Robert Recorde, *The Whetstone of Witte* (1557)

The communication of mathematics

Initially entirely verbal — but usually using a **set form of words**

Scribal **abbreviations** often used

- ▶ e.g., later editors of Diophantus (3rd-century Egypt) used ς as an abbreviation for an unknown quantity
- ▶ e.g., Bhāskara II (12th-century India) used the initial letters of *yāvattāvat* (*unknown*) and *rūpa* (*unit*) as shorthand:
'yā 1 rū 1' denoted ' $x + 1$ '

But these were not symbols that could be manipulated algebraically

Arrangement of signs on the page could carry information

- ▶ e.g., *tiān yuán shù* 天元術 (13th-century China):

$$\begin{array}{r} \text{II} \\ - \text{III} \text{ 元} \\ \equiv - \text{ㄟ} \end{array} \quad (2x^2 + 18x - 316)$$

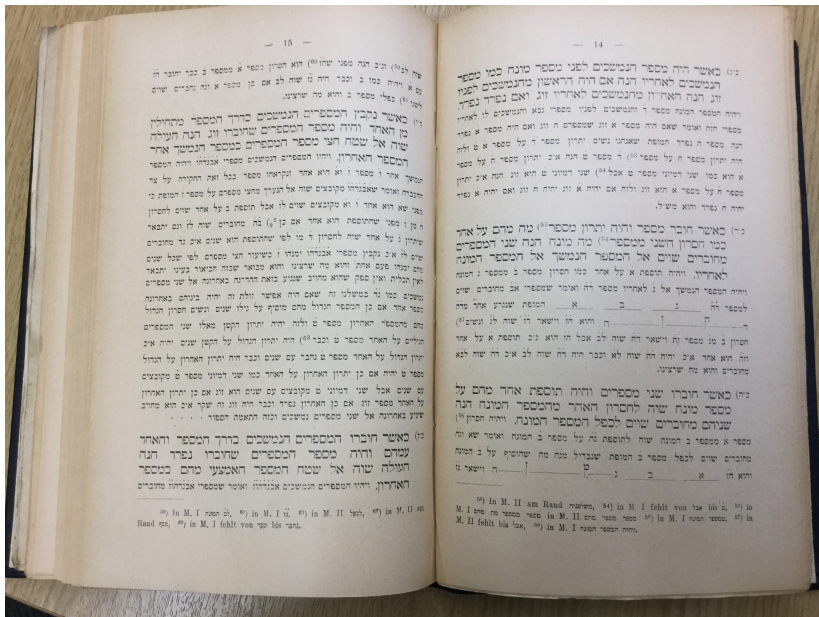
Algebraic symbolism of the form that we use came later

A cautionary tale: Levi Ben Gerson and sums of integers



Levi Ben Gerson (Gersonides), *Ma'aseh Hoshev* (*The Work of the Calculator*), 1321 [picture is of a version printed in Venice in 1716]

A cautionary tale: Levi Ben Gerson and sums of integers



⁸⁰⁾ in M. I. nicht 26, ⁸¹⁾ in M. I. 12. ⁸²⁾ in M. II 522b, ⁸³⁾ in M. II 50
Rand 200, ⁸⁴⁾ in M. I. fehlt von 200 bis 201.

⁸⁰) in M. II am Rand stehen, ⁸¹) in M. I fehlt von *taw* bis ⁸²) in
M. I und so weiter wie in M. II und so weiter ⁸³) in M. I noch steht, ⁸⁴) in
M. II fehlt bis *tak*, ⁸⁵) in M. I noch steht.

A cautionary tale: Levi Ben Gerson and sums of integers

Book I, Proposition 26:

If we add all consecutive numbers from one to any given number and the given number is even, then the addition equals the product of half the number of numbers that are added up times the number that follows the given even number.

Book I, Proposition 27:

If we add all consecutive numbers from one to any given number and the given number is odd, then the addition equals the product of the number at half way times the last number that is added.

(Translations from Hebrew by Leo Corry.)

A cautionary tale: Levi Ben Gerson and sums of integers

Converting these into modern notation, we get:

Book I, Proposition 26:

If n is an even number, then $1 + 2 + 3 + \cdots + n = \frac{n}{2}(n + 1)$.

Book I, Proposition 27:

If n is an odd number, then $1 + 2 + 3 + \cdots + n = \frac{n+1}{2}n$.

The formulae are clearly the same, so why are these treated as separate propositions? The answer lies in the proofs, which, like the results themselves, are **entirely verbal**.

A cautionary tale: Levi Ben Gerson and sums of integers

A fundamental problem here lies in the difficulty of expressing the notion of 'any given number' (our ' n ').

A commonly adopted solution was to outline the proof for a specific example, on the understanding that the reader should then be able to adapt the **method** to any other instance.

Ben Gerson's proof of Proposition 26 takes this approach, and is based on the idea of forming pairs of numbers with equal sums.*

*You might have heard a story about the young Gauss doing the same thing.

A cautionary tale: Levi Ben Gerson and sums of integers

Proof of Proposition 26:

Take the example of 6. If we add 1 and 6, we get 7 ('the number that follows the given even number'). Notice that 2 is obtained from 1 by adding 1, and that 5 is obtained from 6 by subtracting 1, so 2 added to 5 is the same as 1 added to 6, namely 7. The only remaining pair is 3 and 4, which also add to give 7. The number of pairs is half the given even number, hence the total sum is half the number of numbers that are added up times the number that follows the given even number.

This proof is clearly not valid when the given number is odd, since Ben Gerson would have been required to halve it — but he was working only with (positive) integers

A cautionary tale: Levi Ben Gerson and sums of integers

Proposition 27 therefore needs a separate proof, which similarly does not apply when the given number is even (see Leo Corry, *A brief history of numbers*, OUP, 2015, p. 119)

As Corry notes:

For Gersonides, the two cases were really different, and there was no way he could realize that the two situations . . . were one and the same as they are for us.

Moral: take care when converting historical mathematics into modern terms!

Notation: compare Cardano (*Ars magna*, 1545)...



Having raised a third part of the number of things to a cube, to which you add the square of half the number in the equation and take the root of the total, consider the square [root], which you will take twice; and to one of them you add half of the same, and you will have the binome with its apotome, whence taking the cube root of the apotome from the cube root of its binome, the difference that comes from this, is the value of the thing.

(*Mathematics emerging*, p. 327)

... with Viète (c. 1590)...

François Viète
(Francisci Vieta)
Opera mathematica
1646, p. 130

130 DE EMENDATIONE

II.

S^1 A quad. — B in A 2, æquetur Z plano. A — B esto E. Igitur E quad, æquabitur Z plano → B quad.

Confectarium.

Itaque ✓ z^2 plani → B quad. + B fit A, de qua primum quærebatur.

Sit B 1. Z planum 20. A 1 N. 1 Q — 2 N, æquabitur 20. & fit 1 N. ✓ 21 + 1.

III.

S^1 D 2 in A — A quad., æquetur Z plano. D — E, vel D → E esto A. E quad., æquabitur D quad. — Z plano.

Confectarium.

Itaque, D minus, plusve ✓ D quad. — Z plano fit A, de qua primum quærebatur.

Sit D 5. Z planum 20. A 1 N. 10 N — 1 Q, æquatur 20. & fit 1 N. 5 — ✓ 5, vel 5 + ✓ 5.

De reductione cuborum simpliciter adfectuorum sub quadrato, ad cubos simpliciter adfectos sub latere.

Formule tres.

I.

S^1 A cubus → B 3 in A quad., æquetur Z folido. A → B esto E. E cubus — B quad. 3 in E, æquabitur Z folido — B cubo 2.

1 C + 6 Q, æquatur 1600. & fit 1 N 10. 1 C — 12 N, æquatur 1584. & fit 1 N 12.

Ad Arithmetica non incongrue equuor aliquod superimponitur notisalteratæ radicis, ad differentiam notarum ejus, de qua primum quærebatur.

II.

S^1 A cubus — B 3 in A quad., æquetur Z folido. A — B esto E. E cubus — B quad. 3 in E, æquabitur Z folido → B cubo 2.

1 C — 6 Q, æquatur 400. & fit 1 N 10. 1 C — 12 N, æquatur 416. & fit 1 N 8.

III.

S^1 B 3 in A quad. — A cubo, æquetur Z folido. A — B esto E. B quad. 3 in E. — E cubo, æquabitur Z folido — B cubo 2. Vel B — A esto E. B quad. 3 in E. — E cubo, æquabitur B cubo 2 — Z folido.

21 Q — 1 C, æquatur 972. & fit 1 N 9, vel 18. 147 N — 1 C, æquatur 286. & fit 1 N 2, vel 17.

9 Q — 1 C, æquatur 28. & fit 1 N 2. 27 N — 1 C, æquatur 26. & fit 1 N 1.

De reductione cuborum adfectuorum tam sub quadrato quam latere, ad cubos adfectos simpliciter sub latere.

Formule septem.

I.

S^1 A cubus → B 3 in A quad. → D plano in A, æquetur Z folido. A → B esto E. E cubus → D plano — B quad., in E æquabitur Z folido → D plano in B — B cubo 2.

1 C + 30 Q + 330 N, æquatur 788. & fit 1 N 2. 1 C + 30 N, æquatur 2088. & fit 1 N 12.

1 C +

... with Viète (c. 1590)...

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DE EMENDATIONE

II.

SI A quad. — Bin A 2, æquetur Z plano. A — B esto E. Igitur E quad, æquabitur Z plano + B quad.

Confectarium.

Itaque $\sqrt{Z \text{ plani} + B \text{ quad.}}$ + B fit A, de qua primum quærebatur.

Sit B 1. Z planum 20. A 1 N. 1 Q — 2 N, æquabitur 20. & fit 1 N. $\sqrt{21 + 1}$.

III.

SI D 2 in A — A quad., æquetur Z plano. D — E, vel D + E esto A. E quad., æquabitur D quad. — Z plano.

Confectarium.

Itaque, D minus, plusve $\sqrt{D \text{ quad.} - Z \text{ plano}}$ fit A, de qua primum quærebatur.

Sit D 5. Z planum 20. A 1 N. 10 N — 1 Q, æquatur 20. & fit 1 N. $5 - \sqrt{5}$, vel $5 + \sqrt{5}$.

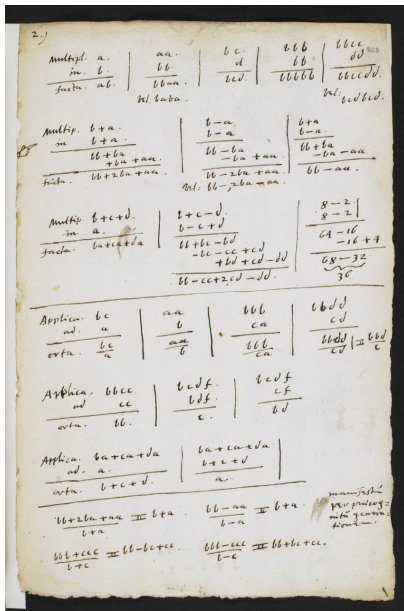
... and with Harriot (c. 1600)

British Library

Add MS 6784 f. 323

available at

[Thomas Harriot Online](https://www.thomasharriot.org/)

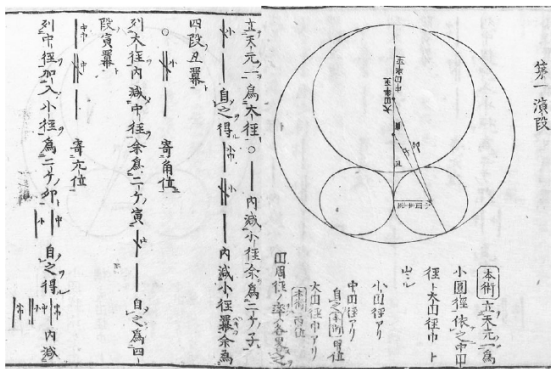


... and with Harriot (c. 1600)

$\begin{array}{r} \text{Applica. } ba+ca+da \\ \text{ad. } a. \\ \hline \text{orta. } b+c+d. \end{array}$	$\left \begin{array}{r} ba+ca+da \\ b+c+d \\ \hline a. \end{array} \right $	
$\frac{bb+2ba+aa}{b+a} \equiv b+a.$	$\frac{bb-aa}{b-a} \equiv b+a.$	<i>manifeste per prolog= niti genera= tionem.</i>
$\frac{bbb+ccc}{b+c} \equiv bb-bc+cc.$	$\frac{bbb-ccc}{b-c} \equiv bb+bc+cc.$	

Elsewhere in the world

Seki Takakazu, *Hatsubi Sanpō* 発微算法 (1674), concerning the solution of equations in several variables:



Equations written using the technique of *bōshohō* 傍書法 ('side-writing'; a.k.a. *tenzan jutsu* 点竄術)

Notation: Viète (Tours, c. 1590)

François Viète (1540–1603, France):

A, E, ... (i.e., vowels) for unknowns

B, C, D, ... (i.e., consonants) for known
or given quantities

symbols + , −

but otherwise verbal descriptions and
connections: quadratum (squared),
cubus (cubed), aequatur (be equal), ...



Notation: Harriot (London, c. 1600)

Thomas Harriot (1560–1621, England):

a, e, ... for unknowns

b, c, d, ... for known or given quantities

+, −

ab, aa, aaa

and many symbols: =, >, ...

(For another example of Harriot's use of notation, see *Mathematics emerging*, §2.2.1.)



[Thomas Harriot Online](#)

Notation: Descartes (Netherlands, 1637)

René Descartes (1596–1650, France and Holland):

x , y , ... for unknowns

a , b , c , ... for known or given quantities

$+$, $-$

xx , x^3 , x^4 , ...

Descartes' notation was widely adopted, although his ' ∞ ' for equality eventually gave way to '=', and his ' $\sqrt{C}$ ' to ' $\sqrt[3]{}$ '.



Descartes' notation

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LA GEOMETRIE.

tirer de cete science. Aussi que ie n'y remarque rien de si difficile, que ceux qui feront vn peu versés en la Geometrie commune, & en l'Algebre, & qui prendront garde a tout ce qui est en ce traité, ne puissent trouver.

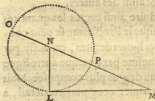
C'est pourquoy ie me contenteray icy de vous avertir, que pourvû qu'en demeslant ces Equations on ne manque point a se servir de toutes les diuisions, qui seront possibles, on aura infalliblement les plus simples termes, auxquels la question puisse estre reduite.

Quels
sont les
problèmes
plans

Et que si elle peut estre resolue par la Geometrie ordinaire, c'est a dire, en ne se servant que de lignes droites & circulaires tracées sur vne superficie plate, lorsque la dernière Equation aura esté entièrement demeslée, il n'y restera tout au plus qu'un quarré inconnu, esgal a ce qui se produist de l'Addition, ou soustraction de sa racine multipliée par quelque quantité connue, & de quelque autre quantité aussi connue.

Comment
ils
se
resolvent.

Et lors cete racine, ou ligne inconnue se trouve aisément. Car si l'ay par exemple



ie fais le triangle rectangle NLM, dont le costé LM est esgal à b racine quarrée de la quantité connue bb, & l'autre LN est $\frac{1}{2}a$, la moitié de l'autre quantité connue, qui estoit multipliée par $\frac{1}{2}a$ que ie suppose estre la ligne inconnue. puis prolongeant MN la baze de ce triangle,

LIVRE PREMIER.

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angle, iusques a O, en sorte qu'N O soit esgale a NL, la toute OM est $\frac{1}{2}a$ la ligne cherchée. Et elle s'exprime en cete sorte

$$x \propto \frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}.$$

Que si i'ay $y \propto -ay + bb$, & qu'y soit la quantité qu'il faut trouver, ie fais le mesme triangle rectangle NLM, & de sa baze MN i'oste N P esgale a NL, & le reste PM est y la racine cherchée. De façon que i'ay $y \propto -\frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}$. Et tout de mesme si i'aurois $x \propto -ax + b$. PM seroit x. & i'aurois

$$x \propto \sqrt{-\frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}}: \text{ \& ainsi des autres.}$$

Enfin si i'ay

$$x \propto ax - bb:$$

ie fais NL esgale à $\frac{1}{2}a$, & LM esgale à b côme deuât, puis, au lieu de ioindre les points MN, ie tire MQR parallele a LN, & du centre N par L ayant descrit vn cercle qui la coupe aux points Q & R, la ligne cherchée x est MQ, oubié MR, car en ce cas elle s'ex-

prime en deux façons, a sçauoir $x \propto \frac{1}{2}a + \sqrt{\frac{1}{4}aa - bb}$, & $x \propto \frac{1}{2}a - \sqrt{\frac{1}{4}aa - bb}$.

Et si le cercle, qui ayant son centre au point N, passe par le point L, ne coupe ny ne touche la ligne droite MQR, il n'y a aucune racine en l'Equation, de façon qu'on peut assurer que la construction du problème proposé est impossible.

Au

Symbolism established in algebra



Frontispiece to: Johannes Faulhaber, *Ingenieurs-Schul, Anderer Theil*, Ulm, 1633 (on fortification)

See: Volker Remmert, 'Antiquity, nobility, and utility: picturing the Early Modern mathematical sciences', in *The Oxford handbook of the history of mathematics* (Eleanor Robson & Jacqueline Stedall, eds.), OUP, 2009, pp. 537–563

‘Analysis’ vs ‘synthesis’

Viète (and others) sought to ‘restore’ ancient Greek mathematical ideas — in particular, those found in the recently rediscovered *Collection* (or *Synagoge*: *Συναγωγή*) of Pappus of Alexandria (4th century AD) [published in Latin by Federico Commandino in 1588]

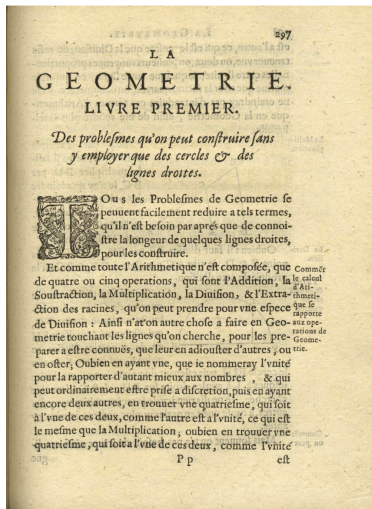
Book VII of Pappus’s *Collection* outlines the methods of **analysis** and **synthesis**:

Synthesis: starting from what is known, we make a sequence of deductions until we arrive at what is sought (constructive method, as, e.g., in Euclid’s *Elements*)

Analysis: starting from what is sought, as if it has already been established, we work backwards until we arrive at what is known (method of discovery or problem-solving, preliminary to synthesis)

“Analysis was thus the working tool of the geometer, but it was with synthesis that one could demonstrate things in an indisputable way.”
(Niccolò Guicciardini, ‘Analysis and synthesis in Newton’s mathematical work’, *The Cambridge Companion to Newton* (ed. I. Bernard Cohen and George E. Smith), CUP, 2002, pp. 308–328 at p. 308)

Analytic (algebraic) geometry



La géométrie (1637)

Solution of geometric problems
by algebraic methods

Appendix to

Discours de la méthode

“by commencing with objects the
simplest and easiest to know, I
might ascend by little and little”

Descartes' analytic geometry

We may label lines (line segments) with letters $a, b, c, \dots$

Then $a + b, a - b, ab, a/b, \sqrt{a}$ may be constructed by ruler and compass.

Descartes' method

- ▶ represent all lines by letters
- ▶ use the conditions of the problem to form equations
- ▶ reduce the equations to a single equation
- ▶ solve
- ▶ construct the solution geometrically

For examples, see Katz (3rd ed.), §14.2

Algebraic methods in geometry: some objections

Pierre de Fermat (1656, France):

I do not know why he has preferred this method with algebraic notation to the older way which is both more convincing and more elegant ...

Thomas Hobbes (1656, England):

... a scab of symbols ...

The beginnings of calculus: tangent methods

Calculus:

- ▶ finding tangents;
- ▶ finding areas.

Descartes' method for finding tangents (1637)

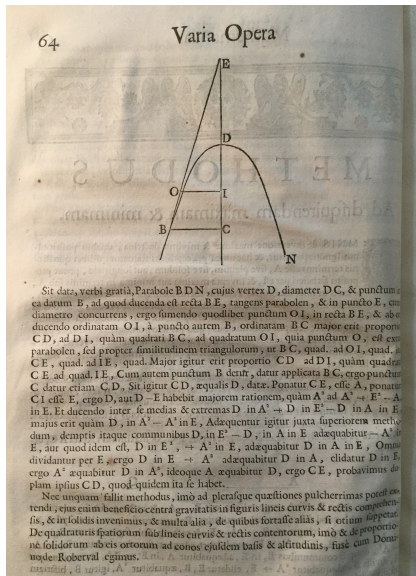
- ▶ based on finding a circle that touches the curve at the given point — a tangent to the circle is then a tangent to the curve
- ▶ used his algebraic approach geometry to find double roots to equation of intersection
- ▶ was in principle a general method — but laborious

Fermat's method for finding tangents

Pierre de Fermat (1601–1665):

- ▶ steeped in classical mathematics
- ▶ like Descartes, investigated problems of Pappus
- ▶ devised a tangent method (1629) quite different from that of Descartes

Fermat's tangent method (1629)

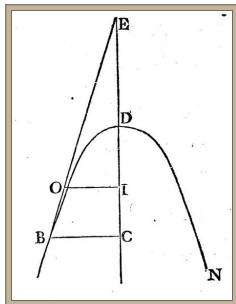


Worked out c. 1629, but only published posthumously in *Varia opera mathematica*, 1679.

See *Mathematics emerging*, §3.1.1.

Fermat's tangent method (1629)

Choose an arbitrary point B on the parabola.



Suppose that the tangent at B exists, and that it crosses the axis of the parabola at E .

Choose any point O on the line BE .

Draw horizontals OI and BC .

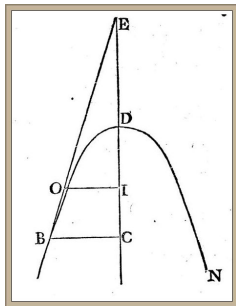
Since O is outside the parabola, we have

$$\frac{CD}{DI} > \frac{(BC)^2}{(OI)^2}.$$

Fermat's tangent method (1629)

Since O is outside the parabola, we have

$$\frac{CD}{DI} > \frac{(BC)^2}{(OI)^2}.$$



By similarity of triangles,

$$\frac{(BC)^2}{(OI)^2} = \frac{(CE)^2}{(IE)^2}.$$

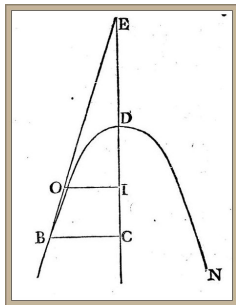
Therefore

$$\frac{CD}{DI} > \frac{(CE)^2}{(IE)^2}.$$

Fermat's tangent method (1629)

Therefore

$$\frac{CD}{DI} > \frac{(CE)^2}{(IE)^2}.$$



Put $CD = d$, $CE = a$, $CI = e$, so that

$$\frac{d}{d-e} > \frac{a^2}{(a-e)^2}.$$

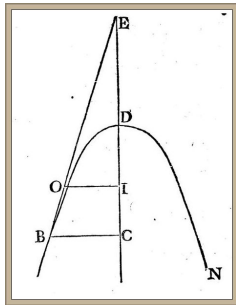
Now (Fermat says), we obtain equality as e decreases (as OI becomes BC):

$$\frac{d}{d-e} = \frac{a^2}{(a-e)^2}.$$

Fermat's tangent method (1629)

We solve the equality

$$\frac{d}{d-e} = \frac{a^2}{(a-e)^2}.$$



Rearranging gives $de^2 + a^2e = 2ade$.

Cancel e : $de + a^2 = 2ad$.

Now e will be small, so we can neglect it, leaving us with $a^2 = 2ad$.

Hence $a = 2d$.

Or $CE = 2 \times CD$.