

BO1.1. History of Mathematics  
Lecture VII  
Infinite series

MT25 Week 4

# Summary

- ▶ A non-Western prelude
- ▶ Newton and the Binomial Theorem
- ▶ Other 17th century discoveries
- ▶ Ideas of convergence
- ▶ Much 18th century progress: power series
- ▶ Doubts — and more on convergence

# The Kerala School

Flourished in Southern India from the 14th to the 16th centuries, working on mathematical and astronomical problems

Names associated with the school: Narayana Pandita, Madhava of Sangamagrama, Vatasseri Parameshvara Nambudiri, Kelallur Nilakantha Somayaji, Jyeṣṭhadeva, Achyuta Pisharati, Melpathur Narayana Bhattathiri, Achyutha Pisharodi, Narayana Bhattathiri

Treatises on arithmetic, algebra, geometry, inc. methods for approximation of roots of equations, discussion of magic squares, infinite series, . . .

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# Tantrasamgraha (1501)

Completed by Kelallur Nilakantha Somayaji (1444–1544) in 1501;  
concerns astronomical computations



## Keralan series

Infinite series for trigonometric functions appear in Sanskrit verse in an anonymous commentary on the *Tantrasamgraha*, entitled the *Tantrasamgraha-vyakhya*, of c. 1530:

दृष्टज्यात्रिज्ययोर्घातात् कोट्याप्तं प्रथमं फलम् ।  
ज्यावर्गं गुणकं कृत्वा कोटिवर्गं च हारकम् ॥  
प्रथमादिफलेभ्योऽथ नेया फलततिर्मुहुः ।  
एकत्रयाद्योज संख्याभिर्भक्ते ष्वेतेष्वनुक्रमात् ॥  
ओजानां संयुतेस्त्यक्त्वा युग्मयोगं धनुर्भवेत् ।  
दोःकोट्योरल्पमेवेह कल्पनीयमिह स्मृतम् ।  
लब्धीनामवसानं स्यान्नान्यथापि मुहुः कृते ॥

Proof supplied by Jyeṣṭhadeva in his *Yuktibhāṣā* (1530)

## Keralan series

From the *Tantrasamgraha-vyakhya*:

The product of the given Sine and the radius divided by the Cosine is the first result. From the first, [and then, second, third] etc., results obtain [successively] a sequence of results by taking repeatedly the square of the Sine as the multiplier and the square of the Cosine as the divisor. Divide [the above results] in order by the odd numbers one, three, etc. [to get the full sequence of terms]. From the sum of the odd terms, subtract the sum of the even terms. [The results] become the arc. In this connection, it is laid down that the [Sine of the] arc or [that of] its complement, which ever is smaller, should be taken here [as the given Sine]; otherwise, the terms obtained by the [above] repeated process will not tend to the vanishing magnitude.

Modern interpretation:

$$R\theta = \frac{R(R \sin \theta)^1}{1(R \cos \theta)^1} - \frac{R(R \sin \theta)^3}{3(R \cos \theta)^3} + \frac{R(R \sin \theta)^5}{5(R \cos \theta)^5} - \dots \quad (R \sin \theta < R \cos \theta)$$

But these results were  
unknown in the West until  
the 1830s

As we will see, the series for  
arctan was reproduced  
independently in Scotland in  
the 1670s

( 509 )

XXXIII. *On the Hindú Quadrature of the Circle, and the infinite Series of the proportion of the circumference to the diameter exhibited in the four Sāstras, the Tantra Sangraham, Yucti Bhāṣhā, Carana Padhati, and Sadratnamāla. By CHARLES M. WILSH, Esq., of the Hon. East-India Company's Civil Service on the Madras Establishment.*

(Communicated by the MADRAS LITERARY SOCIETY and AUXILIARY  
ROYAL ASIATIC SOCIETY.)

Read the 15th of December 1832.

A'RYAB'HATTA, who flourished in the beginning of the thirty-seventh century of the *Cālī Yuga*,\* of which four thousand nine hundred and twenty years have passed, has in his work, the *Aryab'hatiyam*, in which he mentions the period of his birth, exhibited the proportion of the diameter to the circumference of the circle as 20000 to 62832, in the following verse :

*Chaturaddham yatamashagunandwāshashūtatathā sahasrāṇam  
Ayutadwaya vishecābhāsyasannā vritta pariṇāhah†*

Which is thus translated :

" The product of one hundred increased by four and multiplied by eight, added to sixty and two thousands, is the circumference of a circle whose diameter is twice ten thousand."

The author of the *Līlāvatī*, who lived six centuries after A'RYAB'HATTA, states the proportion as 7 to 22, which, he adds, is sufficiently exact for common purposes. As a more correct or precise circumference, he proposes that the diameter be multiplied by 3927, and the product divided by 1250 ; the quotient will be a very precise circumference. This proportion is the same with that of A'RYAB'HATTA, which is less correct than that of

\* Or the sixth century of the Christian era.

† This verse is in the variety of the *Argavartam* measure, called *Tipula*.

## Keralan series

### Warning!

It has sometimes been claimed that there **must** be a link between European and Keralan ideas about infinite series, because the same results occur in both places.

However, there is **no** documentary evidence of such a link.

In general,

conceptual similarities  $\neq$  evidence of transmission

Question: 'what are or should be the criteria for accepting a hypothesis of cross-cultural transmission as plausible, and are those criteria culturally dependent?' (Kim Plofker, *Mathematics in India*, Princeton University Press, 2009, p. 252)

# Infinite series 1600–1900: an overview

## Lecture VII:

- ▶ mid–late 17th century: many discoveries
- ▶ early 18th century: much progress
- ▶ later 18th century: doubts and questions

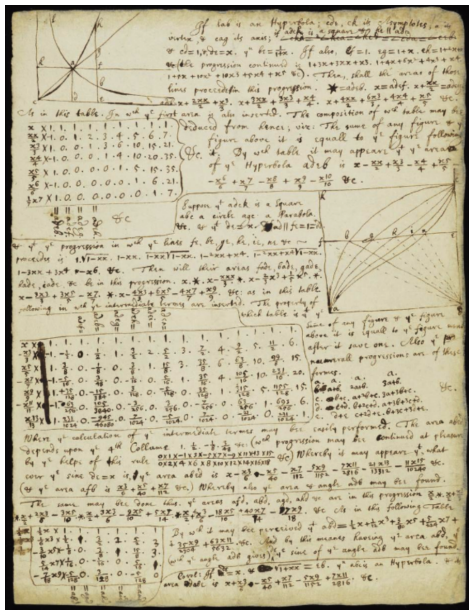
## Lecture VIII:

- ▶ early 19th century: Fourier series
- ▶ early 19th century: convergence better understood

## Newton and the general binomial theorem

CUL Add. MS 3958.3, f. 72

(See lecture IV)



## Recall: Newton's integration of $(1+x)^{-1}$

	$(1+x)^{-1}$	$(1+x)^0$	$(1+x)^1$	$(1+x)^2$	$(1+x)^3$	$(1+x)^4$	$\dots$
$x$	1	1	1	1	1	1	$\dots$
$\frac{x^2}{2}$	-1	0	1	2	3	4	$\dots$
$\frac{x^3}{3}$	1	0	0	1	3	6	$\dots$
$\frac{x^4}{4}$	-1	0	0	0	1	4	$\dots$
$\frac{x^5}{5}$	1	0	0	0	0	1	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The entry in the row labelled  $\frac{x^m}{m}$  and the column labelled  $(1+x)^n$  is the coefficient of  $\frac{x^m}{m}$  in  $\int(1+x)^n dx$ . (NB.

Newton did **not** use the notation  $\int(1+x)^n dx$ .)



# Newton's method of extrapolation

In fact, this method extends easily to any integer  $n$

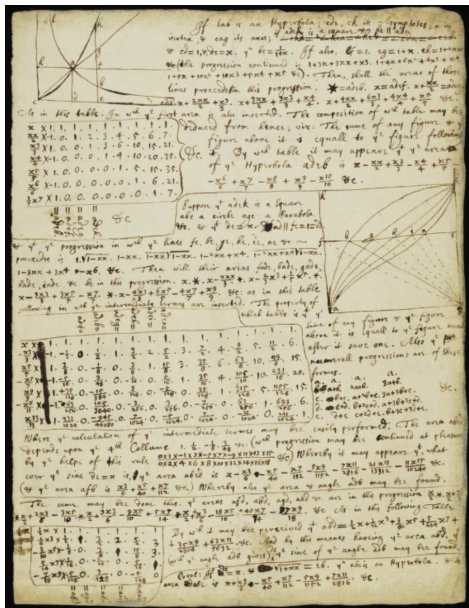
Newton's explanation:

*The property of which table is  $y^t$   $y^e$  sum of any figure and  $y^e$  figure above it is equal to  $y^e$  figure next after it save one. Also  $y^e$  numerall progressions are of these forms.*

$a$	$a$	$a$	$a$	
$b$	$a + b$	$2a + b$	$3a + b$	
$c$	$b + c$	$a + 2b + c$	$3a + 3b + c$	&c.
$d$	$c + d$	$b + 2c + d$	$a + 3b + 3c + d$	
$e$	$d + e$	$c + 2d + e$	$b + 3c + 3d + e$	

(See: *Mathematics emerging*, §8.1.1.)

## Newton and the general binomial theorem



CUL Add. MS 3958.3, f. 72

## Newton's method of interpolation

	$x^0$	$x^1$	$x^2$	$x^3$	$x^4$	$x^5$	$x^6$	$x^7$	$x^8$	$x^9$	$x^{10}$	$x^{11}$	$x^{12}$	$x^{13}$	$x^{14}$	$x^{15}$
$x^{15}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$-x^{14}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2}$	5	$\frac{11}{2}$	6	
$x^{13}$	1	$\frac{3}{8}$	0	$-\frac{1}{8}$	0	$\frac{3}{8}$	1	$\frac{15}{8}$	3	$\frac{35}{8}$	6	$\frac{63}{8}$	10	$\frac{99}{8}$	15	
$-x^{12}$	-1	$-\frac{5}{16}$	0	$\frac{3}{16}$	0	$-\frac{1}{16}$	0	$\frac{5}{16}$	1	$\frac{35}{16}$	4	$\frac{105}{16}$	10	$\frac{231}{16}$	20	
$x^{11}$	1	$+\frac{75}{128}$	0	$-\frac{15}{384}$	0	$\frac{3}{128}$	0	$-\frac{5}{128}$	0	$\frac{35}{128}$	1	$\frac{315}{128}$	5	$\frac{1155}{128}$	15	
$-x^{10}$	-1	$-\frac{63}{256}$	0	$\frac{105}{3840}$	0	$-\frac{3}{256}$	0	$\frac{3}{256}$	0	$-\frac{7}{256}$	0	$\frac{63}{256}$	1	$\frac{693}{256}$	6	
$x^{13}$	1	$\frac{231}{1024}$	0	$-\frac{945}{46080}$	0	$\frac{7}{1024}$	0	$-\frac{5}{1024}$	0	$\frac{7}{1024}$	0	$-\frac{31}{1024}$	0	$\frac{231}{1024}$	1	

# Newton's method of interpolation

	$(1-x^2)^{-1}$	$(1-x^2)^{-\frac{1}{2}}$	$(1-x^2)^0$	$(1-x^2)^{\frac{1}{2}}$	$(1-x^2)^1$	$(1-x^2)^{\frac{3}{2}}$	$(1-x^2)^2$	...
$x$	1	1	1	1	1	1	1	...
$-\frac{x^3}{3}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	...
$\frac{x^5}{5}$	1	$\frac{3}{8}$	0	$-\frac{1}{8}$	0	$\frac{3}{8}$	1	...
$-\frac{x^7}{7}$	-1	$-\frac{5}{16}$	0	$\frac{3}{48}$	0	$-\frac{1}{16}$	0	...
$\frac{x^9}{9}$	1	$\frac{35}{128}$	0	$-\frac{15}{384}$	0	$\frac{3}{128}$	0	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The entry in the row labelled  $\pm \frac{x^m}{m}$  and the column labelled  $(1-x^2)^n$  is the coefficient of  $\pm \frac{x^m}{m}$  in  $\int (1-x^2)^n dx$ .

(NB: possible slips in the last two rows of the original table)

## Newton's method of interpolation

Can fill in some initial values by other methods

Newton applied the formula

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

to fractional  $n$ , so that

$$\binom{1/2}{1} = \frac{1}{2}, \quad \binom{1/2}{2} = \frac{1/2(1/2-1)}{2!} = -\frac{1}{8}$$

and so on

# Newton's and the general binomial theorem

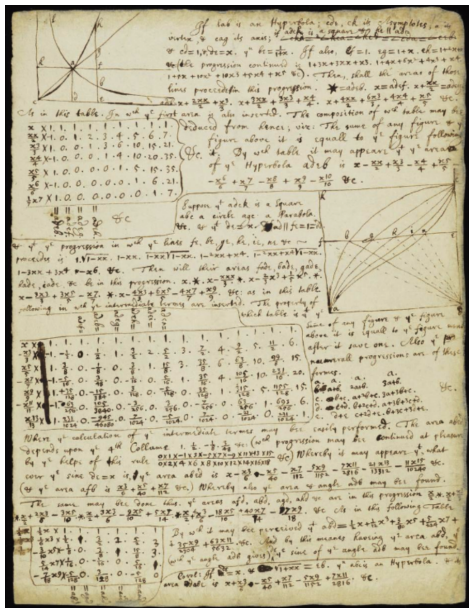
Newton went on to extend this method to other fractional powers, and also to  $(a + bx)^n$ , thereby convincing himself of the truth of the **general binomial theorem** — but this was not **proved** until the 19th century

On Newton and the binomial theorem, see

[https://www.youtube.com/watch?v=xv\\_PWwdDWDk](https://www.youtube.com/watch?v=xv_PWwdDWDk)

## One more table

The table at the bottom of the page gives the interpolations for  $(1+x)^n$  for half-integer  $n$



## Further discoveries by Newton

By further interpolations and integrations (based on strong geometric intuition) Newton found further series for:

- ▶  $(1 + x)^{p/q}$
- ▶ log, antilog
- ▶ sin, tan, ... (NB: cosine was not yet much in use)
- ▶ arcsin, arctan, ...

(See: *Mathematics emerging*, §§8.1.2–8.1.3.)



## Newton on the move from finite to infinite series

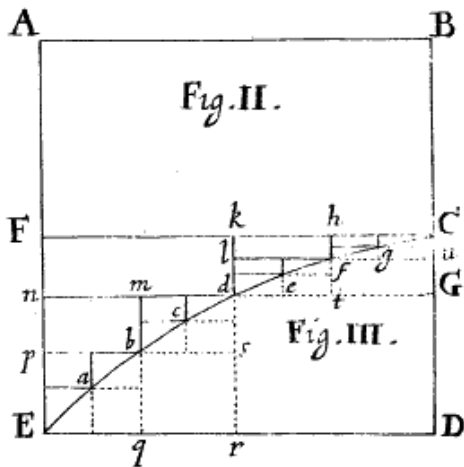
*And whatever common analysis performs by equations made up of a finite number of terms (whenever it may be possible), this method may always perform by infinite equations: in consequence, I have never hesitated to bestow on it also the name of analysis.*

(*De analysi*, 1669; Derek T. Whiteside, *The mathematical papers of Isaac Newton*, CUP, 1967–1981, vol. II, p. 241)

## Other 17th-century discoveries (1a)

Brouncker, c. 1655, published 1668: area under the hyperbola

given by  $\frac{1}{1 \times 2} + \frac{1}{3 \times 4} + \frac{1}{5 \times 6} + \dots$



## Other 17th-century discoveries (1b)

$$\begin{aligned}
 \text{I say } ABCdEA &= \frac{1}{1 \times 2} + \frac{1}{3 \times 4} + \frac{1}{5 \times 6} + \frac{1}{7 \times 8} + \frac{1}{9 \times 10} \&c. \\
 EdCDE &= \frac{1}{2 \times 3} + \frac{1}{4 \times 5} + \frac{1}{6 \times 7} + \frac{1}{8 \times 9} + \frac{1}{10 \times 11} \&c. \\
 EdCyE &= \frac{1}{2 \times 3 \times 4} + \frac{1}{4 \times 5 \times 6} + \frac{1}{6 \times 7 \times 8} + \frac{1}{8 \times 9 \times 10} \&c.
 \end{aligned}
 \left. \vphantom{\begin{aligned} \text{I say } ABCdEA \\ EdCDE \\ EdCyE \end{aligned}} \right\} \text{in infinitum.}$$

(647)

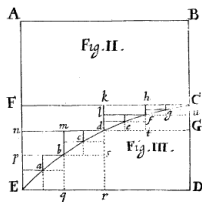
And that therefore in the first series half the first term is greater than the sum of the two next, and half this sum of the second and third greater than the sum of the four next, and half the sum of those four greater than the sum of the next eight, &c. *in infinitum*. For  $\frac{1}{2} dD = br + bn$ ; but  $bn > fg$ , therefore  $\frac{1}{2} dD > br + fg$ , &c. And in the second series half the first term is less than the sum of the two next, and half this sum less than the sum of the four next, &c. *in infinitum*.

That the first series are the even terms, viz. the 2<sup>d</sup>, 4<sup>th</sup>, 6<sup>th</sup>, 8<sup>th</sup>, 10<sup>th</sup>, &c. and the second, the odd, viz. the 1<sup>st</sup>, 3<sup>d</sup>, 5<sup>th</sup>, 7<sup>th</sup>, 9<sup>th</sup>, &c. of the following series, viz.  $\frac{1}{1 \times 2}, \frac{1}{2 \times 3}, \frac{1}{3 \times 4}, \frac{1}{4 \times 5}, \frac{1}{5 \times 6}, \frac{1}{6 \times 7}, \&c.$  *in infinitum* = 1. Whereof  $a$  being put for the number of terms taken at pleasure,  $\frac{1}{a-1}$  is the last,  $\frac{a}{a-1}$  is the sum of all those terms from the begin-

ning, and  $\frac{1}{a-1}$  the sum of the rest to the end.

That  $\frac{1}{4}$  of the first terme in the third series is less than the sum of the two next, and a quarter of this sum, less than the sum of the four next, and one fourth of this last sum less than the next eight, I thus demonstrate.

Let  $a$  = the 3<sup>d</sup> or last number of any term of the first Column, viz. of Divisors,

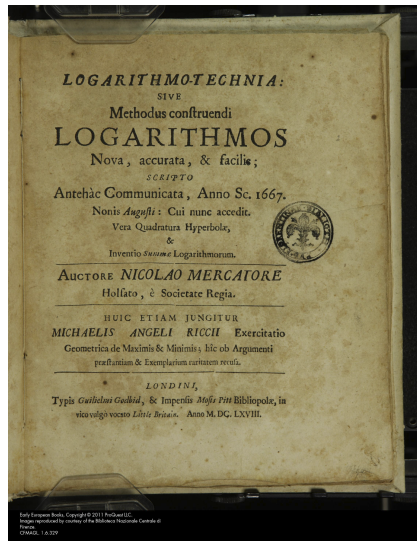


## Other 17th-century discoveries (2)

Mercator's series (1668), found  
by long division:

$$\frac{1}{1+a} = 1 - a + aa - a^3 + a^4 \text{ (&c.)}$$

Gives rise to series for log



## Other 17th-century discoveries (3)



James Gregory (1671):

- ▶ general binomial expansion
- ▶ series for  $\tan$ ,  $\sec$ , and others, including

$$\theta = \tan \theta - \frac{1}{2} \tan^3 \theta + \frac{1}{5} \tan^5 \theta - \dots$$

$$\text{for } -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$$

Gregory to Collins, 23rd November 1670:

*I suppose these series I send here enclosed, may have some affinity with those inventions you advertise me that Mr. Newton had discovered.*

(On Gregory's work, see: *Mathematics emerging*, §8.1.4.)

## Other 17th-century discoveries (4)

Gottfried Wilhelm Leibniz (1675):

The area of a circle with unit diameter is given by

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \&c.$$

The error in the sum is successively less than  $\frac{1}{3}$ ,  $\frac{1}{5}$ ,  $\frac{1}{7}$ , etc.

*Therefore the series as a whole contains all approximations at once, or values greater than correct and less than correct: for according to how far it is understood to be continued, the error will be smaller than a given fraction, and therefore also less than any given quantity. Therefore the series as a whole expresses the exact value.*

(See: *Mathematics emerging*, §8.3.)

## Series in the 17th century: 'convergence'

John Wallis (1656), *Arithmetica infinitorum*:

$$\square = \frac{4}{\pi} = \frac{3 \times 3 \times 5 \times 5 \times 7 \times 7 \times \dots}{2 \times 4 \times 4 \times 6 \times 6 \times 8 \times \dots}$$

(Determined that

$$\square > \sqrt{\frac{3}{2}}, \quad \square < \frac{3}{2} \sqrt{\frac{3}{4}}, \quad \square > \left( \frac{3 \times 3}{2 \times 4} \right) \sqrt{\frac{5}{4}},$$

and so on)

Brouncker (1668): grouping of terms

Leibniz (1675): 'alternating' series

## Power series in the 17th century

Power series (infinite polynomials):

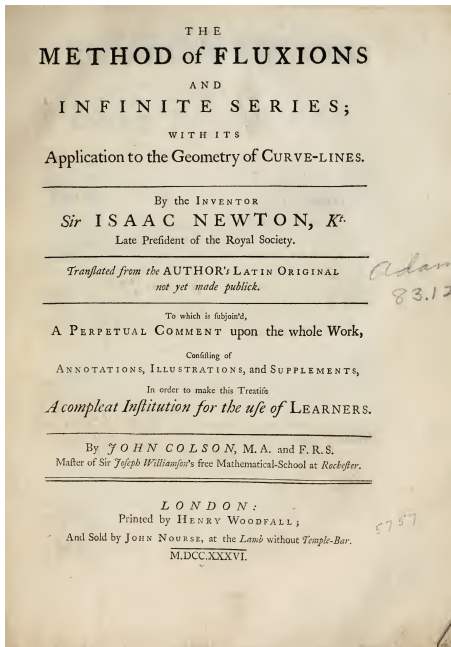
- ▶ enabled term-by-term integration for difficult quadratures;
- ▶ helped establish sine, log, ... as 'functions' (transcendental);
- ▶ encouraged a move from geometric to algebraic descriptions;
- ▶ for Newton (and others) inextricably linked with calculus.

Power series rank with calculus as a major advance of the 17th century



# Calculus and series combined

Newton's treatise of 1671,  
published 1736

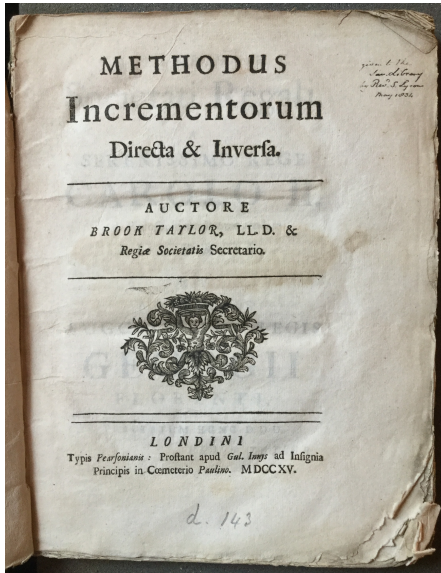


## Move on to the 18th century

Eighteenth century:

- ▶ as in 17th century, much progress;
- ▶ also many questions and doubts

# Taylor series



Brook Taylor,  
*The method of direct and  
inverse increments* (1715)

(See: *Mathematics emerging*, §8.2.1.)



# Taylor series

Taylor denoted a small change in  $x$  by  $\dot{x}$  (our  $\delta x$ ), a small change in  $\dot{x}$  by  $\ddot{x}$  (our  $\delta(\delta x)$ ), and so on

Dependent variable  $x$ ; independent variable  $z$  increases uniformly with time

$x$  increases to  $x + \delta x$  in time  $\delta t$ ; after a further interval of  $\delta t$ ,  $x$  has become  $x + \delta x + \delta(x + \delta x) = x + 2\delta x + \delta(\delta x)$ ; continuing:

$$\begin{aligned} & x + \frac{n}{1}\delta x + \frac{n(n-1)}{1 \cdot 2}\delta(\delta x) + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}\delta(\delta(\delta x)) + \dots \\ &= x + \delta x \frac{n\delta z}{1\delta z} + \delta(\delta x) \frac{n\delta z(n-1)\delta z}{1 \cdot 2 \cdot (\delta z)^2} + \delta(\delta(\delta x)) \frac{n\delta z(n-1)\delta z(n-2)\delta z}{1 \cdot 2 \cdot 3(\delta z)^3} + \dots \end{aligned}$$

# Taylor series

$$x + \delta x \frac{n\delta z}{1\delta z} + \delta(\delta x) \frac{n\delta z(n-1)\delta z}{1 \cdot 2 \cdot (\delta z)^2} + \delta(\delta(\delta x)) \frac{n\delta z(n-1)\delta z(n-2)\delta z}{1 \cdot 2 \cdot 3(\delta z)^3} + \dots$$

Assumptions:

- ▶  $(n-k)\delta z \approx n\delta z$ , since  $\delta z$  is small, so replace each  $(n-k)\delta z$  by  $v$ , a constant
- ▶  $\delta x \propto \dot{x}$  and  $\delta z \propto \dot{z}$ , so in each case the former can be replaced by the latter

In essence (in modern terms):  $\frac{\delta x}{\delta z} \rightarrow \frac{dx}{dz}$ ,  $\frac{\delta(\delta x)}{(\delta z)^2} \rightarrow \frac{d^2x}{dz^2}$ , and so on

Again in modern terms, we arrive at:

$$x + \frac{dx}{dz} v + \frac{d^2x}{dz^2} \frac{v^2}{1 \cdot 2} + \frac{d^3x}{dz^3} \frac{v^3}{1 \cdot 2 \cdot 3} + \dots$$

Cf. Taylor's notation in *Mathematics Emerging*, §8.1.2

# Maclaurin's *Treatise of fluxions*, vol. II, p. 610

610 *Of the inverse method of Fluxions.* Book II.

ties multiplied by  $k + 1x'' + m x^{2n}$  &c. raised to a power of any exponent  $k$ . *De quadrat. curv.* prop. 5. & 6.

751. The following theorem is likewise of great use in this doctrine. Suppose that  $y$  is any quantity that can be expressed by a series of this form  $A + Bz + Cz^2 + Dz^3 + \&c.$  where  $A, B, C, \&c.$  represent invariable coefficients as usual, any of which may be supposed to vanish. When  $z$  vanishes, let  $E$  be the value of  $y$ , and let  $\dot{E}, \ddot{E}, \ddot{\dot{E}}, \&c.$  be then the respective values of  $\dot{y}, \ddot{y}, \ddot{\dot{y}}, \&c.$   $z$  being supposed to flow uniformly.

Then  $y = E + \frac{\dot{E}z}{1} + \frac{\ddot{E}z^2}{1 \times 2} + \frac{\ddot{\dot{E}}z^3}{1 \times 2 \times 3} + \frac{\ddot{\dot{\dot{E}}}z^4}{1 \times 2 \times 3 \times 4} + \&c.$  the law of the continuation of which series is manifest. For since  $y = A + Bz + Cz^2 + Dz^3 + \&c.$  it follows that when  $z = 0$ ,  $A$  is equal to  $y$ ; but (by the supposition)  $E$  is then equal to  $y$ ; consequently  $A = E$ . By taking the fluxions, and dividing by  $\dot{z}$ ,  $\frac{\dot{y}}{\dot{z}} = B + 2Cz + 3Dz^2 + \&c.$  and when

$z = 0$ ,  $B$  is equal to  $\frac{\dot{y}}{\dot{z}}$ , that is to  $\frac{\dot{E}}{z}$ . By taking the fluxions a-

gain, and dividing by  $\dot{z}$  (which is supposed invariable)  $\frac{\ddot{y}}{\dot{z}} =$

$2C + 6Dz + \&c.$  let  $z = 0$ , and substituting  $\ddot{E}$  for  $\ddot{y}$ ,  $\frac{\ddot{E}}{\dot{z}} =$

$2C$ , or  $C = \frac{\ddot{E}}{2\dot{z}}$ . By taking the fluxions again, and dividing by

$\dot{z}$ ,  $\frac{\ddot{\dot{y}}}{\dot{z}} = 6D + \&c.$  and by supposing  $z = 0$ , we have  $D = \frac{\ddot{\dot{E}}}{6\dot{z}}$ .

Thus it appears that  $y = A + Bz + Cz^2 + Dz^3 + \&c. =$

$E + \frac{\dot{E}z}{1} + \frac{\ddot{E}z^2}{1 \times 2} + \frac{\ddot{\dot{E}}z^3}{1 \times 2 \times 3} + \frac{\ddot{\dot{\dot{E}}}z^4}{1 \times 2 \times 3 \times 4} + \&c.$  This pro-

position may be likewise deduced from the binomial theorem. Let

Suppose that  $y$  can be expressed as  $A + Bz + Cz^2 + Dz^3 + \dots$

When  $z$  vanishes,  $y = E$ ,  $\dot{y} = \dot{E}$ ,

$\ddot{y} = \ddot{E}$ ,  $\ddot{\dot{y}} = \ddot{\dot{E}}$ , and so on

$z$  is assumed to flow uniformly, so that  $\dot{z} = \text{const}$

By repeatedly taking fluxions, we may calculate in turn:  $A = E$ ,

$B = \frac{\dot{E}\dot{z}}{\dot{z}^2}$ ,  $C = \frac{\ddot{E}\ddot{z}}{2\dot{z}^2}$ ,  $D = \frac{\ddot{\dot{E}}\ddot{\dot{z}}}{6\dot{z}^3}$ , etc.

'the law of the continuation of [the] series is manifest'

(*Mathematics emerging*, §8.2.2.)

# Euler's *Introductio*

Leonhard Euler, *Introduction  
to analysis of the infinite*  
(1748)

## INTRODUCTIO IN ANALYSIN INFINITORUM.

AUCTORE

LEONHARDO EULERO,

Professore Regio BEROLINENSI, & Academiæ Im-  
perialis Scientiarum PETROPOLITANÆ  
Socio.

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TOMUS PRIMUS.

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LAUSANNÆ,

Apud MARCUM-MICHAELEM BOUSQUET & Socios.

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MDCCXLVIII



## Euler's *Introductio*

Incorporated power series into the definition of a **function**:

*A **function** of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities.*

Euler derived series for sine, cosine, exp, log, etc.;

he also discovered relationships between them, for example:

$$\cos v = \frac{1}{2}(e^{iv} + e^{-iv})$$

# An application of series

THE  
DOCTRINE  
OF  
CHANCES:

OR,  
A METHOD of Calculating the Probabilities  
of Events in PLAY.

THE SECOND EDITION,  
*Fuller, Clearer, and more Correct than the First.*

BY A. DE MOIVRE,  
*Fellow of the ROYAL SOCIETY, and Member of the ROYAL ACADEMY  
of SCIENCES of Berlin.*



LONDON:  
Printed for the AUTHOR, by H. WOODFALL, without Temple-Bar.  
M.DCC.XXXVIII.

Abraham de Moivre posed this problem about confidence intervals:

*What are the Odds that after a certain number of Experiments have been made concerning the happening or failing of Events, the Accidents of Contingency will not afterwards vary from those of Observation beyond certain Limits?*

His answer involved clever (but non-rigorous) summation and manipulation of infinite series.

*(Mathematics emerging, §7.1.3.)*



## XXXV<sup>ME</sup> MÉMOIRE.

*Reflexions sur les Suites & sur les Racines imaginaires.*

### S. I.

*Reflexions sur les suites divergentes ou convergentes.*

1. SI on élève  $1 + \mu$  à la puissance  $m$ , le terme  $n^e$  de la serie sera  $\mu^{n-1} \times \frac{m(m-1)\dots(m-n+2)}{2 \cdot 3 \cdot 4 \dots n-1}$ , & le suivant, c'est-à-dire le  $(n+1)^e$ , sera  $\mu^n \times \frac{m(m-1)\dots(m-n+2)(m-n+1)}{2 \cdot 3 \cdot 4 \dots n-1 \cdot n}$ ; donc le rapport du  $(n+1)^e$  terme au  $n^e$  sera  $\frac{\mu(m-n+1)}{n}$ ; or pour que la serie soit convergente, il faut que ce rapport (abstraction faite du signe qu'il doit avoir) soit < que l'unité.

2. Remarquons d'abord que la formule précédente donnera le moyen de former très-promptement les termes d'une suite: par exemple, si  $m = \frac{1}{2}$ , il faudra multiplier le premier terme par  $\mu \times \frac{1}{2}$  pour avoir le second;

Y ij

D'Alembert, 1761:

*... all reasoning and calculation based on series that do not converge, or that one may suppose not to, always seems to me extremely suspect, even when the results of this reasoning agree with truths known in other ways.*

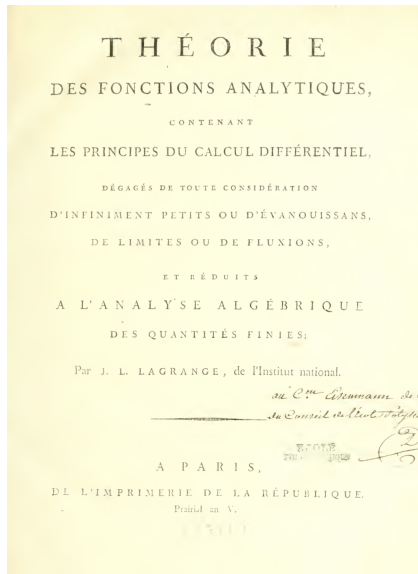
Introduced, without proof, what came to be known (in a more general setting) as **d'Alembert's ratio test**.

(See: *Mathematics emerging*, §8.3.1.)

# Lagrange's use of series

J.-L. Lagrange, *Théorie des fonctions analytiques* (1797)

Lagrange's use of series: an attempt to liberate calculus from infinitely small quantities (essentially by treating only those functions that may be described by power series)



# Lagrange and convergence

*... [one needs] a way of stopping the expansion of the series at any term one wants and of estimating the value of the remainder of the series.*

*This problem, one of the most important in the theory of series, has not yet been resolved in a general way*

Lagrange found bounds for the 'remainder' ...

and applied his findings to the binomial series ...

thus proving what Newton had taken for granted

(See: *Mathematics emerging*, §8.3.2.)