BO1.1. History of Mathematics
Lecture XI
19th-century rigour in real analysis, part 1

MT25 Week 6

Summary

- New difficulties emerge
- Continuity and convergence
- Integration
- The Fundamental Theorem of Calculus
- New ideas about integration

Recall from lecture VIII: Fourier series, 1822

Joseph Fourier, *Théorie analytique de la chaleur* [*Analytic theory of heat*] (1822):

Suppose that $\phi(x) = a \sin x + b \sin 2x + c \sin 3x + \cdots$

and also that $\phi(x) = x\phi'(0) + \frac{1}{6}x^3\phi'''(0) + \cdots$

After many pages of calculations, multiplying and comparing power series, Fourier found that the coefficient of $\sin nx$ must be

$$\frac{2}{\pi}\int_0^\pi \phi(x) \sin nx \, dx$$

Fourier's derivation was based on 'naive' manipulations of infinite series. It was ingenious but non-rigorous by today's standards.

BUT it led to profound results

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- functions — what exactly should they be?
- convergence — what exactly should it be?
- convergence of functions — what properties are preserved?
- integration — what exactly should it be?
- existence of limits — what are the essential properties of real numbers? [Lecture XII]

Recall from Lecture VIII: Cauchy sequences, 1821

Augustin-Louis Cauchy, *Cours d'analyse* (1821), Ch. VI, pp. 124, 125:

> *In order for the series* $u_0, \; u_1, \; u_2, \ldots$ [that is, $\sum u_i$] *to be convergent ... it is necessary and sufficient that the partial sums*
>
> $$s_n = u_0 + u_1 + u_2 + \&c. \ldots + u_{n-1}$$
>
> *converge to a fixed limit* s*: in other words, it is necessary and sufficient that for infinitely large values of the number* n*, the sums*
>
> $$s_n, \; s_{n+1}, \; s_{n+2}, \; \&c. \ldots$$
>
> *differ from the limit* s*, and consequently from each other, by infinitely small quantities.*

Cauchy and continuity revisited

In *Cours d'analyse*, p. 34, Cauchy defined a function f to be continuous between certain limits if, for each x between those limits, the value of $f(x)$ is unique and finite, and $|f(x+\alpha) - f(\alpha)|$, where α is indefinitely small, decreases indefinitely with α.

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So Cauchy defined continuity on an interval, rather than at a point.

He went on to derive basic results concerning continuous functions: that the composition of two continuous functions is continuous, the Intermediate Value Theorem, etc.

A theorem of Cauchy (1821)

Cauchy, *Cours d'analyse*, pp. 131–132:

> *When the various terms of a series are functions of a variable* x*, continuous with respect to this variable in the neighbourhood of a particular value for which the series is convergent, the sum* s *of the series is also, in the neighbourhood of this value, a continuous function of* x*.*

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Not true!

Cauchy's argument

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NB. All notation except '$\sum$' is Cauchy's.

Cauchy's argument

est convergente, la somme de cette série est représentée par

$$u_0 + u_1 + u_2 + u_3 + \&c. \ldots$$

En vertu de cette convention, la valeur du nombre e se trouvera déterminée par l'équation

$$(6) \qquad e = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \&c. \ldots ;$$

et, si l'on considère la progression géométrique

$$1, \; x, \; x^2, \; x^3, \; \&c. \ldots,$$

on aura, pour des valeurs numériques de x inférieures à l'unité,

$$(7) \qquad 1 + x + x^2 + x^3 + \&c. \ldots = \frac{1}{1-x}.$$

La série

$$u_0, \; u_1, \; u_2, \; u_3, \; \&c. \ldots$$

étant supposée convergente, si l'on désigne sa somme par s, et par s_n la somme de ses n premiers termes, on trouvera

$$s = u_0 + u_1 + u_2 + \ldots + u_{n-1} + u_n + u_{n+1} + \&c. \ldots$$
$$= s_n + u_n + u_{n+1} + \&c. \ldots,$$

et par suite

$$s - s_n = u_n + u_{n+1} + \&c. \ldots$$

De cette dernière équation il résulte que les quantités

$$u_n, \; u_{n+1}, \; u_{n+2}, \; \&c. \ldots$$

formeront une nouvelle série convergente dont la somme sera équivalente à $s - s_n$. Si l'on représente cette même somme par r_n, on aura

$$s = s_n + r_n ;$$

et r_n sera ce qu'on appelle le *reste* de la série (1) à partir du n.me terme.

Lorsque, les termes de la série (1) renfermant une même variable x, cette série est convergente, et ses différens termes fonctions continues de x, dans le voisinage d'une valeur particulière attribuée à cette variable ;

$$s_n, \quad r_n \quad \text{et} \quad s$$

sont encore trois fonctions de la variable x, dont la première est évidemment continue par rapport à x dans le voisinage de la valeur particulière dont il s'agit. Cela posé, considérons les accroissemens que reçoivent ces trois fonctions, lorsqu'on fait croître x d'une quantité infiniment petite α. L'accroissement de s_n sera, pour toutes les valeurs possibles de n, une quantité infiniment petite ; et celui de r_n deviendra insensible en même temps que r_n, si l'on attribue à n une valeur très-considérable. Par suite, l'accroissement de la fonction s ne pourra être qu'une quantité infiniment petite. De cette remarque on déduit immédiatement la proposition suivante.

1.er THÉORÈME. *Lorsque les différens termes de la série (1) sont des fonctions d'une même variable x,*

A modern counterexample

For each $n \in \mathbb{N}$, define continuous functions f_n by

$$f_n(x) = \begin{cases} -1 & \text{if } x \leq -\frac{1}{n}; \\ nx & \text{if } -\frac{1}{n} \leq x \leq \frac{1}{n}; \\ +1 & \text{if } x \geq \frac{1}{n}. \end{cases}$$

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Now set $u_1(x) = f_1(x)$, and define new functions u_n recursively by

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But we see that $s_n \to s$ as $n \to \infty$, where

$$s(x) = \begin{cases} -1 & \text{if } x < 0; \\ 0 & \text{if } x = 0; \\ +1 & \text{if } x > 0, \end{cases}$$

which is discontinuous at $x = 0$.

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$$r_n(x) = \begin{cases} -1 - nx & \text{if } -\frac{1}{n} \leq x < 0; \\ 0 & \text{if } x = 0; \\ 1 - nx & \text{if } 0 < x \leq \frac{1}{n}. \end{cases}$$

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For each x, $r_n(x) \to 0$ as $n \to \infty$, but this does not happen simultaneously for all values of x.

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Cauchy clearly didn't make this distinction — but should this really be regarded as a 'mistake'?

Reactions to Cauchy's 'mistake'

Abel (Crelle's Journal, 1826) on the *Cours d'analyse*:

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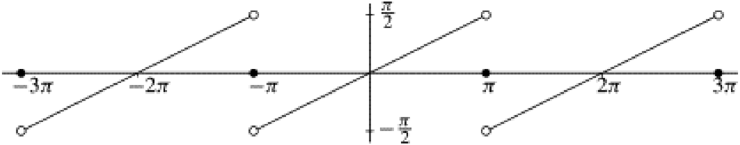
Four pages later, on Cauchy's 'theorem' on sums of continuous functions:

> *it seems to me that the theorem admits exceptions. For example, the series*
>
> $$\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots$$
>
> *is discontinuous for every value* $(2m+1)\pi$ *of* x, m *being a whole number. There are, as one knows, many series of this kind.*

Abel's counterexample

$$\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots$$



Abel's counterexample elsewhere

Abel to Holmboe, January 1826:

> *One applies all operations to infinite series as if they were finite, but is this permissible? I think not. — Where is it proved that one gets the differential of an infinite series by differentiating each term? It is easy to give an example for which this is not true, e.g.*
>
> $$\frac{1}{2}x = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots .$$
>
> *Differentiation gives*
>
> $$\frac{1}{2} = \cos x - \cos 2x + \cos 3x - \cdots \text{etc.}$$
>
> *a result which is quite false because this series is divergent.*

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(See: Henrik Kragh Sørensen, 'Exceptions and counterexamples: Understanding Abel's comment on *Cauchy's Theorem*', *Historia Mathematica* **32** (2005) 453–480)

Abel's interpretation

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Similarly, what did Abel mean by 'continuity' and 'convergence'? The same as Cauchy? Or did he use a similar form of words but with a different meaning?

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- Karl Weierstrass (lectures in Berlin), 1841;

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See: G. H. Hardy, 'Sir George Stokes and the concept of uniform convergence', *Proc. Camb. Phil. Soc.* 19 (1918) 148–156 (also: *Collected Papers of G. H. Hardy*, vol. VII, 505–513)

And: Klaus Viertel, 'The development of the concept of uniform convergence in Karl Weierstrass's lectures and publications between 1861 and 1886', *Arch. Hist. Exact Sci.* 75 (2021), 455–490

Cauchy revisits his theorem (1853)

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Theorem. If the different terms of the series

$$u_0, u_1, u_2, \ldots, u_n, u_{n+1}, \ldots$$

are functions of a real variable x, continuous with respect to this variable within the given limits; and if, in addition, the sum

$$u_n + u_{n+1} + \cdots + u_{n'}$$

always becomes infinitely small for infinitely large values of the whole numbers n and $n' > n$, then the series will be convergent and the sum of the series will be, within the given limits, a continuous function of the variable x.

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But it was becoming clear that the language of infinities and infinitesimals was inadequate for expressing the problems at hand.

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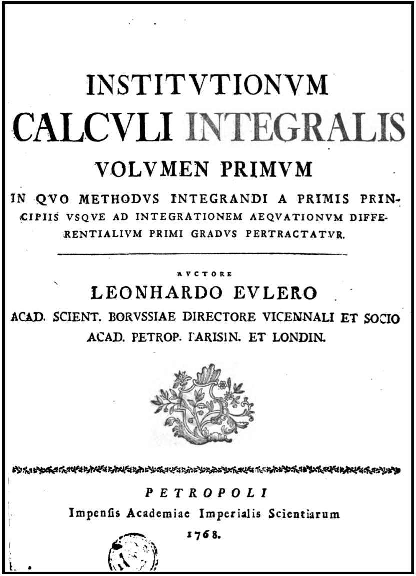
- In the 18th century, ‘integration’ was essentially regarded as the inverse of differentiation.

Integration in the 18th century (1)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AVCTORE
LEONHARDO EVLERO
ACAD. SCIENT. BORVSSIAE DIRECTORE VICENNALI ET SOCIO ACAD. PETROP. PARISIN. ET LONDIN.

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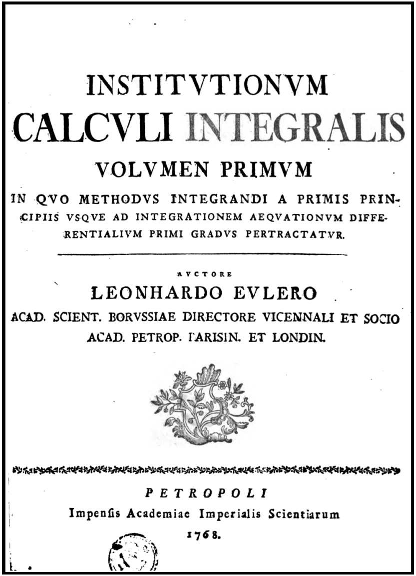

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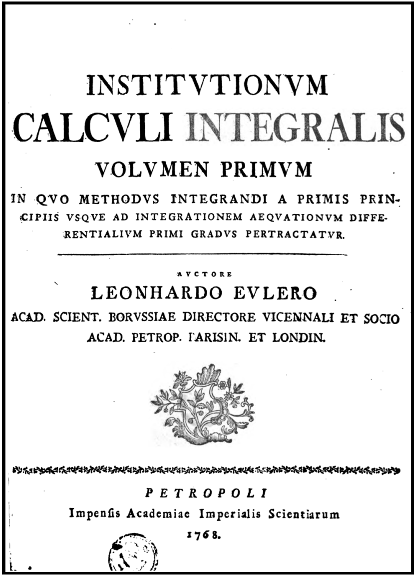
Leonhard Euler, *Foundations of integral calculus* (1768):

Definition 1: Integral calculus is the method of finding, from a given relationship between differentials, a relationship between the quantities themselves: and the operation by which this is carried out is usually called integration.

(See *Mathematics emerging*, §14.2.1.)

Integration in the 18th century (2)



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Corollary 1: Therefore where differential calculus teaches us to investigate the relationship between differentials from a given relationship between variable quantities, integral calculus supplies the inverse method.

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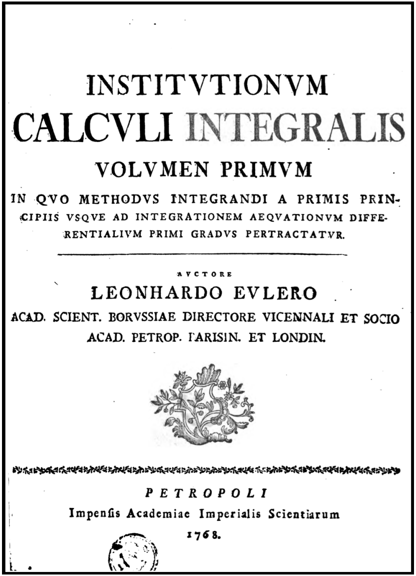

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Corollary 1: Therefore where differential calculus teaches us to investigate the relationship between differentials from a given relationship between variable quantities, integral calculus supplies the inverse method.

Corollary 2: Clearly just as in Analysis two operations are always contrary to each other, as subtraction to addition, division to multiplication, extraction of roots to raising of powers, so also by similar reasoning integral calculus is contrary to differential calculus.

Integration in the 18th century (3)

ratione conſcripti prodierint, huiusmodi conciliatio nullum vſum eſſet habitura.

Definitio 2.

7. Cum functionis cuiuscunque ipſius x differentiale huiusmodi habeat formam Xdx, propoſita tali forma differentiali Xdx, in qua X ſit functio quaecunque ipſius x, illa functio, cuius differentiale eſt $=Xdx$, huius vocatur integrale, et praefixo ſigno $\int$ indicari ſolet, ita vt $\int Xdx$ eam denotet quantitatem variabilem, cuius differentiale eſt $=Xdx$.

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Coroll. 2.

9. Vti ergo littera d ſignum eſt differentiationis, ita littera $\int$ pro ſigno integrationis vtimur, ſicque haec duo ſigna ſibi mutuo opponuntur, et quaſi ſe deſtruunt ſcilicet $\int dX$ erit $=X$, quia ea quantitas denotatur cuius differentiale eſt dX, quae vtique eſt X.

Coroll. 3.

10. Cum igitur harum ipſius x functionum x^2, x^n, $\sqrt{(aa-xx)}$ differentialia ſint $2xdx$, $nx^{n-1}dx$, $\frac{-xdx}{\sqrt{(aa-xx)}}$ ſigno integrationis $\int$ adhibendo patet fore

$\int 2x$

Definition 2: Since the differentiation of any function of x has a form of this kind: $X\,dx$, when such a differential form $X\,dx$ is proposed, in which X is any function of x, that function whose differential $= X\,dx$ is called its integral, and is usually indicated by the prefix $\int$, so that $\int X\,dx$ denotes that variable quantity whose differential $= X\,dx$.

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Corollary 2: Therefore just as the letter d is the sign of differentiation, so we use the letter $\int$ as the sign of integration, and thus these two signs are mutually contrary to each other, as though they destroy each other: certainly $\int dX = X$, ...

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Some 19th-century ideas

Recall that Fourier coefficients are given by $\dfrac{2}{\pi}\displaystyle\int_0^\pi \phi(x)\sin nx\,dx$.

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Fourier thus returned to the idea of integral as area and influenced Cauchy almost immediately...

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Suppose $f(x)$ *continuous between* $x = x_0$ *and* $x = X$. *Choose* $x_1, x_2, \ldots, x_{n-1}$ *between these limits. Define*

$$S = (x_1 - x_0)f(x_0) + (x_2 - x_1)f(x_1) + \cdots + (X - x_{n-1})f(x_{n-1})$$

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If the numerical values of the elements are made to decrease indefinitely by increasing their number, the value of S *will become essentially constant, or in other words, it will finish by attaining a certain limit which will depend only on the form of the function* $f(x)$ *and the boundary values* $x = x_0$, $x = X$ *given to the variable* x. *This limit is what one calls a definite integral.*

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[further issues connected with uniform convergence]

Cauchy and integrals

Observons maintenant que, si l'on désigne par $\Delta x = h = dx$ un accroissement fini attribué à la variable x, les différens termes dont se compose la valeur de S, tels que les produits $(x_1 - x_0) f(x_0)$, $(x_2 - x_1) f(x_1)$, &c... seront tous compris dans la formule générale

$$(8) \qquad h f(x) = f(x)\, dx$$

de laquelle on les déduira l'un après l'autre, en posant d'abord $x = x_0$ et $h = x_1 - x_0$, puis $x = x_1$ et $h = x_2 - x_1$, &c... On peut donc énoncer que la quantité S est une somme de produits semblables à l'expression (8); ce qu'on exprime quelquefois à l'aide de la caractéristique Σ en écrivant

$$(9) \qquad S = \Sigma h f(x) = \Sigma f(x) \Delta x.$$

Quant à l'intégrale définie vers laquelle converge la quantité S, tandis que les élémens de la différence $X - x_0$ deviennent infiniment petits, on est convenu de la représenter par la notation $\int h f(x)$ ou $\int f(x) dx$, dans laquelle la lettre $\int$ substituée à la lettre Σ indique, non plus une somme de produits semblables à l'expression (8), mais la limite d'une somme de cette espèce. De plus, comme la valeur de l'intégrale définie que l'on considère dépend des valeurs extrêmes x_0, X attribuées à la variable x, on est convenu de placer ces deux valeurs, la première au-dessous, la seconde au-dessus de la lettre $\int$, ou de les écrire à côté de l'intégrale, que l'on désigne en conséquence par l'une des notations

$$(10) \qquad \int_{x_0}^{X} f(x)\, dx, \quad \int f(x)\, dx \left[{x_0 \atop X}\right], \quad \int f(x)\, dx \left[{x = x_0 \atop x = X}\right].$$

La première de ces notations, imaginée par M. *Fourier*, est la plus simple. Dans le cas particulier où la fonction $f(x)$ est remplacée par une quantité constante a, on trouve, quel que soit le mode de division de la différence $X - x_0$, $\quad S = a(X - x_0)$, $\quad$ et l'on en conclut

$$(11) \qquad \int_{x_0}^{X} a\, dx = a(X - x_0),$$

Si, dans cette dernière formule on pose $a = 1$, on en tirera

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84 COURS D'ANALYSE.

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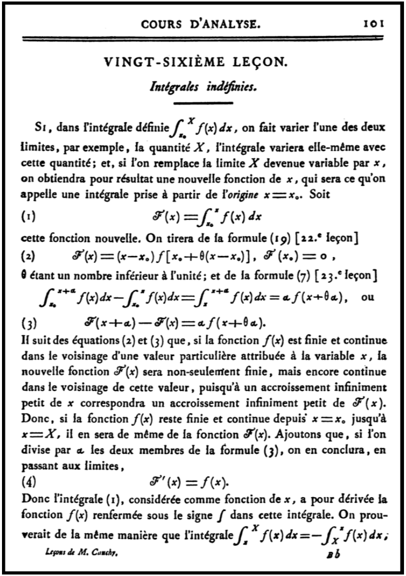
Is it valid to use the symbol $\int$ here?

Cauchy and the Fundamental Theorem of Calculus



VINGT-SIXIÈME LEÇON.

Intégrales indéfinies.

Si, dans l'intégrale définie $\int_{x_0}^{X} f(x)\,dx$, on fait varier l'une des deux limites, par exemple, la quantité X, l'intégrale variera elle-même avec cette quantité; et, si l'on remplace la limite X devenue variable par x, on obtiendra pour résultat une nouvelle fonction de x, qui sera ce qu'on appelle une intégrale prise à partir de l'*origine* $x=x_0$. Soit

(1) $$\mathscr{F}(x)=\int_{x_0}^{x} f(x)\,dx$$

cette fonction nouvelle. On tirera de la formule (19) [22.e leçon]

(2) $$\mathscr{F}(x)=(x-x_0)f[x_0+\theta(x-x_0)], \quad \mathscr{F}(x_0)=0,$$

θ étant un nombre inférieur à l'unité; et de la formule (7) [23.e leçon]

$$\int_{x_0}^{x+\alpha} f(x)\,dx-\int_{x_0}^{x} f(x)\,dx=\int_{x}^{x+\alpha} f(x)\,dx=\alpha f(x+\theta\alpha), \quad \text{ou}$$

(3) $$\mathscr{F}(x+\alpha)-\mathscr{F}(x)=\alpha f(x+\theta\alpha).$$

Il suit des équations (2) et (3) que, si la fonction $f(x)$ est finie et continue dans le voisinage d'une valeur particulière attribuée à la variable x, la nouvelle fonction $\mathscr{F}(x)$ sera non-seulement finie, mais encore continue dans le voisinage de cette valeur, puisqu'à un accroissement infiniment petit de x correspondra un accroissement infiniment petit de $\mathscr{F}(x)$. Donc, si la fonction $f(x)$ reste finie et continue depuis $x=x_0$ jusqu'à $x=X$, il en sera de même de la fonction $\mathscr{F}(x)$. Ajoutons que, si l'on divise par α les deux membres de la formule (3), on en conclura, en passant aux limites,

(4) $$\mathscr{F}'(x)=f(x).$$

Donc l'intégrale (1), considérée comme fonction de x, a pour dérivée la fonction $f(x)$ renfermée sous le signe $\int$ dans cette intégrale. On prouverait de la même manière que l'intégrale $\int_{x}^{X} f(x)\,dx=-\int_{X}^{x} f(x)\,dx$;

If in the definite integral $\int_{x_0}^{X} f(x)\,dx$ one makes one of the two limits vary, for example the quantity X, the integral itself will vary with this quantity; and if one replaces the variable limit X by x, there results a new function of x, . . .

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Cauchy and the Fundamental Theorem of Calculus

VINGT-SIXIÈME LEÇON.

Intégrales indéfinies.

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$$\mathscr{F}(x) = \int_{x_0}^{x} f(x)\,dx \tag{1}$$

cette fonction nouvelle. On tirera de la formule (19) [22.e leçon]

$$\mathscr{F}(x) = (x - x_0) f[x_0 + \theta(x - x_0)], \quad \mathscr{F}(x_0) = 0, \tag{2}$$

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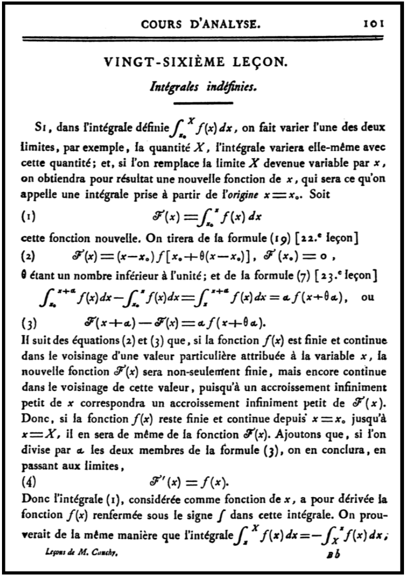

COURS D'ANALYSE. 101

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Leçons de M. Cauchy. Bb

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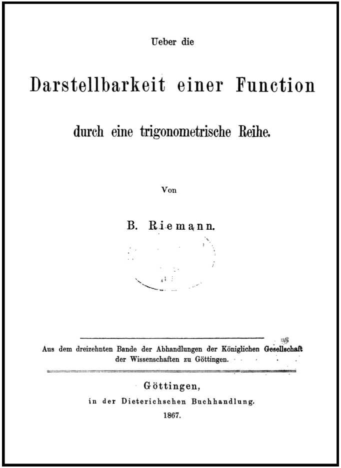
Bernhard Riemann (1826–1866)



Ueber die

Darstellbarkeit einer Function

durch eine trigonometrische Reihe.

Von

B. Riemann.

Aus dem dreizehnten Bande der Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen.

Göttingen,

in der Dieterichschen Buchhandlung.

1867.

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If this has the property that it comes infinitely close to a fixed value A when all the δ_i become infinitely small, then this is the value of $\int_a^b f(x)\,dx$.

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Many variants over the years, all called Riemann integral.

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Results in a notion of integral of wider applicability than Riemann's; for example:

can integrate highly discontinuous functions, such as the Dirichlet function:

$$f(x) := \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$