# BO1.1. History of Mathematics
# Lecture XII
# 19th-century rigour in real analysis, part 2: real numbers and sets

MT25 Week 6

# Summary

- Proofs of the Intermediate Value Theorem revisited
- Convergence and completeness
- Dedekind and the continuum
- Cantor and numbers and sets
- Where and when did sets emerge?
- Early set theory
- Set theory as a language

# The Intermediate Value Theorem (1)

Bolzano's criticisms (1817) of existing proofs:

> *The most common kind of proof depends on a truth borrowed from geometry ... But it is clear that it is an intolerable offense against correct method to derive truths of pure (or general) mathematics (i.e., arithmetic, algebra, analysis) from considerations which belong to a merely applied (or special) part, namely, geometry.*

His own proof includes something close to a proof that Cauchy sequences converge:

> *... the true value of $X$ [the limit] therefore ... can be determined as accurately as required ... There is, therefore, a real quantity which the terms of the series, if it is continued far enough, approach as closely as desired.*

But Bolzano assumed the existence of the limit.

# The Intermediate Value Theorem (2)

Cauchy's 1st proof (*Cours d'analyse*, 1821, p. 44) is geometric (though he didn't provide a diagram):

> *The function $f(x)$ being continuous between the limits $x = x_0$, $x = X$, the curve which has for equation $y = f(x)$ passes first through the point corresponding to the coordinates $x_0$, $f(x_0)$, second through the point corresponding to the coordinates $X$, $f(X)$, will be continuous between these two points: and, since the constant ordinate $b$ of the line which has for equation $y = b$ is to be found between the ordinates $f(x_0)$, $f(X)$ of the two points under consideration, the line will necessarily pass between these two points, which it cannot do without meeting the curve mentioned above in the interval.*

Cauchy's 2nd proof in a different context (p. 460): a numerical method for finding roots of equations — tacitly assumes that bounded monotone sequences of real numbers converge [see Lecture VIII].

# The need for a deeper understanding (1)

Emergence of rigour in Analysis:

- Bolzano, *Rein analytischer Beweis ...*, 1817;
- Cauchy, *Cours d'analyse*, 1821, etc.

  *By 1821, therefore, attempts to prove the intermediate value theorem had brought three important propositions into play:*

  1. *Cauchy sequences are convergent (with an unsuccessful proof by Bolzano in 1817; accepted without proof by Cauchy in 1821).*
  2. *A* [*non-empty*] *set of numbers bounded below has a greatest lower bound (proved by Bolzano in 1817 on the basis of (1)).*
  3. *A monotonic bounded sequence converges to a limit (taken for granted by Cauchy in 1821).*

(*Mathematics emerging*, §16.3.1.)

# The need for a deeper understanding (2)

What's missing here from the modern point of view: completeness

Completeness of the real number system $\mathbb{R}$ in modern teaching:

- non-empty bounded sets of real numbers have least upper bounds
- monotonic bounded sequences converge
- Cauchy sequences converge
- ...

All equivalent

# Equivalence of formulations of completeness

Bolzano–Weierstrass Theorem: A bounded sequence of real numbers has a convergent subsequence.

Implicit in Bolzano (1817); explicit in lectures by Karl Weierstrass (1815–1897) in Berlin 1859/60, 1863/64: a step in proofs from other definitions of completeness that Cauchy sequences converge.

Modern proofs often use the lemma that every infinite sequence of real numbers has an infinite monotonic subsequence.

How to incorporate these ideas into analysis in a rigorous way?

All of the above relies upon an intuitive notion of real number — so perhaps provide a formal definition of these? One that includes the idea of completeness?

# Richard Dedekind (1831–1916)

Stetigkeit

und

irrationale Zahlen.

102194

Von

Richard Dedekind,

Professor der höheren Mathematik am Collegium Carolinum zu Braunschweig.

Braunschweig,

Druck und Verlag von Friedrich Vieweg und Sohn.

1872.

# Dedekind and the foundations of analysis

Teaching calculus in the Zürich Polytechnic (1858), later (from 1862) teaching Fourier series in the Braunschweig Polytechnic, found himself dissatisfied with:

- geometry as a foundation for analysis;
- tacit assumptions about convergence (for monotonic bounded sequences, for example).

Response eventually published in *Stetigkeit und irrationale Zahlen* (1872) [translated as *Continuity and irrational numbers* by Wooster Woodruff Beman, 1901]

# Dedekind and continuity (1)

Intuition suggests that numbers (an arithmetical concept) should have the same completeness and continuity properties as a line (a geometrical concept). But we must define these concepts for numbers without appeal to geometrical intuition.

Geometrically, every point separates a line into two parts.

> *I find the essence of continuity in the converse, i.e., in the following principle:*
>
> *"If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions."*

# Dedekind and continuity (2)

But Dedekind couldn't *prove* this property, so he had to take it as an axiom:

> *The assumption of this property for the line is nothing but an Axiom, through which alone we attribute continuity to the line, through which we understand continuity in the line.*

(See *Mathematics emerging*, §16.3.2.)

# Dedekind and continuity (3)

Next adapt this idea to the arithmetical context:

- every number $x$ separates all other numbers into two classes — those greater than $x$, and those less than $x$;
- conversely, every such separation of numbers defines a number.

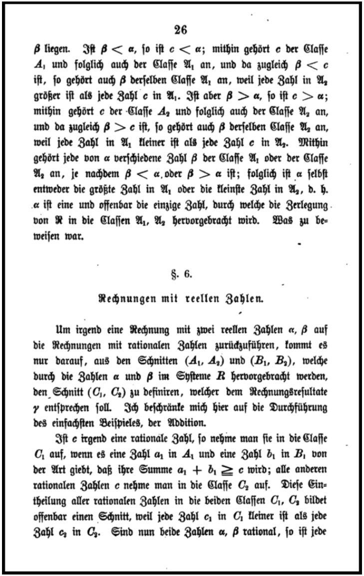
Hence Dedekind cuts (or sections, from the original German Schnitt).

# Dedekind cuts (1)

- Start from the system of rational numbers $R$ (assumed known)
- Separate $R$ into two classes $A_1$ and $A_2$ such that
  - for any $a_1$ in $A_1$, $a_1 < a_2$ for every $a_2$ in $A_2$
  - for any $a_2$ in $A_2$, $a_2 > a_1$ for every $a_1$ in $A_1$
- The cut denoted by $(A_1, A_2)$ defines a number
- Important observation: $(A_1, A_2)$ need not be rational

*Whenever, then, we have to do with a cut produced by no rational number, we create a new irrational number, which we regard as completely defined by this cut . . .*

# Dedekind cuts (2)



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$\beta$ liegen. Ist $\beta < \alpha$, so ist $c < \alpha$; mithin gehört $c$ der Classe $A_1$ und folglich auch der Classe $\mathfrak{A}_1$ an, und da zugleich $\beta < c$ ist, so gehört auch $\beta$ derselben Classe $\mathfrak{A}_1$ an, weil jede Zahl in $\mathfrak{A}_2$ größer ist als jede Zahl $c$ in $\mathfrak{A}_1$. Ist aber $\beta > \alpha$, so ist $c > \alpha$; mithin gehört $c$ der Classe $A_2$ und folglich auch der Classe $\mathfrak{A}_2$ an, und da zugleich $\beta > c$ ist, so gehört auch $\beta$ derselben Classe $\mathfrak{A}_2$ an, weil jede Zahl in $\mathfrak{A}_1$ kleiner ist als jede Zahl $c$ in $\mathfrak{A}_2$. Mithin gehört jede von $\alpha$ verschiedene Zahl $\beta$ der Classe $\mathfrak{A}_1$ oder der Classe $\mathfrak{A}_2$ an, je nachdem $\beta < \alpha$ oder $\beta > \alpha$ ist; folglich ist $\alpha$ selbst entweder die größte Zahl in $\mathfrak{A}_1$ oder die kleinste Zahl in $\mathfrak{A}_2$, d. h. $\alpha$ ist eine und offenbar die einzige Zahl, durch welche die Zerlegung von $\mathfrak{R}$ in die Classen $\mathfrak{A}_1$, $\mathfrak{A}_2$ hervorgebracht wird. Was zu beweisen war.

§. 6.

Rechnungen mit reellen Zahlen.

Um irgend eine Rechnung mit zwei reellen Zahlen $\alpha$, $\beta$ auf die Rechnungen mit rationalen Zahlen zurückzuführen, kommt es nur darauf, aus den Schnitten $(A_1, A_2)$ und $(B_1, B_2)$, welche durch die Zahlen $\alpha$ und $\beta$ im Systeme $R$ hervorgebracht werden, den Schnitt $(C_1, C_2)$ zu definiren, welcher dem Rechnungsresultate $\gamma$ entsprechen soll. Ich beschränke mich hier auf die Durchführung des einfachsten Beispieles, der Addition.

Ist $c$ irgend eine rationale Zahl, so nehme man sie in die Classe $C_1$ auf, wenn es eine Zahl $a_1$ in $A_1$ und eine Zahl $b_1$ in $B_1$ von der Art giebt, daß ihre Summe $a_1 + b_1 \geq c$ wird; alle anderen rationalen Zahlen $c$ nehme man in die Classe $C_2$ auf. Diese Eintheilung aller rationalen Zahlen in die beiden Classen $C_1$, $C_2$ bildet offenbar einen Schnitt, weil jede Zahl $c_1$ in $C_1$ kleiner ist als jede Zahl $c_2$ in $C_2$. Sind nun beide Zahlen $\alpha$, $\beta$ rational, so ist jede

Dedekind showed how to add two cuts, and how to use them in limiting arguments — but did little else with them.

**Significance:** a major step towards

- understanding completeness, and
- giving a rigorous definition of an irrational number, hence
- setting the foundations of analysis onto a sound logical basis.

# Dissemination of Dedekind's ideas

*Stetigkeit und irrationale Zahlen* reprinted many times, often in conjunction with the later essay *Was sind und was sollen die Zahlen?* (1888) [see below].

Translated into English as *Essays on the theory of numbers* by Wooster Woodruff Beman (1901).

Popularised and organised for teaching, starting from Peano axioms for natural numbers, by Edmund Landau in *Grundlagen der Analysis* [*Foundations of analysis*] (1930), a book that contains very few words.

A good modern (historically sensitive) account can be found in: Leo Corry, *A brief history of numbers*, OUP, 2015, §10.6.

# Other approaches

Georg Cantor (1872) and Eduard Heine (1872) created real numbers as equivalence classes of Cauchy sequences of rational numbers. (Also: Charles Méray in 1869.)

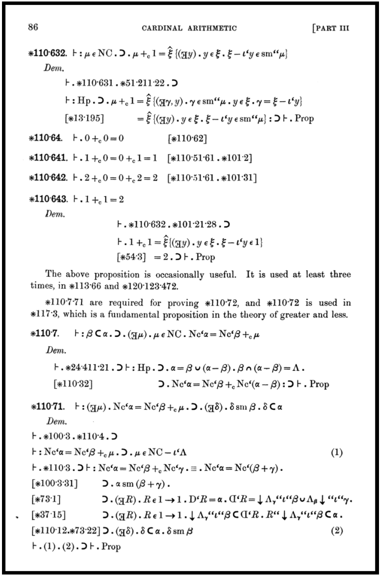
(On Cantor's approach, see *Mathematics emerging*, §16.3.3.)

Heine acknowledged a debt to Cantor and a debt to the lectures of Weierstrass.

Later constructions by many mathematicians and philosophers, often as part of a broader effort to lay down logical foundations for mathematics as a whole — for example:

- Carl Johannes Thomae, 1880, 1890
- Giuseppe Peano, 1889, 1891
- Gottlob Frege, 1884, 1893, 1903
- Otto Hölder, 1901

# Extreme formalism


86 CARDINAL ARITHMETIC [PART III

**∗110·632.** $\vdash : \mu \in \mathrm{NC} \,.\supset .\, \mu +_c 1 = \hat{\xi}\{(\exists y) \,.\, y \in \xi \,.\, \xi - \iota' y \in \mathrm{sm}``\mu\}$

*Dem.*

$\vdash .\, *110{\cdot}631 \,.\, *51{\cdot}211{\cdot}22 \,.\supset$

$\vdash : \mathrm{Hp} \,.\supset .\, \mu +_c 1 = \hat{\xi}\{(\exists \gamma, y) \,.\, \gamma \in \mathrm{sm}``\mu \,.\, y \in \xi \,.\, \gamma = \xi - \iota' y\}$

$[*13{\cdot}195] \qquad = \hat{\xi}\{(\exists y) \,.\, y \in \xi \,.\, \xi - \iota' y \in \mathrm{sm}``\mu\} : \supset \vdash .\, \mathrm{Prop}$

**∗110·64.** $\vdash .\, 0 +_c 0 = 0 \qquad [*110{\cdot}62]$

**∗110·641.** $\vdash .\, 1 +_c 0 = 0 +_c 1 = 1 \qquad [*110{\cdot}51{\cdot}61 \,.\, *101{\cdot}2]$

**∗110·642.** $\vdash .\, 2 +_c 0 = 0 +_c 2 = 2 \qquad [*110{\cdot}51{\cdot}61 \,.\, *101{\cdot}31]$

**∗110·643.** $\vdash .\, 1 +_c 1 = 2$

*Dem.*

$\vdash .\, *110{\cdot}632 \,.\, *101{\cdot}21{\cdot}28 \,.\supset$

$\vdash .\, 1 +_c 1 = \hat{\xi}\{(\exists y) \,.\, y \in \xi \,.\, \xi - \iota' y \in 1\}$

$[*54{\cdot}3] \quad = 2 \,.\supset \vdash .\, \mathrm{Prop}$

The above proposition is occasionally useful. It is used at least three times, in ∗113·66 and ∗120·123·472.

∗110·7·71 are required for proving ∗110·72, and ∗110·72 is used in ∗117·3, which is a fundamental proposition in the theory of greater and less.

**∗110·7.** $\vdash : \beta \subset \alpha \,.\supset .\, (\exists \mu) \,.\, \mu \in \mathrm{NC} \,.\, \mathrm{Nc}'\alpha = \mathrm{Nc}'\beta +_c \mu$

*Dem.*

$\vdash .\, *24{\cdot}411{\cdot}21 \,.\supset \vdash : \mathrm{Hp} \,.\supset .\, \alpha = \beta \cup (\alpha - \beta) \,.\, \beta \cap (\alpha - \beta) = \Lambda \,.$

$[*110{\cdot}32] \qquad \supset .\, \mathrm{Nc}'\alpha = \mathrm{Nc}'\beta +_c \mathrm{Nc}'(\alpha - \beta) : \supset \vdash .\, \mathrm{Prop}$

**∗110·71.** $\vdash : (\exists \mu) \,.\, \mathrm{Nc}'\alpha = \mathrm{Nc}'\beta +_c \mu \,.\supset .\, (\exists \delta) \,.\, \delta \,\mathrm{sm}\, \beta \,.\, \delta \subset \alpha$

*Dem.*

$\vdash .\, *100{\cdot}3 \,.\, *110{\cdot}4 \,.\supset$

$\vdash : \mathrm{Nc}'\alpha = \mathrm{Nc}'\beta +_c \mu \,.\supset .\, \mu \in \mathrm{NC} - \iota'\Lambda \qquad (1)$

$\vdash .\, *110{\cdot}3 \,.\supset \vdash : \mathrm{Nc}'\alpha = \mathrm{Nc}'\beta +_c \mathrm{Nc}'\gamma \,.\equiv .\, \mathrm{Nc}'\alpha = \mathrm{Nc}'(\beta + \gamma) \,.$

$[*100{\cdot}3{\cdot}31] \qquad \supset .\, \alpha \,\mathrm{sm}\, (\beta + \gamma) \,.$

$[*73{\cdot}1] \qquad \supset .\, (\exists R) \,.\, R \in 1 \to 1 \,.\, \mathrm{D}'R = \alpha \,.\, \mathrm{G}'R = \downarrow \Lambda_\gamma``\iota``\beta \cup \Lambda_\beta \downarrow ``\iota``\gamma \,.$

$[*37{\cdot}15] \qquad \supset .\, (\exists R) \,.\, R \in 1 \to 1 \,.\, \downarrow \Lambda_\gamma``\iota``\beta \subset \mathrm{G}'R \,.\, R`` \downarrow \Lambda_\gamma``\iota``\beta \subset \alpha \,.$

$[*110{\cdot}12 . *73{\cdot}22] \supset .\, (\exists \delta) \,.\, \delta \subset \alpha \,.\, \delta \,\mathrm{sm}\, \beta \qquad (2)$

$\vdash .\, (1) \,.\, (2) \,.\supset \vdash .\, \mathrm{Prop}$

Alfred North Whitehead and Bertrand Russell, *Principia mathematica*, 3 vols., Cambridge University Press, 1910, 1912, 1913

Vol. II, p. 86: $1 + 1 = 2$

"The above proposition is occasionally useful."

**NB.** This is not the source of our axioms for the reals.

See: *Logicomix: An epic search for truth* (2009)

# New ideas

An idea that emerged as central to Dedekind's work: that of a set

In fact, naive notions of sets had already appeared all but unremarked earlier in the nineteenth century

- as Gauss' classes, orders, genera (of binary quadratic forms with integer coefficients) [see Lecture XIV];
- as Galois' groupes (of permutations and of substitutions);
- as Cauchy's systèmes (of substitutions);
- as Dedekind's Zahlkörpern (of algebraic numbers).

This is by no means an exhaustive list of examples; see *Mathematics emerging*, §18.2 for others.

# Formalisation of the concept of a set

Georg Cantor: series of articles in *Mathematische Annalen*, 1879–1883

Final one also published separately as *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* [*Foundations of a general theory of aggregates*], Teubner, Leipzig, 1883:

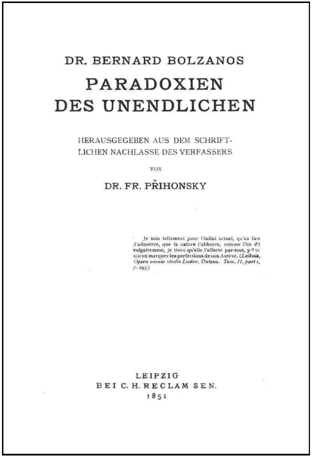
> *By an "aggregate" (Menge) we are to understand any collection into a whole (Zusammenfassung zu einem Ganzen) M of definite and separate objects m of our intuition or our thought.*

# Cantor's inspiration: questions of the infinite

A long history:

- Zeno's Paradoxes, e.g., Achilles and the tortoise (5th century BC): attempt to show that motion does not exist
- Refuted by Aristotle in his *Physics* (4th century BC), leading to ideas of actual infinity vs potential infinity
- Galileo, *Discorsi e dimostrazioni matematiche intorno a due nuove scienze* (1638): in discussing indivisibles, observed that we cannot discuss the infinite in the same language as the finite, particularly when it comes to ordering — there are as many squares as there are roots, yet there should be more numbers than squares

# Bolzano on the infinite



DR. BERNARD BOLZANOS

PARADOXIEN
DES UNENDLICHEN

HERAUSGEGEBEN AUS DEM SCHRIFT-
LICHEN NACHLASSE DES VERFASSERS

VON

DR. FR. PŘIHONSKY

Je suis tellement pour l'infini actuel, qu'au lieu d'admettre, que la nature l'abhorre, comme l'on dit vulgairement, je tiens qu'elle l'affecte par-tout, pour mieux marquer les perfections de son Auteur. (*Leibnis, Opera omnia studio Ludov. Dutens. Tom. II, part 1, p. 243.*)

LEIPZIG
BEI C. H. RECLAM SEN.
1851

*Paradoxien des Unendlichen* (1851):

- Mathematics, not philosophy or theology, is the proper forum for discussing the infinite
- Infinite to be understood as a property of collections of objects, not via growth and change
- Emphasised the role of one-to-one correspondences
- Acknowledged that there may be infinite quantities of different sizes

# Cantor's path to the infinite

Under Weierstrass's influence, asked a question of functions: given a function $f$, can we find values $a_n$, $b_n$ such that $f(x) = \sum_{n=0}^{\infty} a_n \sin nx + b_n \cos nx$?

Answer: we can derive such a series that is valid for all real $x$ apart from an infinite set of exceptional values

Prompted questions about the nature of the real numbers, and about the nature of the infinite

# Cantor and the continuum

Cantor's major interest: the continuum (i.e., the set of real numbers).

How to characterise this set within the collection of all sets? — A question that Cantor never satisfactorily answered.

Cantor's first great insight regarding sets (1873): infinite sets can have different sizes.

# Cantor's first proof that the continuum is uncountable

**Proposition**: Given any sequence of real numbers $\omega_1, \omega_2, \omega_3, \ldots$ and any interval $[\alpha, \beta]$, there is a real number in $[\alpha, \beta]$ that is not contained in the given sequence.

Proof proceeds by construction of a sequence of nested intervals $[\alpha, \beta] \supseteq [\alpha_1, \beta_1] \supseteq [\alpha_2, \beta_2] \supseteq [\alpha_3, \beta_3] \supseteq \cdots$. Cantor considered the different cases where the sequence terminates or does not, but in all instances he constructed a real number in the interval that does not lie in the original sequence.

Next suppose that the continuum is countable, i.e., that the real numbers may be listed $\omega_1, \omega_2, \omega_3, \ldots$. But then there is a real number in any interval $[\alpha, \beta]$ that does not belong to this list — a contradiction.

The more famous diagonal argument came later (1891).

# One-to-one correspondences

Also in the 1874 paper:

The algebraic $\mathbb{A}$ numbers are countable — therefore transcendental numbers exist.

> **NB:** In 1851 Joseph Liouville had already produced a constructive proof of the existence of transcendental numbers.
>
> Charles Hermite proved in 1873 that $e$ is transcendental.
>
> Proof of the transcendence of $\pi$ was finally accomplished by Carl Louis Lindemann in 1882.

Cantor to Dedekind (1877): there is a one-to-one correspondence between a line and the plane — "Je le vois, mais je ne le crois pas!" ("I see it, but I don't believe it!")

# Cantor's *Mengenlehre*

Developed at the end of the nineteenth century (1878–1897): a general theory of sets and of transfinite numbers — infinite cardinals (e.g., $\#\mathbb{N} = \aleph_0$, $\#\mathbb{R} = c$), transfinite ordinals, ...

Arithmetic of infinite cardinals, with associated peculiarities, e.g., $\aleph_0 + 1 = \aleph_0$, $c \times c = c$ (and similarly for transfinite ordinals)

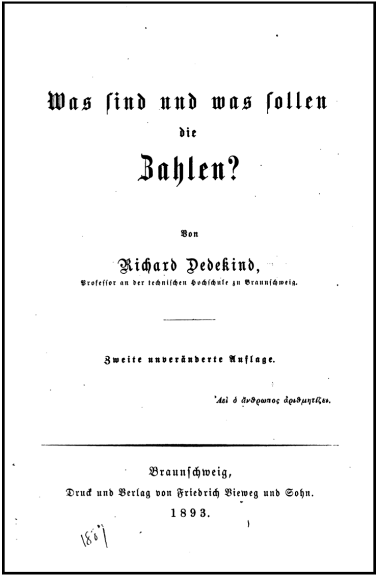
Continuum hypothesis (1878): there is no infinite cardinal strictly between $\aleph_0$ and $c$

Power set construction given in 1890: $\mathscr{P}(S)$ — the set of all subsets of a set $S$

Cantor's Theorem: $\#\mathscr{P}(S) > \#S$
Further: $\#\mathscr{P}(\mathbb{N}) = \#\mathbb{R}$, or $2^{\aleph_0} = c$

But what is the power set of the set of all sets?

# Was sind und was sollen die Zahlen?


Was sind und was sollen
die
Zahlen?

Von

Richard Dedekind,
Professor an der technischen Hochschule zu Braunschweig.

Zweite unveränderte Auflage.

Ἀεὶ ὁ ἄνθρωπος ἀριθμητίζει.

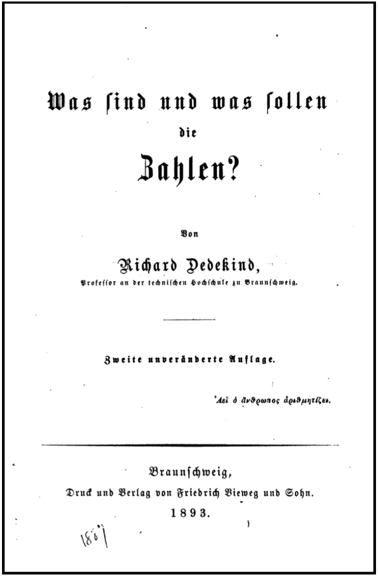
Braunschweig,
Druck und Verlag von Friedrich Vieweg und Sohn.
1893.

Richard Dedekind, *Was sind und was sollen die Zahlen?* Braunschweig, 1893

Contains, amongst other things:

- a definition of infinite sets;
- an axiomatisation of the natural numbers (soon simplified by Peano).

# *Was sind und was sollen die Zahlen?*



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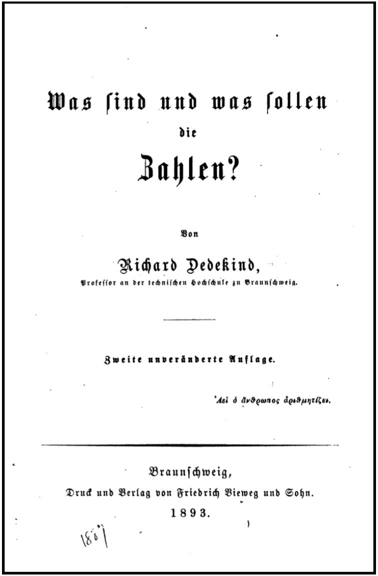
1893.

Also includes a definition of a function as a mapping between sets (p. 6):

"By a mapping of a system $S$ we understand a law according to which every determinate element $s$ of $S$ is associated with a determinate thing which is called the *image* of $s$ and is denoted by $\phi(s)$ ..."

# *Was sind und was sollen die Zahlen?*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1893.

Calls two systems similar if there is a one-to-one mapping between their elements

Goes on to incorporate this into a definition of the infinite (p. 17):

"A system is infinite if it is similar to a proper part of itself."

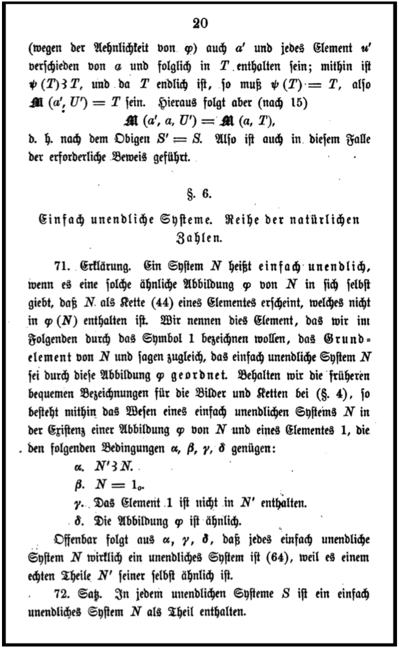
# *Was sind und was sollen die Zahlen?*

Extract from William Ewald, *From Kant to Hilbert: a source book in the foundations of mathematics*, OUP, 1996, vol. II, p. 790:

> *The title of Dedekind's paper is subtle: rigidly translated it asks 'What are, and what ought to be, the numbers?' But sollen here carries several senses—among them, 'What is the best way to regard the numbers?'; 'What is the function of numbers?; 'What are numbers supposed to be?'. But perhaps Dedekind's title is famous enough to be left in the original.*

W. W. Beman translated the essay under the title *The nature and meaning of numbers* (1901).

# Was sind und was sollen die Zahlen?





(wegen der Aehnlichkeit von $\varphi$) auch $a'$ und jedes Element $u'$ verſchieden von $a$ und folglich in $T$ enthalten ſein; mithin iſt $\psi(T) \, 3 \, T$, und da $T$ endlich iſt, ſo muß $\psi(T) = T$, alſo $\mathfrak{M}(a', U') = T$ ſein. Hieraus folgt aber (nach 15)

$$\mathfrak{M}(a', a, U') = \mathfrak{M}(a, T),$$

d. h. nach dem Obigen $S' = S$. Alſo iſt auch in dieſem Falle der erforderliche Beweis geführt.

§. 6.

Einfach unendliche Syſteme. Reihe der natürlichen Zahlen.

71. Erklärung. Ein Syſtem $N$ heißt einfach unendlich, wenn es eine ſolche ähnliche Abbildung $\varphi$ von $N$ in ſich ſelbſt giebt, daß $N$ als Kette (44) eines Elementes erſcheint, welches nicht in $\varphi(N)$ enthalten iſt. Wir nennen dies Element, das wir im Folgenden durch das Symbol 1 bezeichnen wollen, das Grundelement von $N$ und ſagen zugleich, das einfach unendliche Syſtem $N$ ſei durch dieſe Abbildung $\varphi$ geordnet. Behalten wir die früheren bequemen Bezeichnungen für die Bilder und Ketten bei (§. 4), ſo beſteht mithin das Weſen eines einfach unendlichen Syſtems $N$ in der Exiſtenz einer Abbildung $\varphi$ von $N$ und eines Elementes 1, die den folgenden Bedingungen $\alpha, \beta, \gamma, \delta$ genügen:

$\alpha$. $N' \, 3 \, N$.

$\beta$. $N = 1_0$.

$\gamma$. Das Element 1 iſt nicht in $N'$ enthalten.

$\delta$. Die Abbildung $\varphi$ iſt ähnlich.

Offenbar folgt aus $\alpha, \gamma, \delta$, daß jedes einfach unendliche Syſtem $N$ wirklich ein unendliches Syſtem iſt (64), weil es einem echten Theile $N'$ ſeiner ſelbſt ähnlich iſt.

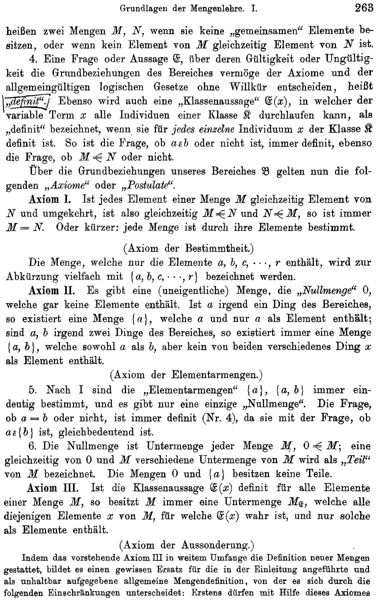
72. Satz. In jedem unendlichen Syſteme $S$ iſt ein einfach unendliches Syſtem $N$ als Theil enthalten.

Written in an explicitly set-theoretic language

(But with slightly different notation from ours.)

For a summary, see: Kathryn Edwards, ‘Richard Dedekind (1831–1916)’, *Mathematics Today* 52(1) (Feb 2016) 212–215

# Sets at the beginning of the twentieth century




heißen zwei Mengen $M$, $N$, wenn sie keine „gemeinsamen" Elemente besitzen, oder wenn kein Element von $M$ gleichzeitig Element von $N$ ist.

4. Eine Frage oder Aussage $\mathfrak{E}$, über deren Gültigkeit oder Ungültigkeit die Grundbeziehungen des Bereiches vermöge der Axiome und der allgemeingültigen logischen Gesetze ohne Willkür entscheiden, heißt „definit". Ebenso wird auch eine „Klassenaussage" $\mathfrak{E}(x)$, in welcher der variable Term $x$ alle Individuen einer Klasse $\mathfrak{K}$ durchlaufen kann, als „definit" bezeichnet, wenn sie für *jedes einzelne* Individuum $x$ der Klasse $\mathfrak{K}$ definit ist. So ist die Frage, ob $a \varepsilon b$ oder nicht ist, immer definit, ebenso die Frage, ob $M \Subset N$ oder nicht.

Über die Grundbeziehungen unseres Bereiches $\mathfrak{B}$ gelten nun die folgenden „*Axiome*" oder „*Postulate*".

**Axiom I.** Ist jedes Element einer Menge $M$ gleichzeitig Element von $N$ und umgekehrt, ist also gleichzeitig $M \Subset N$ und $N \Subset M$, so ist immer $M = N$. Oder kürzer: jede Menge ist durch ihre Elemente bestimmt.

(Axiom der Bestimmtheit.)

Die Menge, welche nur die Elemente $a, b, c, \cdots, r$ enthält, wird zur Abkürzung vielfach mit $\{a, b, c, \cdots, r\}$ bezeichnet werden.

**Axiom II.** Es gibt eine (uneigentliche) Menge, die „*Nullmenge*" 0, welche gar keine Elemente enthält. Ist $a$ irgend ein Ding des Bereiches, so existiert eine Menge $\{a\}$, welche $a$ und nur $a$ als Element enthält; sind $a, b$ irgend zwei Dinge des Bereiches, so existiert immer eine Menge $\{a, b\}$, welche sowohl $a$ als $b$, aber kein von beiden verschiedenes Ding $x$ als Element enthält.

(Axiom der Elementarmengen.)

5. Nach I sind die „Elementarmengen" $\{a\}$, $\{a, b\}$ immer eindeutig bestimmt, und es gibt nur eine einzige „Nullmenge". Die Frage, ob $a = b$ oder nicht, ist immer definit (Nr. 4), da sie mit der Frage, ob $a \varepsilon \{b\}$ ist, gleichbedeutend ist.

6. Die Nullmenge ist Untermenge jeder Menge $M$, $0 \Subset M$; eine gleichzeitig von 0 und $M$ verschiedene Untermenge von $M$ wird als „*Teil*" von $M$ bezeichnet. Die Mengen 0 und $\{a\}$ besitzen keine Teile.

**Axiom III.** Ist die Klassenaussage $\mathfrak{E}(x)$ definit für alle Elemente einer Menge $M$, so besitzt $M$ immer eine Untermenge $M_{\mathfrak{E}}$, welche alle diejenigen Elemente $x$ von $M$, für welche $\mathfrak{E}(x)$ wahr ist, und nur solche als Elemente enthält.

(Axiom der Aussonderung.)

Indem das vorstehende Axiom III in weitem Umfange die Definition neuer Mengen gestattet, bildet es einen gewissen Ersatz für die in der Einleitung angeführte und als unhaltbar aufgegebene allgemeine Mengendefinition, von der es sich durch die folgenden Einschränkungen unterscheidet: Erstens dürfen mit Hilfe dieses Axiomes

After some initial opposition, much investigation of sets at the beginning of the twentieth century — in particular, to try to remove the paradoxes

Following the general trend towards axiomatisation, Ernst Zermelo axiomatised sets in 1908: postulated the existence of $\emptyset$ and infinite sets, and established tools for constructing new sets from old — set of all sets was beyond this framework

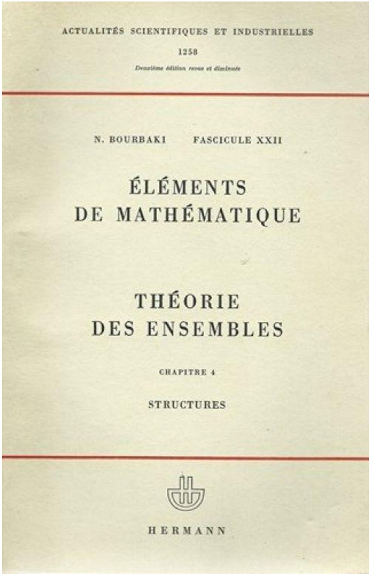
Many later modifications

# Set theory in our lives

Set theory as an effective language for mathematics:

- Set-builder notation
- Unification of ideas concerning functions and relations

# Nicolas Bourbaki (1934–????)




Collective of French mathematicians who set out to reformulate mathematics on extremely formal, abstract, structural lines — the language of sets has a significant role to play.

Association des collaborateurs de Nicolas Bourbaki

# SMP/New Math

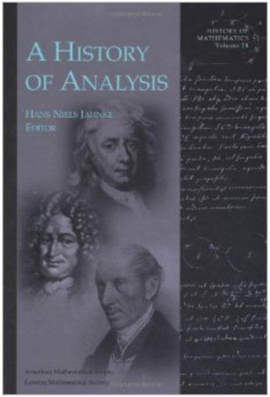
School Mathematics Project (UK)/New Mathematics (USA):

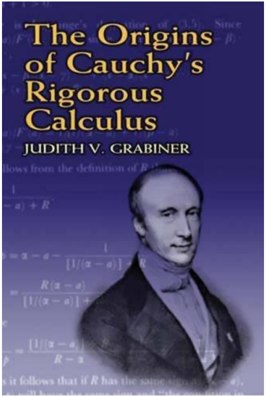
- Response to the launch of Sputnik I in 1957
- Traditional school arithmetic and geometry replaced by abstract algebra, matrices, symbolic logic, different bases, ... — in short, mathematical topics based on set theory
- Much debate — now usually regarded as a passing fad
- Tom Lehrer — 'New Math'

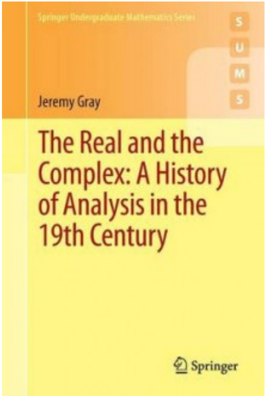
# Conclusions

- Our modern perception of real numbers took well over 2000 years to crystallise, with geometric, arithmetic, set-theoretic intuitions to the fore at different times.
- The concept of set emerged at about the same time as the modern concept of real number, 1870–1890.
- This coincidence is no coincidence.

# Further reading on the development of analysis...



# ...and on set theory and foundations