# BO1.1. History of Mathematics
# Lecture XV
# Geometry and number theory

MT25 Week 8

# Summary

- Euclid's *Elements* revisited
- The parallel postulate
- Non-Euclidean geometry
- Number theory down the centuries

# Euclid's *Elements*

Euclid's Elements, in 13 books, compiled c. 250 BC.

| | |
|---|---|
| Books I–V: | definitions, postulates, plane geometry of lines and circles |
| Book VI: | similarity, proportion |
| Books VII–IX: | number theory |
| Book X: | commensurability, irrational numbers, surds |
| Books XI–XIII: | solid geometry ending with the classification of the regular polyhedra |

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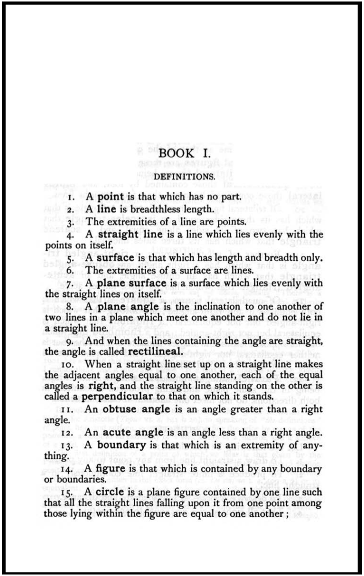
# Euclid in English



BOOK I.

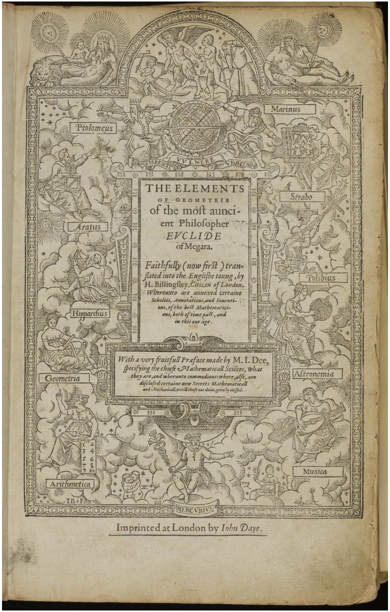
DEFINITIONS.

1. A **point** is that which has no part.
2. A **line** is breadthless length.
3. The extremities of a line are points.
4. A **straight line** is a line which lies evenly with the points on itself.
5. A **surface** is that which has length and breadth only.
6. The extremities of a surface are lines.
7. A **plane surface** is a surface which lies evenly with the straight lines on itself.
8. A **plane angle** is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
9. And when the lines containing the angle are straight, the angle is called **rectilineal.**
10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is **right,** and the straight line standing on the other is called a **perpendicular** to that on which it stands.
11. An **obtuse angle** is an angle greater than a right angle.
12. An **acute angle** is an angle less than a right angle.
13. A **boundary** is that which is an extremity of anything.
14. A **figure** is that which is contained by any boundary or boundaries.
15. A **circle** is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another;

Canonical English edition by Sir Thomas L. Heath, 1908

See also the Reading Euclid Project

# Billingsley's Euclid, 1570



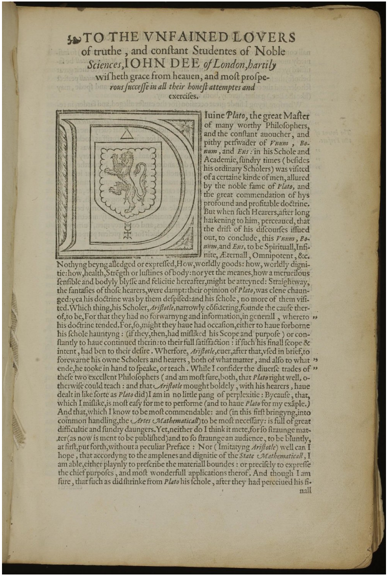

*The Elements of Geometrie*:

"Faithfully (now first) translated into the Englishe toung" by H. Billingsley, London, 1570

Available online

Preface by John Dee

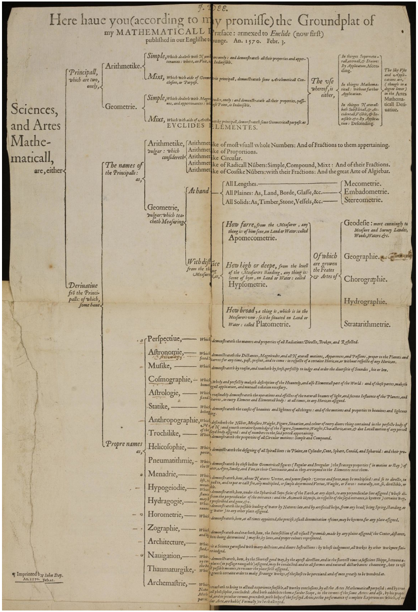
# Dee's Preface



TO THE VNFAINED LOVERS
of truthe, and conſtant Studentes of Noble
*Sciences*, IOHN DEE *of London, hartily*
wiſheth grace from heauen, and moſt proſpe-
*rous ſucceſſe in all their honeſt attemptes and*
exerciſes.

Diuine *Plato*, the great Maſter of many worthy Philoſophers, and the conſtant auoucher, and pithy perſwader of *Vnum*, *Bonum*, and *Ens*: in his Schole and Academie, ſundry times (beſides his ordinary Scholers) was viſited of a certaine kinde of men, allured by the noble fame of *Plato*, and the great commendation of hys profound and profitable doctrine. But when ſuch Hearers, after long harkening to him, perceaued, that the drift of his diſcourſes iſſued out, to conclude, this *Vnum*, *Bonum*, and *Ens*, to be Spirituall, Infinite, Æternall, Omnipotent, &c. Nothyng beyng alledged or expreſſed, How, worldly goods: how, worldly dignitie: how, health, Strẽgth or luſtines of body: nor yet the meanes, how a merueilous ſenſible and bodyly blyſſe and felicitie hereafter, might be atteyned: Straightway, the fantaſies of thoſe hearers, were dampt: their opinion of *Plato*, was clene chaunged: yea his doctrine was by them deſpiſed: and his ſchole, no more of them viſited. Which thing, his Scholer, *Ariſtotle*, narrowly cõſidering, founde the cauſe therof, to be, For that they had no forwarnyng and information, in generall, whereto ” his doctrine tended. For, ſo, might they haue had occaſion, either to haue forborne his ſchole hauntyng: (if they, then, had miſliked his Scope and purpoſe) or conſtantly to haue continued therin: to their full ſatiſfaction: if ſuch his finall ſcope & intent, had ben to their deſire. Wherfore, *Ariſtotle*, euer, after that, vſed in brief, to forewarne his owne Scholers and hearers, both of what matter, and alſo to what ” ende, he tooke in hand to ſpeake, or teach. While I conſider the diuerſe trades of ” theſe two excellent Philoſophers (and am moſt ſure, both, that *Plato* right well, otherwiſe could teach: and that *Ariſtotle* mought boldely, with his hearers, haue dealt in like ſorte as *Plato* did) I am in no little pang of perplexitie: Bycauſe, that, which I miſlike, is moſt eaſy for me to performe (and to haue *Plato* for my exãple.) And that, which I know to be moſt commendable: and (in this firſt bringyng, into common handling, the *Artes Mathematicall*) to be moſt neceſſary: is full of great difficultie and ſundry daungers. Yet, neither do I think it mete, for ſo ſtraunge matter (as now is ment to be publiſhed) and to ſo ſtraunge an audience, to be bluntly, at firſt, put forth, without a peculiar Preface: Nor (Imitatyng *Ariſtotle*) well can I hope, that accordyng to the amplenes and dignitie of the *State Mathematicall*, I am able, either playnly to preſcribe the materiall boundes: or preciſely to expreſſe the chief purpoſes, and moſt wonderfull applications therof. And though I am ſure, that ſuch as did ſhrinke from *Plato* his ſchole, after they had perceiued his finall

# Dee's 'Groundplat'




See: Jennifer M. Rampling, 'The Elizabethan mathematics of everything: John Dee's 'Mathematicall praeface' to Euclid's *Elements*', *BSHM Bulletin: Journal of the British Society for the History of Mathematics* **26**(3) (2011) 135–146

# Billingsley's Preface, pp. 1, 3





*There is (gentle Reader) nothing (the word of God onely set apart) which so much beautifieth and adorneth the soule and minde of mã, as doth the knowledge of good artes and sciences: as the knowledge of naturall and morall Philosophie. The one setteth before our eyes, the creatures of God, both in the heauens aboue, and in the earth beneath: in which as in a glaße, we beholde the exceding maiestie and wisedome of God, in adorning and beautifying them as we see: in geuing vnto them such wonderfull and manifolde proprieties, and naturall workinges, and that so diuersly and in such varietie: farther in maintaining and conseruing them continually, whereby to praise and adore him, as by S. Paule we are taught. The other teacheth vs rules and preceptes of vertue, how, in common life amongest men, we ought to walke vprightly: what dueties pertaine to our selues, what pertaine to the gouernment or good order both of an housholde, and also of a citie or common wealth. The reading likewise of histories, conduceth not a litle, to the adorning of the soule & minde of man, a studie of all men cõmended: by it are seene and knowen the artes and doinges of infinite wise men gone before vs. In histories are contained infinite examples of heroicall vertues to be of vs followed, and horrible examples of vices to be of vs eschewed. Many other artes also there are which beautifie the minde of man: but of all other none do more garnishe & beautifie it, then those artes which are called Mathematicall. Unto the knowledge of which no man can attaine, without the perfecte knowledge and instruction of the principles, groundes, and Elementes of Geometrie. But per-*



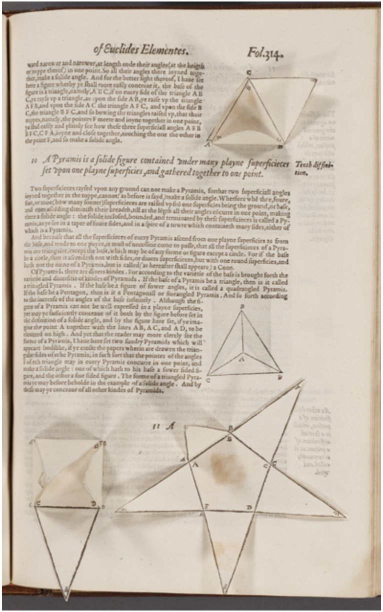
. . .



*well perceaue. The fruite and gaine which I require for these my paines and trauaile, shall be nothing els, but onely that thou gentle reader, will gratefully accept the same: and that thou mayest thereby receaue some profite: and moreouer to excite and stirre vp others learned, to do the like, & to take paines in that behalfe. By meanes wherof, our Englishe tounge shall no lesse be enriched with good Authors, then are other straunge tounges: as the Dutch, French, Italian, and Spanishe: in which are red all good authors in a maner, found amongest the Grekes or Latines. Which is the chiefest cause, that amongest thẽ do florishe so many cunning and skilfull men, in the inuentions of straunge and wonderfull thinges, as in these our daies we see there do. Which fruite and gaine if I attaine vnto, it shall encourage me hereafter, in such like sort to translate, and set abroad some other good authors, both pertaining to religion (as partly I haue already done) and also pertaining to the Mathematicall Artes. Thus gentle reader farewell.*

(?¿)

# Pop-up Euclid



of Euclides Elementes. Fol.314.

ward narower and narower, at length ende their angles (at the heigth or toppe thereof) in one point. So all their angles there ioyned together, make a solide angle. And for the better sight thereof, I haue set here a figure wherby ye shall more easily conceiue it, the base of the figure is a triangle, namely, A B C, if on euery side of the triangle A B C, ye rayse vp a triangle, as vpon the side A B, ye raise vp the triangle A F B, and vpon the side A C the triangle A F C, and vpon the side B C, the triangle B F C, and so bowing the triangles raised vp, that their toppes, namely, the pointes F meete and ioyne together in one point, ye shal easily and plainly see how these three superficiall angles A F B B F C, C F A, ioyne and close together, touching the one the other in the point F, and so make a solide angle.

10 *A Pyramis is a solide figure contained vnder many playne superficieces set vpon one playne superficies, and gathered together to one point.* Tenth diffinition.

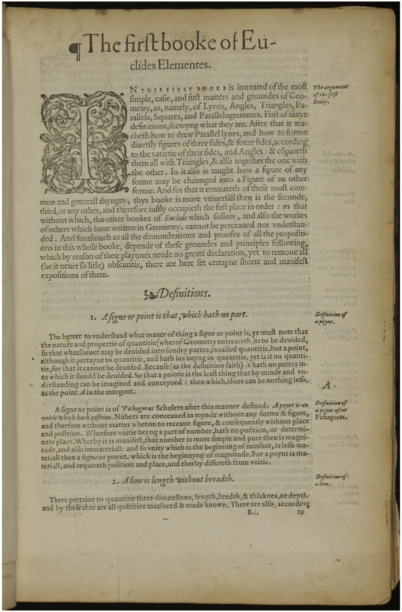
Two superficieces raysed vpon any ground can not make a Pyramis, forthat two superficiall angles ioyned together in the toppe, cannot (as before is sayd) make a solide angle. Wherfore whē thre, foure, fiue, or moe (how many soeuer) superficieces are raised vp frō one superficies being the ground, or base, and euer ascēding diminish their breadth, till at the lēgth all their angles cōcurre in one point, making there a solide angle: the solide inclosed, bounded, and terminated by these superficieces is called a Pyramis, as ye see in a taper of foure sides, and in a spire of a towre which containeth many sides, either of which is a Pyramis.

And because that all the superficieces of euery Pyramis ascend from one playne superficies as from the base, and tende to one poynt, it must of necessitie come to passe, that all the superficieces of a Pyramis are triangular, except the base, which may be of any forme or figure except a circle. For if the base be a circle, then it ascendeth not with sides, or diuers superficieces, but with one round superficies, and hath not the name of a Pyramis, but is called (as hereafter shall appeare) a Cone.

Of Pyramids there are diuers kindes. For according to the varietie of the base is brought forth the varietie and diuersitie of kindes of Pyramids. If the base of a Pyramis be a triangle, then is it called a triangled Pyramis. If the base be a figure of fower angles, it is called a quadrangled Pyramis. If the base be a Pentagon, then is it a Pentagonall or fiuangled Pyramis. And so forth according to the increase of the angles of the base infinitely. Although the figure of a Pyramis can not be well expressed in a playne superficies, yet may ye sufficiently conceaue of it both by the figure before set in the definition of a solide angle, and by the figure here set, if ye imagine the point A together with the lines A B, A C, and A D, to be eleuated on high. And yet that the reader may more clerely see the forme of a Pyramis, I haue here set two sundry Pyramids which will appeare bodilike, if ye erecte the papers wherin are drawen the triangular sides of eche Pyramis, in such sort that the pointes of the angles F of ech triangle may in euery Pyramis concurre in one point, and make a solide angle: one of which hath to his base a fower sided figure, and the other a fiue sided figure. The forme of a triangled Pyramis ye may before beholde in the example of a solide angle. And by these may ye conceaue of all other kindes of Pyramids.

11 A

# Book I: definitions



## ¶ The firſt booke of Euclides Elementes.

*The argument of the firſt Booke.*

IN THIS FIRST BOOKE is intreated of the moſt ſimple, eaſie, and firſt matters and groundes of Geometry, as, namely, of Lynes, Angles, Triangles, Parallels, Squares, and Parallelogrammes. Firſt of theyr definitions, ſhewyng what they are. After that it teacheth how to draw Parallel lynes, and how to forme diuerſly figures of three ſides, & foure ſides, according to the varietie of their ſides, and Angles: & cōpareth them all with Triangles, & alſo together the one with the other. In it alſo is taught how a figure of any forme may be chaunged into a Figure of an other forme. And for that it entreateth of theſe moſt common and generall thynges, thys booke is more vniuerſall then is the ſeconde, third, or any other, and therefore iuſtly occupieth the firſt place in order: as that without which, the other bookes of *Euclide* which follow, and alſo the workes of others which haue written in Geometry, cannot be perceaued nor vnderſtanded. And foraſmuch as all the demonſtrations and proofes of all the propoſitions in this whole booke, depende of theſe groundes and principles following, which by reaſon of their playnnes neede no greate declaration, yet to remoue all (be it neuer ſo litle) obſcuritie, there are here ſet certayne ſhorte and manifeſt expoſitions of them.

### Definitions.

*Definition of a poynt.*

1. *A ſigne or point is that, which hath no part.*

The better to vnderſtand what maner of thing a ſigne or point is, ye muſt note that the nature and propertie of quantitie (wherof Geometry entreateth) is to be deuided, ſo that whatſoeuer may be deuided into ſundry partes, is called quantitie. But a point, although it pertayne to quantitie, and hath his beyng in quantitie, yet is it no quantitie, for that it cannot be deuided. Becauſe (as the definition ſaith) it hath no partes into which it ſhould be deuided. So that a pointe is the leaſt thing that by minde and vnderſtanding can be imagined and conceyued: then which, there can be nothing leſſe, as the point *A* in the margent.

A .

*Definition of a poynt after* Pithagoras.

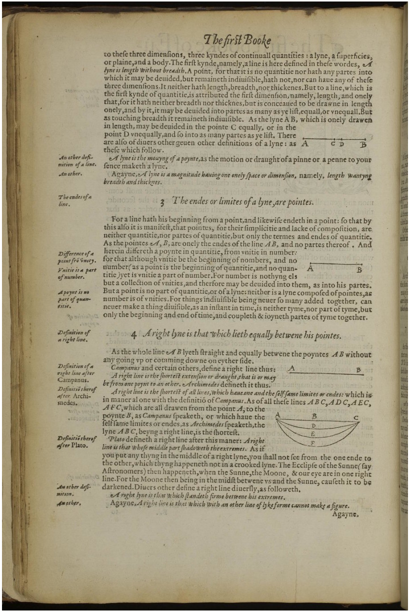
A ſigne or point is of *Pithagoras* Scholers after this manner defined: *A poynt is an vnitie which hath poſition.* Nūbers are conceaued in mynde without any forme & figure, and therfore without matter wheron to receaue figure, & conſequently without place and poſition. Wherfore vnitie beyng a part of number, hath no poſition, or determinate place. Wherby it is manifeſt, that number is more ſimple and pure then is magnitude, and alſo immateriall: and ſo vnity which is the beginning of number, is leſſe materiall then a ſigne or poynt, which is the beginnyng of magnitude. For a poynt is materiall, and requireth poſition and place, and therby differeth from vnitie.

*Definition of a line.*

2. *A line is length without breadth.*

There pertaine to quantitie three dimenſions, length, bredth, & thicknes, or depth: and by theſe thre are all quātities meaſured & made known. There are alſo, according

B.i. to





to theſe three dimenſions, three kyndes of continuall quantities: a lyne, a ſuperficies, or plaine, and a body. The firſt kynde, namely, a line is here defined in theſe wordes, *A lyne is length without breadth.* A point, for that it is no quantitie nor hath any partes into which it may be deuided, but remaineth indiuiſible, hath not, nor can haue any of theſe three dimenſions. It neither hath length, breadth, nor thickenes. But to a line, which is the firſt kynde of quantitie, is attributed the firſt dimenſion, namely, length, and onely that, for it hath neither breadth nor thicknes, but is conceaued to be drawne in length onely, and by it, it may be deuided into partes as many as ye liſt, equall, or vnequall. But as touching breadth it remaineth indiuiſible. As the lyne A B, which is onely drawen in length, may be deuided in the pointe C equally, or in the point D vnequally, and ſo into as many partes as ye liſt. There are alſo of diuers other geuen other definitions of a lyne: as theſe which follow.

A C D B

*An other definition of a line.*

*A lyne is the mouyng of a poynte*, as the motion or draught of a pinne or a penne to your ſence maketh a lyne.

*An other.*

Agayne, *A lyne is a magnitude hauing one onely ſpace or dimenſion*, namely, *length wantyng breadth and thicknes.*

*The endes of a line.*

3 *The endes or limites of a lyne, are pointes.*

For a line hath his beginning from a point, and likewiſe endeth in a point: ſo that by this alſo it is manifeſt, that pointes, for their ſimplicitie and lacke of compoſition, are neither quantitie, nor partes of quantitie, but only the termes and endes of quantitie. As the pointes *A*, *B*, are onely the endes of the line *A B*, and no partes thereof. And herein differeth a poynte in quantitie, from vnitie in number: for that although vnitie be the beginning of nombers, and no number (as a point is the beginning of quantitie, and no quantitie) yet is vnitie a part of number. For number is nothyng els but a collection of vnities, and therfore may be deuided into them, as into his partes. But a point is no part of quantitie, or of a lyne: neither is a lyne compoſed of pointes, as number is of vnities. For things indiuiſible being neuer ſo many added together, can neuer make a thing diuiſible, as an inſtant in time, is neither tyme, nor part of tyme, but only the beginning and end of time, and coupleth & ioyneth partes of tyme together.

*Difference of a point frō vnity.*

*Vnitie is a part of number.*

*A poynt is no part of quantitie.*

A B

*Definition of a right line.*

4 *A right lyne is that which lieth equally betwene his pointes.*

As the whole line *A B* lyeth ſtraight and equally betwene the poyntes *A B* without any going vp or comming downe on eyther ſide.

*Definition of a right line after* Campanus.

*Campanus* and certain others, define a right line thus: *A right line is the ſhorteſt extenſion or draught, that is or may be from one poynt to an other.* *Archimedes* defineth it thus.

A B

*Definitiō therof after* Archimedes.

*A right line is the ſhorteſt of all lines, which haue one and the ſelf ſame limites or endes:* which is in maner al one with the definitiō of *Campanus*. As of all theſe lines *A B C*, *A D C*, *A E C*, *A F C*, which are all drawen from the point *A*, to the poynte *B*, as *Campanus* ſpeaketh, or which haue the ſelf ſame limites or endes, as *Archimedes* ſpeaketh, the lyne *A B C*, beyng a right line, is the ſhorteſt.

A B C D E F

*Definitiō therof after* Plato.

*Plato* defineth a right line after this maner: *A right line is that whoſe middle part ſhadoweth the extremes.* As if you put any thyng in the middle of a right lyne, you ſhall not ſee from the one ende to the other, which thyng happeneth not in a crooked lyne. The Ecclipſe of the Sunne (ſay Aſtronomers) then happeneth, when the Sunne, the Moone, & our eye are in one right line. For the Moone then being in the midſt betwene vs and the Sunne, cauſeth it to be darkened. Diuers other define a right line diuerſly, as followeth.

*An other definition.*

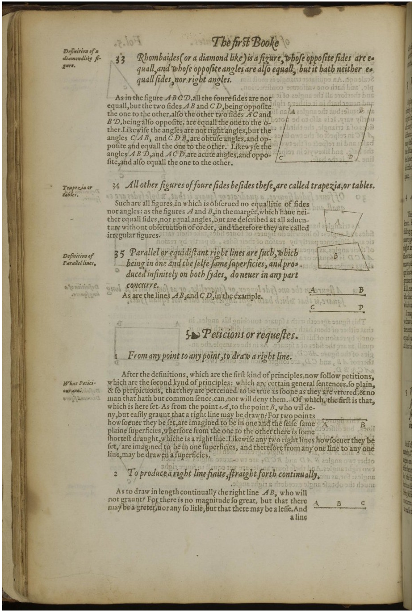
*A right lyne is that which ſtandeth firme betwene his extremes.*

*Another.*

Agayne, *A right line is that which with an other line of lyke forme cannot make a figure.*

Agayne,

# Book I: postulates





*Definition of a diamondlike figure.*

33 *Rhombaides (or a diamond like) is a figure, whoſe oppoſite ſides are equall, and whoſe oppoſite angles are alſo equall, but it hath neither equall ſides, nor right angles.*

As in the figure *ABCD*, all the foure ſides are not equall, but the two ſides *AB* and *CD*, being oppoſite the one to the other, alſo the other two ſides *AC* and *BD*, being alſo oppoſite, are equall the one to the other. Likewiſe the angles are not right angles, but the angles *CAB*, and *CDB*, are obtuſe angles, and oppoſite and equall the one to the other. Likewyſe the angles *ABD*, and *ACD*, are acute angles, and oppoſite, and alſo equall the one to the other.

*Trapezia or tables.*

34 *All other figures of foure ſides beſides theſe, are called trapezia, or tables.*

Such are all figures, in which is obſerued no equallitie of ſides nor angles: as the figures *A* and *B*, in the margét, which haue neither equall ſides, nor equal angles, but are deſcribed at all aduenture without obſeruation of order, and therefore they are called irregular figures.

*Definition of Parallel lines.*

35 *Parallel or equidiſtant right lines are ſuch, which being in one and the ſelfe ſame ſuperficies, and produced infinitely on both ſydes, do neuer in any part concurre.*

As are the lines *AB*, and *CD*, in the example.

## Peticions or requeſtes.

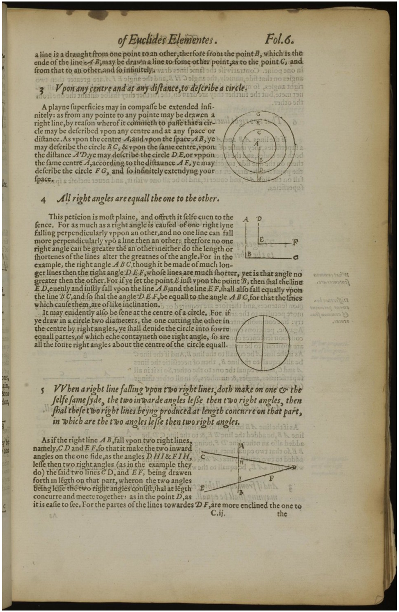
1 *From any point to any point, to draw a right line.*

*What Peticions are.*

After the definitions, which are the firſt kind of principles, now follow petitions, which are the ſecond kynd of principles: which are certain general ſentences, ſo plain, & ſo perſpicuous, that they are perceiued to be true as ſoone as they are vttered, & no man that hath but common ſence, can, nor will deny them. Of which, the firſt is that, which is here ſet. As from the point *A*, to the point *B*, who wil deny, but eaſily graunt that a right line may be drawn? For two points howſoeuer they be ſet, are imagined to be in one and the ſelfe ſame plaine ſuperficies, wherfore from the one to the other there is ſome ſhorteſt draught, whiche is a right line. Likewiſe any two right lines howſoeuer they be ſet, are imagined to be in one ſuperficies, and therefore from any one line to any one line, may be drawen a ſuperficies.

2 *To produce a right line finite, ſtraight forth continually.*

As to draw in length continually the right line *AB*, who will not graunt? For there is no magnitude ſo great, but that there may be a greter, nor any ſo litle, but that there may be a leſſe. And a line





a line is a draught from one point to an other, therfore from the point *B*, which is the ende of the line *AB*, may be drawn a line to ſome other point, as to the point *C*, and from that to an other, and ſo infinitely.

3 *Vpon any centre and at any diſtance, to deſcribe a circle.*

A playne ſuperficies may in compaſſe be extended infinitely: as from any pointe to any pointe may be drawen a right line, by reaſon wherof it commeth to paſſe that a circle may be deſcribed vpon any centre and at any ſpace or diſtance. As vpon the centre *A*, and vpon the ſpace *AB*, ye may deſcribe the circle *BC*, & vpon the ſame centre, vpon the diſtance *AD*, ye may deſcribe the circle *DE*, or vppon the ſame centre *A*, according to the diſtaunce *AF*, ye may deſcribe the circle *FG*, and ſo infinitely extendyng your ſpace.

4 *All right angles are equall the one to the other.*

This peticion is moſt plaine, and offreth it ſelfe euen to the ſence. For as much as a right angle is cauſed of one right lyne falling perpendicularly vppon an other, and no one line can fall more perpendicularly vpō a line then an other: therfore no one right angle can be greater thē an other: neither do the length or ſhortenes of the lines alter the greatnes of the angle. For in the example, the right angle *ABC*, though it be made of much longer lines then the right angle *DEF*, whoſe lines are much ſhorter, yet is that angle no greater then the other. For if ye ſet the point *E* iuſt vpon the point *B*, then ſhal the line *ED*, euenly and iuſtly fall vpon the line *AB*, and the line *EF*, ſhall alſo fall equally vpon the line *BC*, and ſo ſhal the angle *DEF*, be equall to the angle *ABC*, for that the lines which cauſe them, are of like inclination.

It may euidently alſo be ſene at the centre of a circle. For if ye draw in a circle two diameters, the one cutting the other in the centre by right angles, ye ſhall deuide the circle into fowre equall partes, of which eche contayneth one right angle, ſo are all the foure right angles about the centre of the circle equall.

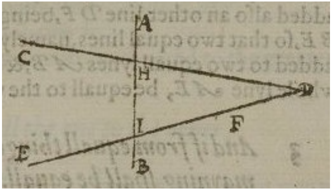
5 *When a right line falling vpon two right lines, doth make on one & the ſelfe ſame ſyde, the two inwarde angles leſſe then two right angles, then ſhal theſe two right lines beyng produced at length concurre on that part, in which are the two angles leſſe then two right angles.*

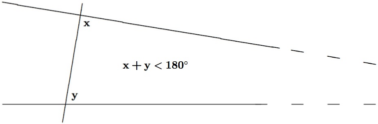
As if the right line *AB*, fall vpon two right lines, namely, *CD* and *EF*, ſo that it make the two inward angles on the one ſide, as the angles *DHI* & *FIH*, leſſe then two right angles (as in the example they do) the ſaid two lines *CD*, and *EF*, being drawen forth in lēgth on that part, wheron the two angles being leſſe thē two right angles conſiſt, ſhal at lēgth concurre and meete together: as in the point *D*, as it is eaſie to ſee. For the partes of the lines towardes *DF*, are more enclined the one to the

C.ij.

# Postulate 5

5 VVhen a right line falling vpon two right lines, doth make on one & the ſelfe ſame ſyde, the two inwarde angles leſſe then two right angles, then ſhal theſe two right lines beyng produced at length concurre on that part, in which are the two angles leſſe then two right angles.




Equivalent formulation (Proclus, 5th century; John Playfair, 1795): given a straight line $L$ and a point $P$ not on $L$ there is one and only one straight line through $P$ that is parallel to $L$.

# Classical disquiet about the fifth postulate

Original to Euclid? Less ‘self-evident’ than the other postulates?

Euclid used it (e.g., in the proof of Proposition 29 of Book I), so the property is necessary — but does it in fact follow from the other postulates?

Proclus in commentary on Euclid, 5th century (after citing Ptolemy’s attempted proof of the parallel postulate, and discussing the nature of truth, with reference to Aristotle and Plato):

> *It is then clear from this that we must seek a proof of the present theorem, and that it is alien to the special character of postulates.*

Attempted (unsuccessfully) to prove the fifth postulate on the basis of the others

See Heath, pp. 202–220

# Mediaeval disquiet about the fifth postulate

In the Islamic world:

Ibn al-Haytham (Alhazen) (965–1039) attempted (unsuccessfully) to prove the parallel postulate by contradiction

Omar Khayyám (1050–1123) attempted to prove the fifth postulate on the basis of the following alternative:

> *two convergent straight lines intersect and it is impossible for two convergent straight lines to diverge in the direction in which they converge*

Described the situations that may occur if the postulate is omitted

Nasir al-Din al-Tusi (1201–1274) criticised Khayyám's attempted proof, offered his own

Al-Tusi's thoughts found their way into Europe via the writings (1298) of his son Sadr al-Tusi

# Early modern disquiet about the fifth postulate

After reading al-Tusi, John Wallis showed that the parallel postulate is equivalent to the following:

> *on a given finite straight line it is always possible to construct a triangle similar to a given triangle*

He lectured on this in Oxford in 1663

Attempts to prove the fifth postulate on the basis of Euclid's other axioms had resulted only in equivalent forms — so can we have a consistent geometry in which it the parallel postulate fails?

# Early hints of non-Euclidean geometry

Giovanni Girolamo Saccheri (1667–1733): sought to establish the validity of Euclidean geometry — negated the parallel postulate in search of a contradiction; two cases:

- internal angles of a triangle add up to less than two right angles — contradicts Euclid's second postulate
- internal angles of a triangle add up to more than two right angles — leads to non-intuitive ideas

Similar results derived by Johann Heinrich Lambert (1728–1777) in his *Theorie der Parallellinien* (1766)

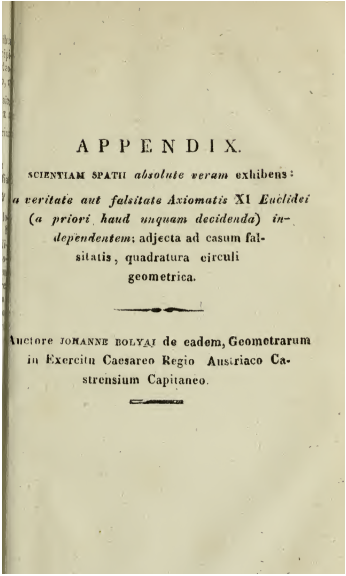
# Non-Euclidean geometries

Consistent non-Euclidean geometry probably first constructed (tentatively) by Gauss, c. 1817–1830, but remained unpublished

Problem pursued independently (without success) by Gauss' friend Farkas Bolyai (1775–1856)

Pursued (against paternal advice) and solved by János Bolyai (1802–1860): "I have created a new and different world out of nothing" (1823)

# Bolyai's geometry


APPENDIX.

SCIENTIAM SPATII *absolute veram* exhibens:

*a veritate aut falsitate Axiomatis* XI *Euclidei* (*a priori haud unquam decidenda*) *independentem*; adjecta ad casum falsitatis, quadratura circuli geometrica.

Auctore JOHANNE BOLYAI de eadem, Geometrarum in Exercitu Caesareo Regio Austriaco Castrensium Capitaneo.

Published as appendix ‘The science absolute of space: independent of the truth or falsity of Euclid's axiom XI (which can never be decided a priori)’ to father's textbook *Tentamen iuventutem studiosam in elementa matheosos introducendi* (1832)

English translation by George Bruce Halstead (1896)

# Meanwhile in Russia...

Non-Euclidean geometry developed independently by Nikolai Ivanovich Lobachevskii [Николай Иванович Лобачевский] (1792–1856) using the negation of Playfair's axiom

# Lobachevskii's works


Geometrische Untersuchungen

zur

Theorie der Parallellinien

von

Nicolaus Lobatschewsky,
Kaiserl. russ. wirkl. Staatsrathe und ord. Prof. der Mathematik
bei der Universität Kasan.

Berlin. 1840.
In der G. Fincke'schen Buchhandlung

Complicated story of dissemination...

*Geometriya* [Геометрия] written in 1823 but not published until 1909

Ideas presented in Kazan in 1826, published there 1829 — but rejected by St Petersburg Academy

Other works in Russian, French and German, including *Geometrische Untersuchungen zur Theorie der Parallellinien* (1840), *Pangéométrie* (1855)

(See Tom Lehrer for an unfair characterisation of Lobachevskii: https://youtu.be/IL4vWJbwmqM)

# Acceptance and impact of non-Euclidean geometries

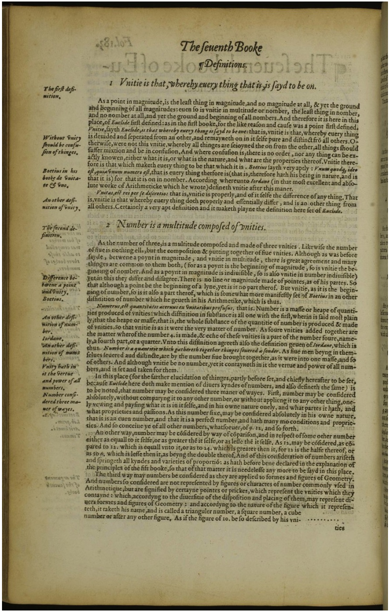
Slow to gain acceptance due to

- obscurity of publications
- lack of intuitive understanding

But non-Euclidean geometries

- overturned old ideas of mathematical certainty
- introduced new ideas about space
- helped drive the late 19th-century move towards axiomatisation

# Euclid on numbers (positive integers)





¶Definitions.

*The first definition.*

1 *Vnitie is that, whereby euery thing that is, is sayd to be on.*

As a point in magnitude, is the least thing in magnitude, and no magnitude at all, & yet the ground and beginning of all magnitudes: euen so is vnitie in multitude or nomber, the least thing in nomber, and no nomber at all, and yet the ground and beginning of all nombers. And therefore it is here in this place, of *Euclide* first defined: as in the first booke, for the like reason and cause was a point first defined. *Vnitie*, sayth *Euclide, is that whereby euery thing is sayd to be one*: that is, vnitie is that, whereby euery thing is deuided and seperated from an other, and remayneth on in it selfe pure and distinct frō all others. Otherwise, were not this vnitie, whereby all thinges are seioyned the on from the other, all things should suffer mixtion and be in confusion. And where confusion is, there is no order, nor any thing can be exactly knowen, either what it is, or what is the nature, and what are the properties thereof. Vnitie therefore is that which maketh euery thing to be that which it is. *Boetius* sayth very aptly: *Vnum quodq; ideo est, quia vnum numero est*, that is euery thing therfore is (that is, therefore hath his being in nature, and is that it is) for that it is on in nomber. According whereunto *Iordane* (in that most excellent and absolute worke of Arithmeticke which he wrote) defineth vnitie after this maner.

*Without Vnity should be confusion of thinges.*

*Boetius in his booke de Vnitate & Vno.*

*Vnitas, est rei per se discretio*: that is, vnitie is properly, and of it selfe the difference of any thing. That is, vnitie is that whereby euery thing doth properly and essentially differ, and is an other thing from all others. Certainely a very apt definition and it maketh playne the definition here set of *Euclide*.

*An other definition of Vnity.*

*The second definition.*

2 *Number is a multitude composed of vnities.*

As the number of three, is a multitude composed and made of three vnities. Likewise the number of fiue is nothing ells, but the composition & putting together of fiue vnities. Although as was before sayde, betwene a poynt in magnitude, and vnitie in multitude, there is great agreement and many thinges are common to them both, (for as a poynt is the beginning of magnitude, so is vnitie the beginning of nomber. And as a poynt in magnitude is indiuisible, so is also vnitie in number indiuisible) yet in this they differ and disagree. There is no line or magnitude made of pointes, as of his partes. So that although a point be the beginning of a lyne, yet is it no part therof. But vnitie, as it is the beginning of number, so is it also a part therof, which is somewhat more manifestly set of *Boetius* in an other diffinition of number which he geueth in his Arithmetike, which is thus.

*Difference betwene a point and Vnity. Boetius.*

*Numerus, est quantitatis acervus ex Vnitatibus profusus*, that is. Number is a masse or heape of quantities produced of vnities: which diffinition in substance is all one with the first, wherin is said most plainly, that the heape or masse, that is, the whole substance of the quantitie of number is produced & made of vnities. So that vnitie is as it were the very matter of number. As foure vnities added together are the matter wherof the number 4. is made, & eche of these vnities is a part of the number foure, namely, a fourth part, or a quarter. Vnto this diffinition agreeth also the definition geuen of *Iordane*, which is thus. *Number is a quantitie which gathereth together thinges seuered a sonder.* As fiue men beyng in themselues seuered aud distincte, are by the number fiue brought together, as it were into one masse, and so of others. And although vnitie be no number, yet it contayneth in it the vertue and power of all numbers, and is set and taken for them.

*An other definition of number. Iordane.*

*An other definition of number.*

*Vnity hath in it the vertue and power of all numbers.*

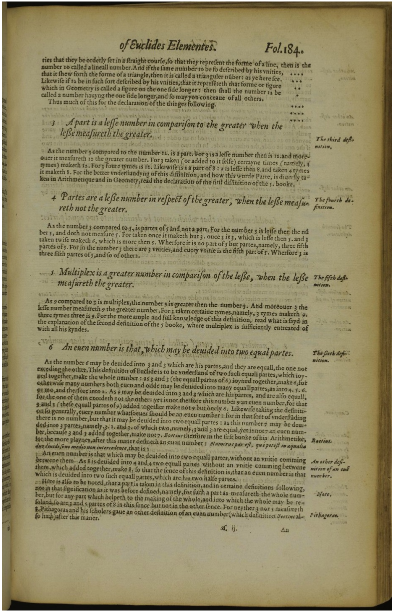
In this place (for the farther elucidation of thinges, partly before set, and chiefly hereafter to be set, because *Euclide* here doth make mention of diuers kyndes of numbers, and also defineth the same) is to be noted, that number may be considered three maner of wayes. First, number may be considered absolutely, without comparyng it to any other number, or without applieng it to any other thing, onely vewing and paysing what it is in it selfe, and in his owne nature onely, and what partes it hath, and what proprieties and passions. As this number sixe, may be considered absolutely in his owne nature, that it is an euen number, and that it is a perfect number, and hath many mo conditions and proprieties. And so conceiue ye of all other numbers, whatsoeuer, of 9. 12. and so forth.

*Number considered three maner of wayes.*

An other way, number may be cōsidered by way of cōparison, and in respect of some other number either as equall to it selfe, or as greater thē it selfe, or as lesse thē it selfe. As 12. may be cōsidered, as cōpared to 12. which is equall vnto it, or as to 24. which is greater then it, for 12 is the halfe thereof, or as to 6. which is lesse then it, as beyng the double therof. And of this consideration of numbers ariseth and springeth all kyndes and varieties of proportiō: as hath before bene declared in the explanation of the principles of the fift booke, so that of that matter it is needelesse any more to be sayd in this place.

The third way may numbers be considered as they are applied to formes and figures of Geometry. And numbers so considered are not represented by figures or characters of number commonly vsed in Arithmetique, but are signified by certayne pointes or prickes, which represent the vnities which they contayne: which, accordyng to the diuersitie of the disposition and placing of them, may represent diuers formes and figures of Geometry: and accordyng to the nature of the figure which it representeth, it taketh his name, and is called a trianguler number, a square number, a cube number or after any other figure. As if the figure of 10. be so described by his vni- . . . . . . . . . .

ties





ties that they be orderly set in a straight course, so that they represent the forme of a line, then is the number 10 called a lineall number. And if the same number to be so described by his vnities, that it shew forth the forme of a triangle, then it is called a trianguler nūber: as ye here see. Likewise if 12 be in such sort described by his vnities, that it represēteth that forme or figure which in Geometry is called a figure on the one side longer: then shall the number 12 be called a number hauyng the one side longer, and so may you conceaue of all others.

Thus much of this for the declaration of the thinges following.

3 *A part is a lesse number in comparison to the greater when the lesse measureth the greater.*

*The third definition.*

As the number 3 compared to the number 12. is a part. For 3 is a lesse number then is 12. and moreouer it measureth 12 the greater number. For 3 taken (or added to it selfe) certayne times (namely, 4 tymes) maketh 12. For 3 foure tymes is 12. Likewise is 2 a part of 8: 2 is lesse then 8, and taken 4 tymes it maketh 8. For the better vnderstandyng of this diffinition, and how this worde Parte, is diuersly taken in Arithmetique and in Geometry, read the declaration of the first diffinition of the 5. booke.

4 *Partes are a lesse number in respect of the greater, when the lesse measureth not the greater.*

*The fourth definition.*

As the number 3 compared to 5, is partes of 5 and not a part. For the number 3 is lesse then the nūber 5, and doth not measure 5. For taken once it maketh but 3. once 3 is 3, which is lesse then 5. and 3 taken twise maketh 6, which is more then 5. Wherfore it is no part of 5 but partes, namely, three fifth partes of 5. For in the number 3 there are 3 vnities, and euery vnitie is the fifth part of 5. Wherfore 3 is three fifth partes of 5, and so of others.

5 *Multiplex is a greater number in comparison of the lesse, when the lesse measureth the greater.*

*The fifth definition.*

As 9 compared to 3 is multiplex, the number 9 is greater then the number 3. And moreouer 3 the lesse number measureth 9 the greater number. For 3 taken certaine tymes, namely, 3 tymes maketh 9. three tymes three is 9. For the more ample and full knowledge of this definition, read what is sayd in the explanation of the second definition of the 5 booke, where multiplex is sufficiently entreated of with all his kyndes.

6 *An euen number is that, which may be deuided into two equal partes.*

*The sixth definition.*

As the number 6 may be deuided into 3 and 3 which are his partes, and they are equall, the one not exceding the other. This definition of Euclide is to be vnderstand of two such equall partes, which ioyned together, make the whole number: as 3 and 3 (the equall partes of 6) ioyned together, make 6, for otherwise many numbers both euen and odde may be deuided into many equall partes, as into 4. 5. 6. or mo, and therfore into 2. As 9 may be deuided into 3 and 3 and 3 which are his partes, and are also equall, for the one of them exceedeth not the other: yet is not therfore this number 9 an euen number, for that 3 and 3 (these equall partes of 9) added together make not 9 but onely 6. Likewise taking the definition so generally, euery number whatsoeuer should be an euen number: for in that sort of vnderstāding there is no number, but that it may be deuided into two equall partes: as this number 7 may be deuided into 3 partes, namely, 3. 1. and 3. of which two, namely, 3 and 3 are equal, yet is not 7 an euen number, because 3 and 3 added together, make not 7. *Boetius* therfore in the first booke of his Arithmetike, for the more playnes, after this maner defineth an euen number: *Numerus par est, qui potest in aequalia duo diuidi, vno medio non intercedente*, that is:

*Boetius.*

An euen number is that which may be deuided into two equall partes, without an vnitie comming betwene them. As 8 is deuided into 4 and 4 two equall partes without an vnitie comming betwene them, which added together, make 8, so that the sence of this definition is, that an euen number is that which is deuided into two such equall partes, which are his two halfe partes.

*An other definition of an euen number.*

Here is also to be noted, that a part is taken in this definition, and in certaine definitions following, not in that signification as it was before defined, namely, for such a part as measureth the whole number, but for any part which helpeth to the making of the whole, and into which the whole may be resolued, so are 3 and 5 partes of 8 in this sence but not in the other sence. For neyther 3 nor 5 measureth 8. Pithagoras and his scholers gaue an other definition of an euen number (which definition *Boetius* also hath) after this maner.

*Note.*

*Pithagoras.*



An

# The Euclidean algorithm (Proposition VII.2)



whole number B A, wherfore it also measureth that which remayneth, namely, the number F A (by the 4. cōmon sentence of the seuenth). But the number A F measureth the number D G, wherfore E also measureth D G. And it measureth also the whole D C, wherfore it also measureth that which remayneth, namely, the number G C (by the same common sentence): but G C measureth the number F H, wherfore also E measureth F H, and it measureth the whole number F A, wherfore (by the former common sentence) it also measureth that which remayneth H A, which is vnitie, it selfe being a number, which is impossible. Wherfore no number doth measure the numbers A B and C D, wherfore the numbers A B and C D are prime numbers the one to the other: which was required to be proued.

### The conuerse of this proposition after *Campane*.

*The conuerse of this proposition.*

And if the two numbers, namely A B and C D be prime the one to the other. Then the lesse being continually taken from the greater there shalbe no stay of that substraction, till that you come to vnitie. For if in the continuall substraction there be a stay before you come to vnitie. Suppose that H A be the number whereat the stay is made, which also being subtrahed out of G C leaueth nothing. Wherfore H A measureth G C wherfore also it measureth F H by the 5. common sentence of the seuenth, And for as much as it also measureth it selfe, therefore it also measureth the whole A F by the sixth common sentence of the seuenth, wherfore also it measureth D G by the 5. common sentence. But it is before proued that it measureth G C, wherfore it measureth the whole C D, by the sixth common sentence of the seuenth: wherfore also it measureth B F by the 5. common sentence of the seuenth. And it is also proued that it measureth F A, wherfore also it measureth the whole number A B by the sixth common sentence of the seuenth, Now for as much as the number H A measureth the numbers A B and C D, therfore the numbers AB and CD are numbers composed: wherfore they are not prime the one to the other: which is contrary to the supposition.

D . . . G . . C
B . . . . . F . . H . A

*How to know whether two numbers geuen be prime the one to the other.*

And by this proposition if there be two numbers geuen. It is easy to finde out, whether they be prime the one to the other or no. For if by such continual substraction of the lesse from the greater, you come at the length to vnitie. Then are those numbers geuen prime the one to the other. But if there be a stay before you come to vnitie, then are the numbers geuen, numbers composed the one to the other.

### ¶ The 1. Probleme. The 2. Proposition.

*Two numbers being geuen not prime the one to the other, to finde out their greatest common measure.*

*Two cases in this probleme.*
*The first case.*

*Suppose the two numbers geuen not prime the one to the other to be* A B *and* C D. *It is required to finde out the greatest common measure of the said numbers* A B *and* C D. *Now the number* C D *either measureth the number* A B *or not. If* C D *measure* A B *it also measureth it selfe. Wherefore* C D *is a common measure to the numbers* C D *and* A B. *And it is manifest also that it is the greatest common measure: for there is no number greater then* C D *that will measure* C D.

A . . . . . . . . B
C . . . . D

*The second case.*

*But if* C D *do not measure* A B, *then if of the numbers* A B *and* C D, *the lesse be continually taken away from the greater, there will before you come to vnitie, be left a number, which will measure the number going before (by the 1. of the seuenth). For if there should not, then should the numbers* A B *and* C D *be prime the one to the other, which is contrary to the supposition. Let the sayd number left by the continuall substraction of the lesse number out of the greater be* F C. *So that let the number* C D *measuring* A B, *and subtrahed out of it as often as you can leue a lesse number then it selfe, namely* A E. *And let* A E *measuring* C D, *and subtrahed out of it*

A . . . . E . . . . . . . B
C . . F . . . . D

45



*as often as you can leaue a lesse then it selfe namely,* C F. *And suppose that* C F *do so measure* A E *that there remayne nothing. Then I say that* C F *is a common measure to the numbers* A B *and* C D. *For forasmuch as* C F *measureth* AE, *and* A E *measureth* D F, *therefore* C F *also measureth* D F *(by the fifth common sentence of the seuenth) and it likewise measureth it selfe, wherfore it also measureth the whole* C D *(by the sixth common sentence of the seuenth) but* C D *measureth* B E, *wherefore* C F *also measureth* B E *(by the fifte common sentence of the seuenth). And it measureth also* E A: *wherefore it also measureth the whole* B A *(by the sixth common sentence of the seuenth): and it also measureth* C D *as we haue before proued: wherefore the number* C F *measureth the numbers* AB *&* C D: *wherfore the number* C F *is a commō measure to the numbers* A B *&* C D.

*Demonstratiō of the second case.*

*That C F is a common measure to the numbers A B and C D.*

*I say also that it is the greatest common measure. For if* C F *be not the greatest commō measure to* A B *and* C D, *let there be a number greater then* C F, *which measureth* A B *and* C D: *which let be* G. *And forasmuch as* G *measureth* C D, *and* C D *measureth* B E, *therefore* G *also measureth* B E *(by the fift common sentence of the seuenth). And it measureth the whole* A B, *wherefore also it measureth the residue, namely,* A E *(by the 4. common sentence of the seuenth). But* A E *measureth* D F, *wherefore* G *also measureth* D F *(by the foresayd 5. common sentence of the seuenth). And it measureth the whole* C D. *Wherefore it also measureth the residue* F C: *namely, the greater number the lesse: which is impossible. No number therefore greater then* C F *shall measure those numbers* A B *and* C D: *wherefore* C F *is the greatest common measure to* AB *and* CD: *which was required to be done.*

A . . . . E . . . . . . B
G . . .
C . . F . . . . D

*That C F is the greatest common measure to A B and C D.*

### Corrolary.

*Hereby it is manifest, that if a number measure two numbers it shall also measure their greatest common measure.* For if it measure the whole & the part taken away, it shall alwayes measure the residue also, which residue is at the length, the greatest common measure of the two numbers geuen.

### ¶ The 2. Probleme. Th 3. Proposition.

*Thre numbers being geuē, not prime the one to the other: to finde out their greatest common measure.*

*Suppose the three numbers geuen not prime the one to the other to be* A, B, C. *Now it is required vnto the sayd numbers* A, B, C *to finde out the greatest common measure. Take the greatest common measure of the two numbers* A *and* B *(by the 2 of the seuenth) which let be* D: *which number* D *either measureth the number* C *or not.*

A . . . . . . . .
B . . . . . .
C . . . .
D . .
E . . .

*Two cases in this Proposition.*
*The first case.*

*First let* D *measure* C. *And it also measureth the numbers* A *and* B, *wherfore* D *measureth the numbers* A, B, C. *Wherefore* D *is a common measure vnto the numbers* A, B, C. *Then I say also, that it is the greatest common measure vnto them. For if* D *be not the greatest common measure vnto the numbers* A, B, C, *let some number greater then* D *measure the numbers* A, B, C. *And let the same number be* E. *Now forasmuch as* E *measureth the numbers* A, B, C, *it measureth also the numbers* A, B. *Wherefore it measureth also*

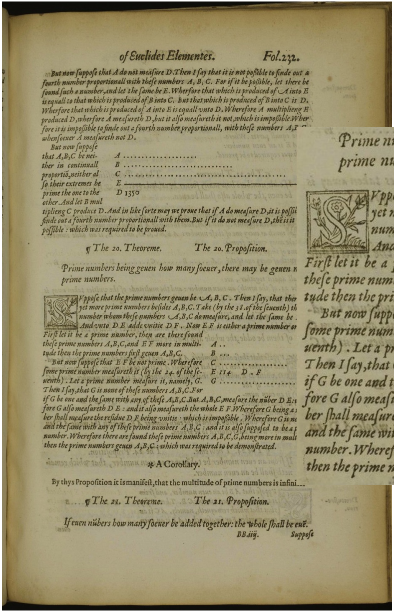
Tt.i. the

# Euclid on prime numbers

*12 A prime (or first) number is that, which onely vnitie doth measure.*

As 5.7.11.13. For no number measureth 5, but onely vnitie. For v. vnities make the number 5. So no number measureth 7, but onely vnitie. 2. taken 3. times maketh 6. which is lesse then 7: and 2. taken 4. times is 8, which is more then 7. And so of 11.13. and such others. So that all prime numbers, which also are called first numbers, and numbers vncomposed, haue no part to measure thē, but onely vnitie.

# Euclid on prime numbers (Proposition IX.20)





But now suppose that A do not measure D. Then I say that it is not possible to finde out a fourth number proportionall with these numbers A, B, C. For if it be possible, let there be found such a number, and let the same be E. Wherfore that which is produced of A into E is equall to that which is produced of B into C. But that which is produced of B into C is D. Wherfore that which is produced of A into E is equall vnto D. Wherefore A multiplieng E produced D, wherfore A measureth D, but it also measureth it not, which is impossible. Wherfore it is impossible to finde out a fourth number proportionall, with these numbers A, B, C whensoeuer A measureth not D.

But now suppose that A, B, C be neither in continuall proportion, neither also their extremes be prime the one to the other. And let B multiplieng C produce D. And in like sorte may we proue that if A do measure D, it is possi[ble to] finde out a fourth number proportionall with them. But if it do not measure D, then is it [im]possible: which was required to be proued.

A
B
C
E
D 1350

¶ The 20. Theoreme. The 20. Proposition.

Prime numbers being geuen how many soeuer, there may be geuen more prime numbers.

Suppose that the prime numbers geuen be A, B, C. Then I say, that there are yet more prime numbers besides A, B, C. Take (by the 38. of the seuenth) the lest number whom these numbers A, B, C do measure, and let the same be D E. And vnto D E adde vnitie D F. Now E F is either a prime number or not. First let it be a prime number, then are there found these prime numbers A, B, C, and E F more in multitude then the prime numbers first geuen A, B, C.

But now suppose that E F be not prime. Wherefore some prime number measureth it (by the 24. of the seuenth). Let a prime number measure it, namely, G. Then I say, that G is none of these numbers A, B, C. For if G be one and the same with any of these A, B, C. But A, B, C, measure the nūber D E: wherfore G also measureth D E: and it also measureth the whole E F. Wherefore G being a number shall measure the residue D F being vnitie: which is impossible. Wherefore G is not one and the same with any of these prime numbers A, B, C: and it is also supposed to be a prime number. Wherefore there are found these prime numbers A, B, C, G, being more in multitude then the prime numbers geuen A, B, C: which was required to be demonstrated.

A ..
B ...
C ..................
E 114 D . F
G ......................

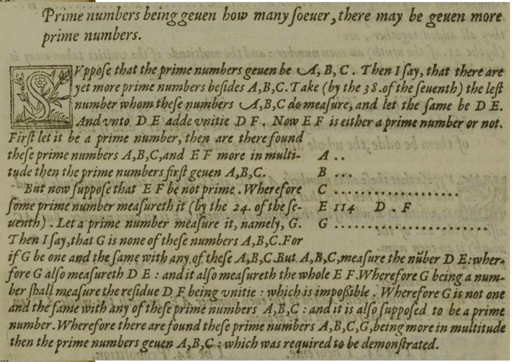
❋ A Corollary.

By thys Proposition it is manifest, that the multitude of prime numbers is infini[te].

¶ The 21. Theoreme. The 21. Proposition.

If euen nūbers how many soeuer be added together: the whole shall be euē.





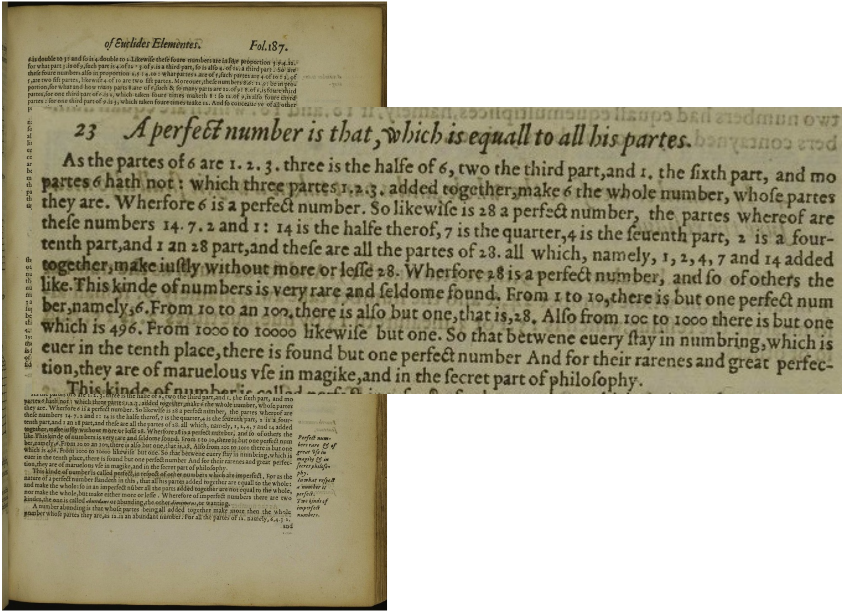
Prime numbers being geuen how many soeuer, there may be geuen more prime numbers.

Suppose that the prime numbers geuen be A, B, C. Then I say, that there are yet more prime numbers besides A, B, C. Take (by the 38. of the seuenth) the lest number whom these numbers A, B, C do measure, and let the same be D E. And vnto D E adde vnitie D F. Now E F is either a prime number or not. First let it be a prime number, then are there found these prime numbers A, B, C, and E F more in multitude then the prime numbers first geuen A, B, C.

But now suppose that E F be not prime. Wherefore some prime number measureth it (by the 24. of the seuenth). Let a prime number measure it, namely, G. Then I say, that G is none of these numbers A, B, C. For if G be one and the same with any of these A, B, C. But A, B, C, measure the nūber D E: wherfore G also measureth D E: and it also measureth the whole E F. Wherefore G being a number shall measure the residue D F being vnitie: which is impossible. Wherefore G is not one and the same with any of these prime numbers A, B, C: and it is also supposed to be a prime number. Wherefore there are found these prime numbers A, B, C, G, being more in multitude then the prime numbers geuen A, B, C: which was required to be demonstrated.

A ..
B ...
C ..................
E 114 D . F
G ......................

# Euclid on perfect numbers





6.is double to 3: and so is 4.double to 2. Likewise these foure numbers are in like proportion 3.9.4.12. for what part 3.is of 9, such part is 4.of 12 · 3.of 9.is a third part, so is also 4. of 12. a third part. So are these foure numbers also in proportion 2.5 : 4.10 : what partes 2.are of 5, such partes are 4.of 10 : 2, of 5, are two fift partes, likewise 4.of 10 are two fift partes. Moreouer, these numbers 8.6: 12.9: be in proportion, for what and how many parts 8.are of 6, such & so many parts are 12.of 9: 8.of 6, is foure third partes, for one third part of 6.is 2, which taken foure times maketh 8 : so 12.of 9, is also foure thyrd partes : for one third part of 9.is 3, which taken foure times make 12. And so conceaue ye of all other

23 *A perfect number is that, which is equall to all his partes.*

As the partes of 6 are 1. 2. 3. three is the halfe of 6, two the third part, and 1. the sixth part, and mo partes 6 hath not: which three partes 1.2.3. added together, make 6 the whole number, whose partes they are. Wherfore 6 is a perfect number. So likewise is 28 a perfect number, the partes whereof are these numbers 14. 7. 2 and 1: 14 is the halfe therof, 7 is the quarter, 4 is the seuenth part, 2 is a fourtenth part, and 1 an 28 part, and these are all the partes of 28. all which, namely, 1, 2, 4, 7 and 14 added together, make iustly without more or lesse 28. Wherfore 28 is a perfect number, and so of others the like. This kinde of numbers is very rare and seldome found. From 1 to 10, there is but one perfect number, namely, 6. From 10 to an 100, there is also but one, that is, 28. Also from 100 to 1000 there is but one which is 496. From 1000 to 10000 likewise but one. So that betwene euery stay in numbring, which is euer in the tenth place, there is found but one perfect number And for their rarenes and great perfection, they are of maruelous vse in magike, and in the secret part of philosophy.

*Perfect numbers rare & of great vse in magike & in secret philosophy.*

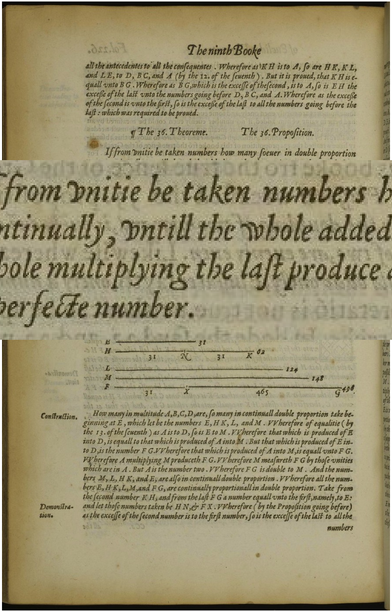
This kinde of number is called perfect, in respect of other numbers which are imperfect. For as the nature of a perfect number standeth in this, that all his partes added together are equall to the whole: and make the whole: so in an imperfect nũber all the parts added together are not equal to the whole, nor make the whole, but make either more or lesse. Wherefore of imperfect numbers there are two kindes, the one is called *abundans* or abunding, the other *diminutus*, or wanting.

*In what respect a number is perfect.*

*Two kinds of imperfect numbers.*

A number abunding is that whose partes being all added together make more then the whole number whose partes they are, as 12. is an abundant number. For all the partes of 12. namely, 6.4.3 2. and

# Euclid on perfect numbers (Proposition IX.36)



The ninth Booke

all the antecedentes to all the conſequentes . Wherefore as K H is to A, ſo are H K, K L, and L E, to D, B C, and A (by the 12. of the ſeuenth) . But it is proued, that K H is equall vnto B G . Wherefore as B G, which is the exceſſe of the ſecond, is to A, ſo is E H the exceſe of the laſt vnto the numbers going before D, B C, and A. Wherefore as the exceſe of the ſecond is vnto the firſt, ſo is the exceſe of the laſt to all the numbers going before the laſt : which was required to be proued.

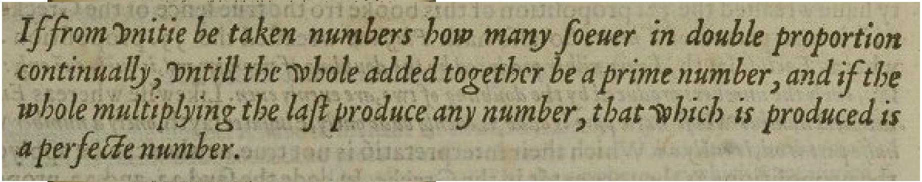
¶ The 36. Theoreme. The 36. Propoſition.

If from vnitie be taken numbers how many ſoeuer in double proportion



Conſtruction. How many in multitude A,B,C,D, are, ſo many in continuall double proportion take beginning at E, which let be the numbers E, H K, L, and M . VVherefore of equalitie (by the 13. of the ſeuenth) as A is to D, ſo is E to M . VVherefore that which is produced of E into D, is equall to that which is produced of A into M . But that which is produced of E into D, is the number F G. VVherefore that which is produced of A into M, is equall vnto F G. VVherefore A multiplying M produceth F G. VVherefore M meaſureth F G by thoſe vnities which are in A . But A is the number two . VVherefore F G is double to M . And the numbers M, L, H K, and E, are alſo in continuall double proportion . VVherefore all the numbers E, H K, L, M, and F G, are continually proportionall in double proportion. Take from the ſecond number K H, and from the laſt F G a number equall vnto the firſt, namely, to E: and let thoſe numbers taken be H N, & F X . VVherefore (by the Propoſition going before)

Demonſtration. as the exceſſe of the ſecond number is to the firſt number, ſo is the exceſſe of the laſt to all the numbers



If from vnitie be taken numbers how many ſoeuer in double proportion continually, vntill the whole added together be a prime number, and if the whole multiplying the laſt produce any number, that which is produced is a perfecte number.

In modern terms: if $2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is perfect

# Number theory after Euclid

Very little for many centuries...

Recall that Diophantus' *Arithmetica* (13 books, c. AD 250) featured number problems; for example [from Lecture IX]:

> Problem I.27: *Find two numbers such that their sum and product are given numbers*

The *Arithmetica* also features problems and ideas that we would now classify as number-theoretic; for example:

> Problem III.19: *To find four numbers such that the square of their sum plus or minus any one singly gives a square*
>
> Problem V.9: *To divide unity into two parts such that, if a given number is added to either part, the result will be a square*

Restrictions on the permitted form of solutions to problems eventually gave rise to the notion of Diophantine equations

# Number theory outside Europe

*Sūnzǐ Suànjīng* 孙子算经 (*The Mathematical Classic of Master Sun*) (3rd–5th century BC) contains a statement, but no proof, of the Chinese Remainder Theorem for the solution of simultaneous congruences

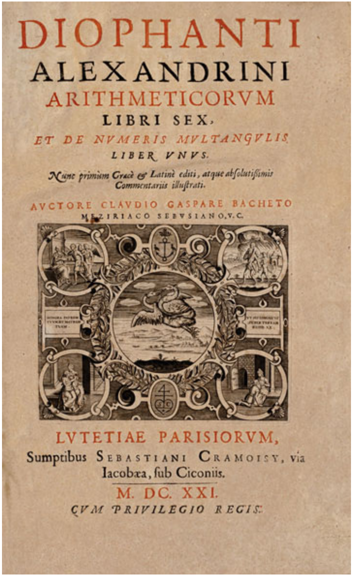
An algorithm for the solution was provided by Aryabhata in 6th-century India

In 7th-century India, Brahmagupta studied Diophantine equations (including Pell's equation — see later, and also: Toke Knudsen and Keith Jones, 'The Pell Equation in India', 2017)

These works were unknown in Europe until the 19th century

See: Eva Caianiello, 'Indeterminate linear problems from Asia to Europe', *Lettera Matematica* 6 (2018), 233–243

# 17th-century number theory


DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX,
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.
Nunc primùm Græcè & Latinè editi, atque absolutissimis Commentariis illustrati.
AVCTORE CLAVDIO GASPARE BACHETO MEZIRIACO SEBVSIANO, V.C.
LVTETIAE PARISIORVM,
Sumptibus SEBASTIANI CRAMOISY, via Iacobæa, sub Ciconiis.
M. DC. XXI.
CVM PRIVILEGIO REGIS.

Bachet's Latin edition of Diophantus' *Arithmetica* (1621)

Pierre de Fermat owned a 1637 edition, which he studied and annotated

# Fermat on number theory

Fermat's Little Theorem: if $a$ is any integer and $p$ is prime then $p$ divides $a^p - a$

Studies of 'Pell's Equation' $x^2 - Dy^2 = 1$

Conjectures on perfect numbers [more in a moment]

Studies of diophantine problems leading to 'Fermat's Last Theorem' [more in a moment]

Published nothing — had to be exhorted to write his ideas down

(See *Mathematics emerging*, §§6.1–6.3)

# The 'Last Theorem'

*Arithmetica* Problem II.8 concerns the splitting of a given square number into two other squares

Fermat's marginal note:

> *It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second, into two like powers. I have discovered a truly marvelous proof of this, which this margin is too narrow to contain.*

(See: Simon Singh, *Fermat's Last Theorem*, Fourth Estate, 1998)

# Perfect numbers

Euclid's Theorem: if $2^n - 1$ is prime then $2^{n-1}(2^n - 1)$ is perfect

Fermat to Mersenne (1640): if $2^n - 1$ is prime then $n$ must be prime

Mersenne (1644): if $p \leq 257$ and $2^p - 1$ is prime then $p$ is one of 2, 3, 5, 7, 13, 17, 67 (a misprint for 61 perhaps?), 127, 257. Not quite right: $2^{89} - 1$, $2^{107} - 1$ are prime and $2^{257} - 1$ is composite.

Euler: proof that all even perfect numbers are of Euclid's form (proved 1749, but published posthumously)

(See *Mathematics emerging*, §6.1.2)

NB. 52 Mersenne primes are currently known, the largest being $2^{136,279,841} - 1$ (found in October 2024)

# 17th-century attitudes to number theory

Fermat failed to spark an interest in number theory in his contemporaries

Pascal to Fermat (1655):

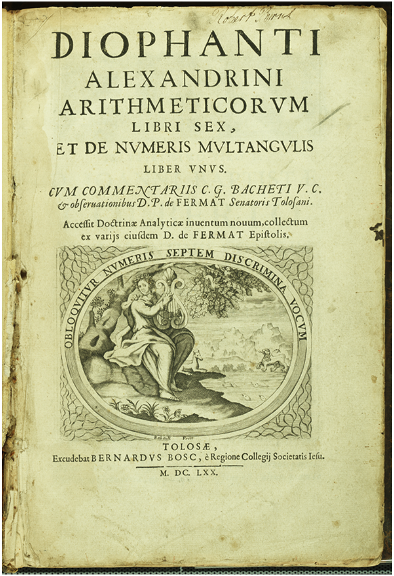
> *... seek elsewhere those who can follow you in your numerical discoveries ... I confess to you that this goes far beyond me ...*

Number-theoretic investigations were widely regarded as trivial and uninteresting

Huygens to Wallis:

> *Good hours should not be spent on such things unless more important things are lacking ...*

# The ‘rebirth’ of number theory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& obseruationibus D. P. de FERMAT Senatoris Tolosani.
Accessit Doctrinæ Analyticæ inuentum nouum, collectum
ex varijs eiusdem D. de FERMAT Epistolis.
OBLOQVITVR NVMERIS SEPTEM DISCRIMINA VOCVM
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Excudebat BERNARDVS BOSC, è Regione Collegij Societatis Iesu.
M. DC. LXX.

1670 edition of Bachet, published by Samuel Fermat, including his father’s notes

The ‘Last Theorem’ was not the only result for which Fermat failed to provide a proof

Number theory was ‘reborn’ from the attempts of Euler (and later Lagrange and Legendre) to fill the gaps left by Fermat

# Euler on number theory

Euler (1747):

*Nor is the author disturbed by the authority of the greatest mathematicians when they sometimes pronounce that number theory is altogether useless and does not deserve investigation. In the first place, knowledge is always good in itself, even when it seems to be far removed from common use. Secondly, all the aspects of the truth which are accessible to our mind are so closely related to one another that we dare not reject any of them as being altogether useless. . . .*

*Consequently, the present author considers that he has by no means wasted his time and effort in attempting to prove various theorems concerning integers and their divisors. . . . Moreover, there is little doubt that the method used here by the author will turn out to be of no small value in other investigations of greater import.*

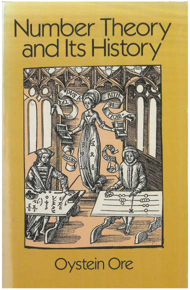
# 19th-century number theory

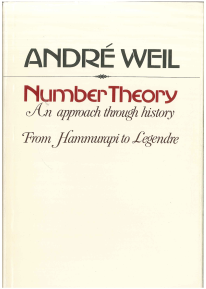
Gauss's *Disquisitiones arithmeticae* (1801) became a key text for many years to come: modular arithmetic, quadratic forms, cyclotomy, ...

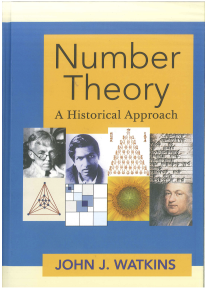
Number-theoretic problems (especially attempts to prove Fermat's Last Theorem) led to the development of ideal theory, and the linking of number theory and abstract algebra in algebraic number theory

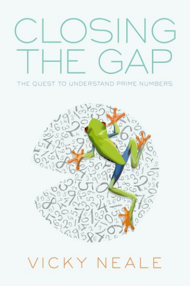
By the end of the 19th century, a new branch, analytic number theory, had also emerged (e.g., Riemann hypothesis, Prime Number Theory $\pi(x) \sim \frac{x}{\log x}, \ldots$)

# The history of number theory




Leonard Eugene Dickson, *History of the theory of numbers*, 3 vols., Carnegie Institution of Washington, 1919–1923: I, II, III