# Infinite groups: Sheet 1

September 17, 2025

**Section A**

*Exercise* 1. (i) If $N \trianglelefteq H$ and $H \trianglelefteq G$, with $N$ of finite index in $H$ and $H$ finitely generated, then $N$ contains a finite-index subgroup $K$ which is normal in $G$.

(ii) Show that if $H$ is characteristic in $G$ then we can take $K$ to be characteristic in $G$ as well.

**Solution.**
(i) This is Lemma 2.7 of the revision notes.

(ii) Since $H$ is finitely generated it has fintely many subgroups of index $|H : N|$ in $H$ (see prop. 2.8 of the revision notes). If we denote the intersection of all these subgroups by $K$ then clearly $K$ is of finite index in $H$ and as $H$ is characteristic in $G$, $K$ is also caracteristic in $G$.

*Exercise* 2. Find the commutator subgroup and the abelianization for the finite dihedral group $D_{2n}$ and for the infinite dihedral group $D_\infty$.

**Solution.** If we denote as usual by $r$ the rotation generator of $D_{2n}$ and by $s$ the reflection we have $srs^{-1} = r^{-1}$ and $[s, r] = r^{-2}$. It follows that $< r^2 >$ is contained in the commutator subgroup of $D_{2n}$. By calculating explicitly all commutators $[sr^i, sr^j]$ we see that in fact the commutator subgroup is $< r^2 >$. So if $n$ is odd the commutator subgroup is $< r >$ and the abelianization is $\mathbb{Z}_2$. If $n$ is even the commutator subgroup is $< r^2 >$ and the abelianization is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Similarly for $D_\infty$ we have that commutator subgroup is $< r^2 >$ and the abelianization is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

*Exercise* 3. Show that

(i) The dihedral group $D_{2n}$ is isomorphic to a semi-direct product $\mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2$ and the infinite dihedral group $D_\infty$ is isomorphic to a semi-direct product $\mathbb{Z} \rtimes_\varphi \mathbb{Z}_2$.

(ii) The permutation group $S_n$ is the semidirect product of $A_n$ and $\mathbb{Z}_2 = \{\mathrm{Id}, (12)\}$.

**Solution.**

(i) Note that $\mathbb{Z}_n, \mathbb{Z}$ are normal subgroups of $D_{2n}, D_\infty$ respectively.

So we have the short exact sequences $0 \to \mathbb{Z}_n \to D_{2n} \to \mathbb{Z}_2 \to 0$ and $0 \to \mathbb{Z} \to D_\infty \to \mathbb{Z}_2 \to 0$.

Both these sequences split: simply map the generator of $\mathbb{Z}_2$ to the reflection generator of the dihedral group.

Hence these two groups split as semi-direct products (see Revision notes).

(ii) Note the the sign of a permutation gives a homomorphism $\sigma : S_n \to \mathbb{Z}_2$ with kernel $A_n$. So we have the split short exact sequence

$1 \to A_n \to S_n \to \mathbb{Z}_2 \to 1$ and $S_n$ splits as a semidirect product of $A_n$ and $\mathbb{Z}_2 = \{\mathrm{Id}, (12)\}$.

*Exercise* 4. (i) Show that the torsion-free abelian group $\mathbb{Q}$ is not a free abelian group.

(ii) Let $0 \to A \to B \xrightarrow{r} C \to 0$ be a short exact sequence of abelian groups, where $C$ is free abelian. Then the sequence splits and $B \cong A \oplus C$.

(iii) Let $F$ be a free abelian group of rank $n$ and let $B = \{x_1, \dots, x_n\}$ be a generating set of $F$. Show that $B$ is a basis of $F$. Conclude that $n$ equals the minimal cardinality of all generating sets of $F$.

**Solution.** (i) Assume that $\mathbb{Q}$ is free abelian and let $x = m/n, y = p/q$ be two elements of a basis with $m, n, p, q \in \mathbb{Z}$. Then $pnx - mqy = 0$ which is a contradiction as $pnx - mqy$ is a non-trivial element of $< x > \oplus < y >$.

(ii) This is Corollary 2.24 of the revision notes.

(iii) Let $Y = \{y_1, \dots, y_n\}$ be a basis of of $F$. By the universal property of free abelian groups the map $f : Y \to B$, $f(y_i) = x_i$ $(i = 1, ..., n)$ extends to a homomorphism $f : F \to F$. $f$ is surjective since $B$ is a generating set. Let $N$ be the kernel of $f$. By theorem 2.26 of the revision notes $N$ is free abelian. So by part (ii) $F \cong F \oplus N$. By comparing the rank of the two sides we have that $N = 0$ and $f$ is an isomorphism, so $B$ is a basis of $F$.