Infinite groups: Sheet 3

November 14, 2024

Section A

Exercise 1. Prove that if $K \leq H \leq G$ and of H is a normal subgroup of G and K is a characteristic subgroup of H then K is a normal subgroup of G.

Is the same true if K is only normal in H? Provide a proof in the affirmative case, a counterexample in the negative case.

Answer. Every $g \in G$ defines an automorphism for H by

$$C_g : H \to H,\ C_g(h) = ghg^{-1}.$$

Since K is a characteristic subgroup, $C_g(K) = K$. As g is an arbitrary element in G, we conclude that K is normal in G.

The statement is not true if we weaken the assumption to 'K is normal in H'.

For instance in $H = \mathbb{Z}^2$ every subgroup is normal, because the group is abelian.

Let $G = \mathbb{Z}^2 \rtimes \mathbb{Z}$, defined by

$$\varphi : \mathbb{Z} \to \mathrm{Au}t(\mathbb{Z}^2),$$

$$\varphi(m) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^m .$$

We have that H is normal in G, but the subgroup K spanned by $(1,0)^T$ is not normal in G, because $\varphi(1)(1,0)^T = (2,1)^T$.

Exercise 2. Let G be a nilpotent group of class k and let A be an abelian group. Write the upper and the lower central series of $G \times A$ in terms of A and of the upper, respectively lower central series of G.

Answer. Let $H = G \times A$. For $k \geq 2$, one can prove by induction that $C^k H = C^k G \times \{1\}$. On the other hand, one can prove by induction that $Z_i(H) = Z_i(G) \times A$.

Exercise 3. Consider the *upper central series* of G

$$Z_0(G) = \{1\} \lhd Z_1(G) \lhd \ldots \lhd Z_i(G) \lhd Z_{i+1}(G) \lhd \ldots$$

Let g be an arbitrary element in G. Prove that the map $\tilde{\varphi} : Z_2(G) \to Z_1(G)$ defined by:

$$\tilde{\varphi}(y) := [g, y],$$

is a group homomorphism.

Answer. We have a general formula

$$[g, yz] = [g, y]\,[y, [g, z]]\,[g, z]\,.$$

Since $z \in Z_2(G)$, it commutes with g modulo $Z_1(G)$, equivalently $[g, z] \in Z_1(G)$. Since $Z_1(G) = Z(G)$, $[y, [g, z]] = 1$.

Exercise 4. Prove the following properties.

1. Any subgroup H of a polycyclic group G is polycyclic (hence, finitely generated).

2. If $N \lhd G$ and G is polycyclic then G/N is polycyclic.

Answer. (1). Given a cyclic series for G

$$G = N_0 \rhd N_1 \rhd \ldots \rhd N_k \rhd N_{k+1} = \{1\}, \tag{1}$$

the intersections $H \cap N_i$ define a cyclic series for H.

(2). Proof by induction *on the length* $\ell(G) = n$.
For $n = 1$, G cyclic, any quotient of G is cyclic.
Assume true for all $k \leqslant n$, consider G with $\ell(G) = n + 1$.

Let N_1 be the first term $\neq G$ in a cyclic series of length $n + 1$.

By induction, $N_1/(N_1 \cap N) \simeq N_1N/N$ is polycyclic.

$N_1N/N \lhd G/N$ and $(G/N)/(N_1N/N) \simeq G/N_1N$ is cyclic, as quotient of G/N_1.
It follows that G/N is polycyclic.