Professor Arguz **B3.2 Geometry of Surfaces** **MT 2025**

Initial problem sheet (Not to be handed in)

1. Define an equivalence relation on the unit interval $[0,1]$ by $x \sim y$ if either $x = y$ or $x = 0$ and $y = 1$. Show that the set of equivalence classes with the quotient topology is homeomorphic to the unit circle.

2. Let X be the space of equivalence classes of points in $\mathbb{R}^2 \backslash \{0\}$ under the equivalence relation $(x_1, x_2) \sim (\lambda x_1, \lambda^{-1} x_2)$ for some $\lambda \in \mathbb{R} \backslash \{0\}$. Show by considering the equivalence classes of $(0,1)$ and $(1,0)$ that the space X is not Hausdorff in the quotient topology.

3. A connected surface X is obtained by taking n copies of a sphere with two disjoint open discs removed, and identifying the $2n$ boundary circles in pairs. Show that the Euler characteristic of X must vanish.

4. A connected surface Y is obtained by taking $2n$ copies of a sphere with *three* disjoint open discs removed, and identifying the $6n$ boundary circles in pairs. What values can the Euler characteristic take?

Can you make this surface this way?

5. Let S be the set of all straight lines in $\mathbb{R}^2$ (not necessarily through 0). Show that there is a natural way to make S into a topological surface. Show that S is homeomorphic to the open Möbius band M.

6. Consider the quotient

$$S = \mathbb{R}^2 / G$$

where $G = \mathbb{Z}^2$ acts[1] by $(n,m) \bullet (x,y) = ((-1)^m x + n, y + m)$ on $\mathbb{R}^2$, where $n, m \in \mathbb{Z}$. Show that S is homeomorphic to the Klein bottle.

7. A figure 8 loop consists of two circles touching at a point. Show that a torus can be obtained by attaching a disc onto a figure 8 loop.

[1] $G = \mathbb{Z}^2$ as a set, but as a group $G = \mathbb{Z} \rtimes \mathbb{Z}$ is a *semi-direct product.*