Example Sheet 0. C5.7, Topics in Fluids. MT2025.

1. **Vector identities and the divergence theorem.**

   (a) Prove the following identities for any differentiable vector fields $\mathbf{u}$ and $\mathbf{v}$:

   (i) $(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = \boldsymbol{\nabla}\left(\frac{1}{2}|\mathbf{u}|^2\right) + (\boldsymbol{\nabla}\wedge\mathbf{u})\wedge\mathbf{u}$;

   (ii) $\boldsymbol{\nabla}\wedge(\mathbf{u}\wedge\mathbf{v}) = (\boldsymbol{\nabla}\cdot\mathbf{v})\mathbf{u} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{u} - (\boldsymbol{\nabla}\cdot\mathbf{u})\mathbf{v} - (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{v}$.

   (iii) $\boldsymbol{\nabla}^2\mathbf{u} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) - \boldsymbol{\nabla}\wedge(\boldsymbol{\nabla}\wedge\mathbf{u})$.

   (b) The faces of a tetrahedron lie in the planes $x_1 = 0, x_2 = 0, x_3 = 0$ and $\mathbf{n}\cdot\mathbf{x} = 1$, where $\mathbf{n} = (n_1, n_2, n_3)$ is a unit vector such that $n_j > 0$ for $j = 1, 2, 3$ and $\mathbf{x} = (x_1, x_2, x_3)$. Let A_j be the area of the face in the plane $x_j = 0$ and let A be the area of the face with unit normal $\mathbf{n}$. By applying the divergence theorem, show that $A_j = n_j A$.

2. **The convective derivative and Reynolds' transport theorem**

   (i) Let $f(\mathbf{x}, t)$ be a differentiable function of position $\mathbf{x}$ and time t, defined in a fluid whose velocity is $\mathbf{u}(\mathbf{x}, t)$. Show that the rate of change of f following a material fluid element is given by

   $$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + (\mathbf{u}\cdot\nabla)f$$

   (ii) Let $V(t)$ be a time-dependent closed region of $\mathbb{R}^3$ that is convected with velocity $\mathbf{u}(\mathbf{x}, t)$. Prove Reynolds' Transport Theorem, namely

   $$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} f \,\mathrm{d}V = \iiint_{V(t)} \frac{\partial f}{\partial t} + \nabla\cdot(f\mathbf{u})\mathrm{d}V$$

   for any continuously differentiable function $f(\mathbf{x}, t)$.

3. **The continuity equation and incompressibility.**

   (a) By applying the principle of conservation of mass to a material volume $V(t)$, derive the continuity equation

   $$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0$$

   for a compressible fluid with density ρ, stating any assumptions that you make about the smoothness of ρ and u.

   (b) When the fluid is incompressible, i.e. $\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0$, derive the incompressibility condition

   $$\nabla\cdot\mathbf{u} = 0$$

   and deduce that the transport theorem for an incompressible fluid may be written in the form

   $$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} f \,\mathrm{d}V = \iiint_{V(t)} \frac{\mathrm{D}f}{\mathrm{D}t}\,\mathrm{d}V$$

   for any continuously differentiable function $f(\mathbf{x}, t)$.

4. **Derivation of the incompressible Navier-Stokes equations.**

   (a) Define the Cauchy stress tensor σ_{ij} and show that the force per unit area exerted on a surface element with unit normal $\mathbf{n} = n_j \mathbf{e}_j$ by the fluid towards which $\mathbf{n}$ is directed is given by the stress vector

$$\mathbf{t} = \mathbf{e}_i \sigma_{ij} n_j$$

   where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are unit vectors along the axes Ox_1, Ox_2, Ox_3.

   (b) Newton's second law for a material volume $V(t)$ with boundary $\partial V(t)$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V(t)} \rho \mathbf{u} \mathrm{d}V = \iint_{\partial V(t)} \mathbf{t} \mathrm{d}S + \iiint_{V(t)} \rho \mathbf{F} \mathrm{d}V$$

   where ρ is the density, $\mathbf{u} = u_i \mathbf{e}_i$ is the velocity and $\mathbf{F} = F_i \mathbf{e}_i$ is an external body force acting per unit mass. Explain the physical significance of each term in this expression.

   (c) Use Reynolds' transport theorem and the divergence theorem to derive Cauchy's momentum equation in the form

$$\rho \frac{\mathrm{D}u_i}{\mathrm{D}t} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho F_i.$$

   (d) Define the rate-of-strain tensor e_{ij}. State the physical assumptions that are needed for an incompressible fluid to be Newtonian, that is

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij},$$

   where p is the pressure and μ is the viscosity.

   (e) For an incompressible, constant viscosity, Newtonian fluid, deduce the Navier-Stokes equations in the form

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{F}, \quad \nabla \cdot \mathbf{u} = 0.$$

5. **Symmetry of the stress tensor.**

   For the Cauchy stress tensor, deduce that $\sigma_{ij} = \sigma_{ji}$.

Q1 Vector identities and the divergence theorem.

(a) Prove the following identities for any differentiable vector fields $\mathbf{u}$ and $\mathbf{v}$:
(i) $(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = \boldsymbol{\nabla}\left(\frac{1}{2}|\mathbf{u}|^2\right) + (\boldsymbol{\nabla}\wedge\mathbf{u})\wedge\mathbf{u}$;
(ii) $\boldsymbol{\nabla}\wedge(\mathbf{u}\wedge\mathbf{v}) = (\boldsymbol{\nabla}\cdot\mathbf{v})\mathbf{u} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{u} - (\boldsymbol{\nabla}\cdot\mathbf{u})\mathbf{v} - (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{v}$.
(iii) $\boldsymbol{\nabla}^2\mathbf{u} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) - \boldsymbol{\nabla}\wedge(\boldsymbol{\nabla}\wedge\mathbf{u})$.

(b) The faces of a tetrahedron lie in the planes $x_1 = 0, x_2 = 0, x_3 = 0$ and $\mathbf{n}\cdot\mathbf{x} = 1$, where $\mathbf{n} = (n_1, n_2, n_3)$ is a unit vector such that $n_j > 0$ for $j = 1, 2, 3$ and $\mathbf{x} = (x_1, x_2, x_3)$. Let A_j be the area of the face in the plane $x_j = 0$ and let A be the area of the face with unit normal $\mathbf{n}$. By applying the divergence theorem, show that $A_j = n_j A$.

Solution

(a) This part uses the summation convention (of summing over repeated indices in an expression) and the following identities for a differentiable scalar field $f(\mathbf{x})$, a differentiable vector field $\mathbf{G}(\mathbf{x}) = G_i(\mathbf{x})\mathbf{e}_i$ and vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$:

$$\boldsymbol{\nabla} f = \mathbf{e}_i \frac{\partial f}{\partial x_i} \tag{1}$$

$$\boldsymbol{\nabla}\cdot\mathbf{G} = \frac{\partial G_j}{\partial x_j} \tag{2}$$

$$\boldsymbol{\nabla}\wedge\mathbf{G} = \mathbf{e}_j \wedge \frac{\partial \mathbf{G}}{\partial x_j} \tag{3}$$

$$(\mathbf{u}\cdot\boldsymbol{\nabla}) f = u_k \frac{\partial f}{\partial x_k} \tag{4}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_l \partial x_l} \tag{5}$$

$$\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \tag{6}$$

(i) By (1), (3), (4) and (6),

$$\begin{aligned}
(\boldsymbol{\nabla}\wedge\mathbf{u})\wedge\mathbf{u} &= \left(\mathbf{e}_j \wedge \frac{\partial \mathbf{u}}{\partial x_j}\right)\wedge\mathbf{u} \\
&= (\mathbf{u}\cdot\mathbf{e}_j)\frac{\partial \mathbf{u}}{\partial x_j} - \left(\mathbf{u}\cdot\frac{\partial \mathbf{u}}{\partial x_j}\right)\mathbf{e}_j \\
&= u_j \frac{\partial \mathbf{u}}{\partial x_j} - \mathbf{e}_j \frac{\partial}{\partial x_j}\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) \\
&= (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla\left(\frac{1}{2}|\mathbf{u}|^2\right).
\end{aligned}$$

(ii) By (2), (3), (4) and (6),

$$
\begin{aligned}
\boldsymbol{\nabla} \wedge (\mathbf{u} \wedge \mathbf{v}) &= \mathbf{e}_j \wedge \frac{\partial}{\partial x_j} (\mathbf{u} \wedge \mathbf{v}) \\
&= \frac{\partial}{\partial x_j} \left(\mathbf{e}_j \wedge (\mathbf{u} \wedge \mathbf{v}) \right) \\
&= \frac{\partial}{\partial x_j} \left((\mathbf{e}_j \cdot \mathbf{v})\, \mathbf{u} - (\mathbf{e}_j \cdot \mathbf{u})\, \mathbf{v} \right) \\
&= \frac{\partial}{\partial x_j} \left(v_j \mathbf{u} - u_j \mathbf{v} \right) \\
&= v_j \frac{\partial \mathbf{u}}{\partial x_j} + \frac{\partial v_j}{\partial x_j} \mathbf{u} - u_j \frac{\partial \mathbf{v}}{\partial x_j} - \frac{\partial u_j}{\partial x_j} \mathbf{v} \\
&= (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{u} + (\boldsymbol{\nabla} \cdot \mathbf{v})\mathbf{u} - (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{v} - (\boldsymbol{\nabla} \cdot \mathbf{u})\mathbf{v}
\end{aligned}
$$

(iii) By (1), (2), (3), (5) and (6),

$$
\begin{aligned}
\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \wedge \mathbf{u}) &= \mathbf{e}_i \wedge \frac{\partial}{\partial x_i} \left(\mathbf{e}_j \wedge \frac{\partial \mathbf{u}}{\partial x_j} \right) \\
&= \frac{\partial^2}{\partial x_i \partial x_j} \left(\mathbf{e}_i \wedge (\mathbf{e}_j \wedge \mathbf{u}) \right) \\
&= \frac{\partial^2}{\partial x_i \partial x_j} \left((\mathbf{e}_i \cdot \mathbf{u})\, \mathbf{e}_j - (\mathbf{e}_i \cdot \mathbf{e}_j)\, \mathbf{u} \right) \\
&= \frac{\partial^2}{\partial x_i \partial x_j} \left(u_i \mathbf{e}_j - \delta_{ij} \mathbf{u} \right) \\
&= \mathbf{e}_j \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_i} \right) - \frac{\partial^2 \mathbf{u}}{\partial x_j \partial x_j} \\
&= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) - \nabla^2 \mathbf{u}
\end{aligned}
$$

where Kronecker's delta δ_{ij} is defined by

$$
\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}
$$

(b) The divergence theorem states that if the region V in $\mathbb{R}^3$ is bounded by a piecewise smooth surface ∂V with outward pointing unit normal $\mathbf{n} = \mathbf{e}_k n_k$ and $\mathbf{G}(\mathbf{x}) = \mathbf{e}_k G_k$ is a differentiable vector field on V, then

$$
\iiint_V \boldsymbol{\nabla} \cdot \mathbf{G} \ \mathrm{d}V = \iint_{\partial V} \mathbf{G} \cdot \mathbf{n} \ \mathrm{d}S \quad \text{or} \quad \iiint_V \frac{\partial G_k}{\partial x_k} \ \mathrm{d}V = \iint_{\partial V} G_k n_k \ \mathrm{d}S \tag{7}
$$

Let V be the tetrahedron and for clarity replace $\mathbf{n}$ in the question with $\hat{\mathbf{n}}$. The face with area A_i has outward unit normal $-\mathbf{e}_i$ and the (slanted) face with area A has outward unit normal $\hat{\mathbf{n}}$(since $\boldsymbol{\nabla}(\hat{\mathbf{n}} \cdot \mathbf{x} - 1) = \hat{\mathbf{n}}, |\hat{\mathbf{n}}| = 1$ and $\hat{\mathbf{n}}$ points out of the tetrahedron). Let $\mathbf{G} = \mathbf{e}_j$, so that $\boldsymbol{\nabla} \cdot \mathbf{G} = 0$. By the divergence theorem,

$$
\begin{aligned}
0 &= \iiint_V \boldsymbol{\nabla} \cdot \mathbf{G} \, \mathrm{d}V \\
&= \iint_{\partial V} \mathbf{G} \cdot \mathbf{n} \, \mathrm{d}S \\
&= \iint_{\partial V \cap \{\mathbf{x} \cdot \hat{\mathbf{n}} = 1\}} \mathbf{e}_j \cdot \hat{\mathbf{n}} \, \mathrm{d}S + \sum_{i=1}^{3} \iint_{\partial V \cap \{x_i = 0\}} \mathbf{e}_j \cdot (-\mathbf{e}_i) \, \mathrm{d}S \\
&= (\mathbf{e}_j \cdot \hat{\mathbf{n}}) \iint_{\partial V \cap \{\mathbf{x} \cdot \mathbf{n} = 1\}} \mathrm{d}S - \sum_{i=1}^{3} (\mathbf{e}_i \cdot \mathbf{e}_j) \iint_{\partial V \cap \{x_i = 0\}} \mathrm{d}S \\
&= \hat{n}_j A - \delta_{ij} A_i \\
&= \hat{n}_j A - A_j
\end{aligned}
$$

Q2 The convective derivative and Reynolds' transport theorem.

(a) Let $f(\mathbf{x}, t)$ be a differentiable function of position $\mathbf{x}$ and time t, defined in a fluid whose velocity is $\mathbf{u}(\mathbf{x}, t)$. Show that the rate of change of f following a material fluid element is given by

$$
\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + (\mathbf{u} \cdot \nabla) f.
$$

(b) Let $V(t)$ be a time-dependent closed region of $\mathbb{R}^3$ that is convected with velocity $\mathbf{u}(\mathbf{x}, t)$. Prove Reynolds' Transport Theorem, namely

$$
\frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V(t)} f \, \mathrm{d}V = \iiint_{V(t)} \frac{\partial f}{\partial t} + \nabla \cdot (f\mathbf{u}) \mathrm{d}V
$$

for any continuously differentiable function $f(\mathbf{x}, t)$.

Solution

We choose the label $\mathbf{X}$ to be the initial position of a fluid particle at time $t = 0$ (say), and denote by $\mathbf{x}(\mathbf{X}, t)$ its position at time $t \geq 0$, thus

$$
\mathbf{x}(\mathbf{X}, 0) = \mathbf{X}, \quad \left. \frac{\partial}{\partial t} \right|_{\mathbf{X}} \mathbf{x}(\mathbf{X}, t) = \mathbf{u}(\mathbf{x}(\mathbf{X}, t), t).
$$

(a) The convective derivative

$$
\frac{\mathrm{D}}{\mathrm{D}t} \equiv \left. \frac{\partial}{\partial t} \right|_{\mathbf{X}}
$$

is related to the Eularian time derivative

$$
\frac{\partial}{\partial t} \equiv \left. \frac{\partial}{\partial t} \right|_{\mathbf{x}}
$$

using the chain rule. The time rate of change of a differentiable scalar field $f(\mathbf{x}, t)$ following the fluid is the convective derivative

$$
\begin{aligned}
\frac{\mathrm{D}f}{\mathrm{D}t} &= \left.\frac{\partial}{\partial t}\right|_{\mathbf{X}} f(\mathbf{x}(\mathbf{X},t),t) \\
&= \left.\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t}\right|_{\mathbf{X}} + \left.\frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t}\right|_{\mathbf{X}} + \left.\frac{\partial f}{\partial x_3}\frac{\partial x_3}{\partial t}\right|_{\mathbf{X}} + \frac{\partial f}{\partial t} \\
&= \frac{\partial f}{\partial t} + \left.\frac{\partial \mathbf{x}}{\partial t}\right|_{\mathbf{X}} \cdot \nabla f \\
&= \frac{\partial f}{\partial t} + (\mathbf{u}\cdot\nabla) f \\
&= \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right) f
\end{aligned}
$$

(b) The Jacobian $J(\mathbf{X},t)$ is defined by

$$
J(\mathbf{X},t) = \frac{\partial\left(x_1,x_2,x_3\right)}{\partial\left(X_1,X_2,X_3\right)} = \varepsilon_{ijk}\frac{\partial x_1}{\partial X_i}\frac{\partial x_2}{\partial X_j}\frac{\partial x_3}{\partial X_k},
$$

where the Levi-Civita symbol ε_{ijk} is defined by

$$
\varepsilon_{ijk} = \begin{cases} 1 \text{ if } i,j,k \text{ in cyclic order }, \\ -1 \text{ if } i,j,k \text{ in acyclic order }, \\ 0 \text{ otherwise.} \end{cases}
$$

Hence, the rate of change of J following the fluid is given by

$$
\begin{aligned}
\frac{\mathrm{DJ}}{\mathrm{D}t} &= \frac{\mathrm{D}}{\mathrm{D}t}\left(\varepsilon_{ijk}\frac{\partial x_1}{\partial X_i}\frac{\partial x_2}{\partial X_j}\frac{\partial x_3}{\partial X_k}\right) \\
&= \varepsilon_{ijk}\left(\frac{\partial \dot{x}_1}{\partial X_i}\frac{\partial x_2}{\partial X_j}\frac{\partial x_3}{\partial X_k} + \frac{\partial x_1}{\partial X_i}\frac{\partial \dot{x}_2}{\partial X_j}\frac{\partial x_3}{\partial X_k} + \frac{\partial x_1}{\partial X_i}\frac{\partial x_2}{\partial X_j}\frac{\partial \dot{x}_3}{\partial X_k}\right) \\
&= \varepsilon_{ijk}\left(\frac{\partial u_1}{\partial x_m}\frac{\partial x_m}{\partial X_i}\frac{\partial x_2}{\partial X_j}\frac{\partial x_3}{\partial X_k} + \frac{\partial x_1}{\partial X_i}\frac{\partial u_2}{\partial x_m}\frac{\partial x_m}{\partial X_j}\frac{\partial x_3}{\partial X_k} + \frac{\partial x_1}{\partial X_i}\frac{\partial x_2}{\partial X_j}\frac{\partial u_3}{\partial x_m}\frac{\partial x_m}{\partial X_k}\right) \\
&= \frac{\partial u_1}{\partial x_m}\frac{\partial\left(x_m,x_2,x_3\right)}{\partial\left(X_1,X_2,X_3\right)} + \frac{\partial u_2}{\partial x_m}\frac{\partial\left(x_1,x_m,x_3\right)}{\partial\left(X_1,X_2,X_3\right)} + \frac{\partial u_3}{\partial x_m}\frac{\partial\left(x_1,x_2,x_m\right)}{\partial\left(X_1,X_2,X_3\right)} \\
&= \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)\frac{\partial\left(x_1,x_2,x_3\right)}{\partial\left(X_1,X_2,X_3\right)}
\end{aligned}
$$

where in the second line we set $\dot{} = \mathrm{D}/\mathrm{D}t$ and used the product rule; in the third line we used $\dot{x}_n = u_n$ and the chain rule; and in the fifth line we used the fact a determinant is zero if it has repeated rows. Thus, we have derived Euler's identity

$$
\frac{\mathrm{D}J}{\mathrm{D}t} = J\nabla\cdot\mathbf{u}.
$$

Assuming the map from $\mathbf{X}$ to $\mathbf{x}(\mathbf{X},t)$ is invertible and continuous, $J(\mathbf{X},t)$ is positive and bounded. We can therefore transform from Eularian coordinates $\mathbf{x}$ to Lagrangian coordinates $\mathbf{X}$ in order to "differentiate under the integral sign: For a continuously differentiable scalar field $f(\mathbf{x},t)$[1]

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} f(\mathbf{x},t)\mathrm{d}x_1\ \mathrm{d}x_2\ \mathrm{d}x_3 &= \iiint_{V(0)} f(\mathbf{x}(\mathbf{X},t),t)J(\mathbf{X},t)\mathrm{d}X_1\ \mathrm{d}X_2\ \mathrm{d}X_3 \\
&= \iiint_{V(0)} \left.\frac{\partial}{\partial t}\right|_{\mathbf{X}} (fJ)\mathrm{d}X_1\ \mathrm{d}X_2\ \mathrm{d}X_3 \\
&= \iiint_{V(0)} \frac{\mathrm{D}}{\mathrm{D}t}(fJ)\mathrm{d}X_1\ \mathrm{d}X_2\ \mathrm{d}X_3 \\
&= \iiint_{V(0)} \left(\frac{\mathrm{D}f}{\mathrm{D}t}J + f\frac{\mathrm{D}J}{\mathrm{D}t}\right)\mathrm{d}X_1\ \mathrm{d}X_2\ \mathrm{d}X_3 \\
&= \iiint_{V(0)} \left(\frac{\mathrm{D}f}{\mathrm{D}t} + f\nabla\cdot\mathbf{u}\right)J\ \mathrm{d}X_1\ \mathrm{d}X_2\ \mathrm{d}X_3 \\
&= \iiint_{V(t)} \left(\frac{\partial f}{\partial t} + \nabla\cdot(f\mathbf{u})\right)\mathrm{d}x_1\ \mathrm{d}x_2\ \mathrm{d}x_3 \qquad (8)
\end{aligned}
$$

where on the fourth line we used Euler's identity and on the last line we set

$$
\frac{\mathrm{D}f}{\mathrm{D}t} + f\boldsymbol{\nabla}\cdot\mathbf{u} = \frac{\partial f}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})f + f\boldsymbol{\nabla}\cdot\mathbf{u} = \frac{\partial f}{\partial t} + \boldsymbol{\nabla}\cdot(f\mathbf{u})
$$

while transforming back to Eulerian coordinates.

Q3 The continuity equation and incompressibility.

(a) By applying the principle of conservation of mass to a material volume $V(t)$, derive the continuity equation

$$
\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0
$$

for a compressible fluid with density ρ, stating any assumptions that you make about the smoothness of ρ and u.

(b) When the fluid is incompressible, i.e. $\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0$, derive the incompressibility condition

$$
\nabla\cdot\mathbf{u} = 0
$$

and deduce that the transport theorem for an incompressible fluid may be written in the form

$$
\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} f\ \mathrm{d}V = \iiint_{V(t)} \frac{\mathrm{D}f}{\mathrm{D}t}\ \mathrm{d}V
$$

for any continuously differentiable function $f(\mathbf{x},t)$.

Solution

(a) Since a material volume $V(t)$ always consists of the same fluid particles, its mass must be preserved, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} \rho \mathrm{d}V = 0$$

Assume $\rho \in C^1$ (i.e. ρ is continuously differentiable) and apply Reynolds Transport Theorem (8) with $f = \rho$ to obtain

$$\iint_{V(t)} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \mathrm{d}V = 0.$$

We assume that ρ and $\mathbf{u}$ are sufficiently smooth that the integrand is continuous (e.g. $\rho, \mathbf{u} \in C^1$ will do). Then we may use the fact the volume $V(t)$ is arbitrary to deduce that the integrand is zero, i.e. the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{9}$$

(b) By (9) and the product rule,

$$0 = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = \frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla)\rho + \rho \nabla \cdot \mathbf{u} = \frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \mathbf{u}.$$

Hence, for an incompressible flow,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \quad \Leftrightarrow \quad \nabla \cdot \mathbf{u} = 0,$$

in which case

$$\frac{\partial f}{\partial t} + \boldsymbol{\nabla} \cdot (f \mathbf{u}) = \frac{\mathrm{D}f}{\mathrm{D}t} + f \boldsymbol{\nabla} \cdot \mathbf{u} = \frac{\mathrm{D}f}{\mathrm{D}t},$$

so that Reynolds Transport Theorem (8) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} f(\mathbf{x}, t) \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 = \iiint_{V(t)} \frac{\mathrm{D}f}{\mathrm{D}t} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3.$$

Q4 Derivation of the incompressible Navier-Stokes equations.

(a) Define the stress tensor σ_{ij} and show that the force per unit area exerted on a surface element with unit normal $\mathbf{n} = n_j \mathbf{e}_j$ by the fluid towards which $\mathbf{n}$ is directed is given by the stress vector

$$\mathbf{t} = \mathbf{e}_i \sigma_{ij} n_j,$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are unit vectors along the axes Ox_1, Ox_2, Ox_3.

(b) Newton's second law for a material volume $V(t)$ with boundary $\partial V(t)$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} \rho \mathbf{u} \mathrm{d}V = \iint_{\partial V(t)} \mathbf{t} \mathrm{d}S + \iiint_{V(t)} \rho \mathbf{F} \mathrm{d}V,$$

where ρ is the density, $\mathbf{u} = u_i\mathbf{e}_i$ is the velocity and $\mathbf{F} = F_i\mathbf{e}_i$ is an external body force acting per unit mass. Explain the physical significance of each term in this expression.

(c) Use Reynolds' transport theorem and the divergence theorem to derive Cauchy's momentum equation in the form

$$\rho\frac{\mathrm{D}u_i}{\mathrm{D}t} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho F_i.$$

(d) Define the rate-of-strain tensor e_{ij}. State the physical assumptions that are needed for an incompressible fluid to be Newtonian, that is

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij},$$

where p is the pressure and μ is the viscosity.

(e) For an incompressible, constant viscosity, Newtonian fluid, deduce the Navier-Stokes equations in the form

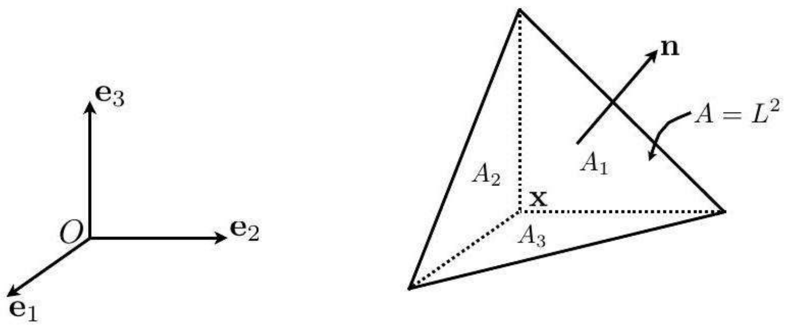
$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u} + \rho\mathbf{F}, \quad \nabla\cdot\mathbf{u} = 0. \tag{10}$$

Solution

(a) The stress tensor $\sigma_{ij}(\mathbf{x}, t)$ is the component of stress (i.e. force per unit area) in the x_i-direction exerted on a surface element with normal in the x_j-direction by the fluid toward which $\mathbf{e}_j$ points. The stress vector $\mathbf{t}(\mathbf{x}, \mathrm{t}, \mathbf{n})$ is the stress exerted on a surface element by the fluid toward which its unit normal $\mathbf{n}$ points. Hence,

$$\sigma_{ij}(\mathbf{x}, t) = \mathbf{e}_i\cdot\mathbf{t}\left(\mathbf{x}, t, \mathbf{e}_j\right) \quad \Leftrightarrow \quad \mathbf{t}\left(\mathbf{x}, t, \mathbf{e}_j\right) = \mathbf{e}_i\sigma_{ij}(\mathbf{x}, t). \tag{11}$$

Consider a material volume $V(t)$ having at time t the configuration of a small tetrahedron as shown.



Let the slanting face have area $A = L^2$ and outward unit normal $\mathbf{n} = \mathbf{e}_j n_j$, with $n_j > 0$.

Begin with conservation of momentum for $V(t)$ from part (c) in the form

$$\iiint_{V(t)} \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} - \rho\mathbf{F}\mathrm{d}V = \iint_{\partial V(t)} \mathbf{t}\mathrm{d}S. \tag{12}$$

Assuming the integrand is continuous, the integral mean value theorem implies that

$$\iiint_{V(t)} \rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} - \rho \mathbf{F} \mathrm{d}V = \mathrm{O}\left(L^3\right) \quad \text{as} \quad L \to 0. \tag{13}$$

Since the face with area $A_j = n_j A = n_j L^2$ (by Q1(b)) has outward unit normal $-\mathbf{e}_j$ and the (slanted) face with area $A = L^2$ has outward unit normal $\mathbf{n}$,

$$\iint_{\partial V(t)} \mathbf{t} \mathrm{d}S = \left(\mathbf{t}(\mathbf{x}, t, \mathbf{n}) + \mathbf{t}\left(\mathbf{x}, t, -\mathbf{e}_j\right) n_j\right) L^2 + \mathrm{O}\left(L^3\right) \quad \text{as} \quad L \to 0. \tag{14}$$

By Newton's third law, $\mathbf{t}\left(\mathbf{x}, t, -\mathbf{e}_j\right) = -\mathbf{t}\left(\mathbf{x}, t, \mathbf{e}_j\right)$, so we we deduce from (12)-(14) that

$$\left(\mathbf{t}(\mathbf{x}, t, \mathbf{n}) - \mathbf{t}\left(\mathbf{x}, t, \mathbf{e}_j\right) n_j\right) L^2 = \mathrm{O}\left(L^3\right) \quad \text{as} \quad L \to 0$$

This expression pertains for arbitrarily small L, so there is a local equilibrium of the surface stresses, with

$$\mathbf{t}(\mathbf{x}, t, \mathbf{n}) = \mathbf{t}\left(\mathbf{x}, t, \mathbf{e}_j\right) n_j = \mathbf{e}_i \sigma_{ij}(\mathbf{x}, t) n_j \tag{15}$$

by (11). Note that the argument may be readily generalized to an arbitrarily oriented unit normal $\mathbf{n}$.

Since the the face with area $A_j = n_j A = n_j L^2$ (by Q1(b)) has outward unit normal - $\mathbf{e}_j$, the (slanted) face with area $A = L^2$ has outward unit normal $\mathbf{n}$ and by Newton's third law $\mathbf{t}\left(-\mathbf{e}_j\right) = -\mathbf{t}\left(\mathbf{e}_j\right)$, the net surface force on the tetrahedron is given by

$$\mathbf{t}(\mathbf{n}) A + \mathbf{t}\left(-\mathbf{e}_j\right) A_j = \left(\mathbf{t}(\mathbf{n}) - \mathbf{t}\left(\mathbf{e}_j\right) n_j\right) L^2$$

to a first approximation as $L \to 0$, with all dependent variables (i.e. $\mathbf{t}$ here) evaluated at $\mathbf{x}$ at time t here and hereafter. To a first approximation Newton's second law for the small material tetrahedron is given by

$$\underbrace{\rho \frac{\mathrm{Du}}{\mathrm{D}t} \delta V}_{\text{Mass} \times \text{acceleration}} = \underbrace{\left(\mathbf{t}(\mathbf{n}) - \mathbf{t}\left(\mathbf{e}_j\right) n_j\right) L^2}_{\text{Net surface force}} + \underbrace{\rho \mathbf{F} \delta V}_{\text{Net body force}}$$

where δV is the volume of the tetrahedron. Since $\delta V = \mathrm{O}\left(L^3\right)$ as $L \to 0$, we deduce that

$$\mathbf{t}(\mathbf{n}) = \mathbf{t}\left(\mathbf{e}_j\right) n_j = \mathbf{e}_i \sigma_{ij} n_j$$

by (11), provided the acceleration and body force are finite. Note that the argument may be readily generalized to an arbitrarily oriented unit normal $\mathbf{n}$.

(b) Conservation of momentum for a material volume $V(t)$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V(t)} \rho \mathbf{u} \mathrm{d}V = \iint_{\partial V(t)} \mathbf{t} \mathrm{d}S + \iiint_{V(t)} \rho \mathbf{F} \mathrm{d}V \tag{16}$$

The term on the left is the time rate of change of the linear momentum of $V(t)$. The first term on the RHS is the net surface force exerted by fluid outside $V(t)$ on fluid inside $V(t)$ via $\partial V(t)$. The second term on the RHS is the net body force exerted on $V(t)$.

(c) By Reynolds Transport Theorem (8) with $f = \rho u_i$ (assuming $\rho, u_i \in C^1$) and the continuity equation (9), we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} \rho u_i \,\mathrm{d}V &= \iiint_{V(t)} \frac{\partial}{\partial t}(\rho u_i) + \boldsymbol{\nabla}\cdot(\rho u_i \mathbf{u})\,\mathrm{d}V \\
&= \iiint_{V(t)} u_i\left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u})\right) + \rho\left(\frac{\partial u_i}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})u_i\right)\mathrm{d}V \\
&= \iiint_{V(t)} \rho\frac{\mathrm{D}u_i}{\mathrm{D}t}\,\mathrm{d}V
\end{aligned}$$

By Cauchy's stress theorem (5) and the divergence theorem,

$$\iint_{\partial V(t)} \mathbf{t}(\mathbf{n})\mathrm{d}S = \mathbf{e}_i \iint_{\partial V(t)} \sigma_{ij} n_j \,\mathrm{d}S = \mathbf{e}_i \iiint_{V(t)} \frac{\partial \sigma_{ij}}{\partial x_j}\,\mathrm{d}V.$$

Hence, the x_i-component of (16) may be written in the form

$$\iiint_{V(t)} \rho\frac{\mathrm{D}u_i}{\mathrm{D}t} - \frac{\partial \sigma_{ij}}{\partial x_j} - \rho F_i \,\mathrm{d}V = 0.$$

Since $V(t)$ is arbitrary, the integrand must be zero (if it is continuous), and we deduce Cauchy's momentum equation in the form

$$\rho\frac{\mathrm{D}u_i}{\mathrm{D}t} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho F_i. \tag{17}$$

(d) The rate-of-strain tensor e_{ij} is given by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Let $\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$, where p is the pressure and τ_{ij} is the deviatoric stress tensor, due to the presence of viscosity. The two physical assumptions defining a Newtonian fluid are
(1) τ_{ij} is a linear function of the velocity gradients $\partial u_\alpha/\partial x_\beta$;
(2) the relation between τ_{ij} and the velocity gradients is isotropic, i.e. invariant to rotations of the coordinate axes (so that there is no preferred direction).
Together with symmetry of the stress tensor, these conditions are sufficient to determine the form of τ_{ij} completely (outline proof in lecture notes not examinable):

$$\tau_{ij} = \lambda(\boldsymbol{\nabla}\cdot\mathbf{u})\delta_{ij} + 2\mu e_{ij},$$

where λ is the bulk viscosity and μ is the dynamic (shear) viscosity. Hence, for an incompressible flow, with $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$, the Newtonian constitutive law is given by

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} = -p\delta_{ij} + 2\mu e_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

(e) If μ is constant, then

$$
\begin{aligned}
\frac{\partial \sigma_{ij}}{\partial x_j} &= \frac{\partial}{\partial x_j}\left(-p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right) \\
&= -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \mu\frac{\partial^2 u_j}{\partial x_j \partial x_i} \\
&= -\frac{\partial p}{\partial x_i} + \mu\nabla^2 u_i + \mu\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) \\
&= -\frac{\partial p}{\partial x_i} + \mu\nabla^2 u_i + \mu\frac{\partial}{\partial x_j}(\nabla \cdot \mathbf{u}) \\
&= -\frac{\partial p}{\partial x_i} + \mu\nabla^2 u_i
\end{aligned} \tag{18}
$$

since $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ for an incompressible flow. By (17) and (18), the incompressible Navier-Stokes equations are given in component form by

$$\rho\left(\frac{\partial u_i}{\partial t} + (\mathbf{u}\cdot\nabla)u_i\right) = -\frac{\partial p}{\partial x_i} + \mu\nabla^2 u_i + \rho F_i, \quad \nabla \cdot \mathbf{u} = 0,$$

and in vector form by

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u} + \rho\mathbf{F}, \quad \nabla \cdot \mathbf{u} = 0. \tag{19}$$

Q5 Symmetry of the Cauchy stress tensor.

For the Cauchy stress tensor, deduce that $\sigma_{ij} = \sigma_{ji}$.

Solution

For a material volume $V(t)$, conservation of angular momentum about the origin O is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} \mathbf{x} \wedge \rho\mathbf{u}\,\mathrm{d}V = \iint_{\partial V(t)} \mathbf{x} \wedge \mathbf{t}\,\mathrm{d}S + \iiint_{V(t)} \mathbf{x} \wedge \rho\mathbf{F}\,\mathrm{d}V.$$

Use Reynolds transport theorem and the divergence theorem to write this expression in the form

$$\iiint_{V(t)} \mathbf{x} \wedge \left(\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} - \mathbf{e}_i\frac{\partial \sigma_{ij}}{\partial x_j} - \rho\mathbf{F}\right)\mathrm{d}V = \iiint_{V(t)} \mathbf{e}_i \wedge \mathbf{e}_j\,\sigma_{ij}\,\mathrm{d}V.$$

We can then deduce

Dotting this with e_k gives the kth component of the momentum conservation equation, which is zero. Hence this zero.

$$\iiint_{V(t)} \mathbf{e}_i \wedge \mathbf{e}_j\sigma_{ij}\,\mathrm{d}V = \mathbf{0}.$$

Since $V(t)$ is arbitrary, the integrand must be zero (if it is continuous), *i.e.*

$$\mathbf{0} = \mathbf{e}_i \wedge \mathbf{e}_j\,\sigma_{ij} = \mathbf{e}_1(\sigma_{23} - \sigma_{32}) + \mathbf{e}_2(\sigma_{31} - \sigma_{12}) + \mathbf{e}_3(\sigma_{12} - \sigma_{21}),$$

which implies that the stress tensor is symmetric, *i.e.*

$$\sigma_{ij} = \sigma_{ji}.$$