C4.1 Further Functional Analysis

Sheet 1 — MT 2025

For classes in week 3 or 4

This example sheet is based on the material in sections 2, 3 and 4 of the notes, together with Appendix A.

Section A

1. Let X be a Banach space and Y a normed space. Let $T \in \mathcal{B}(X,Y)$ be such that there exists $\delta > 0$ such that $\|Tx\| \geq \delta\|x\|$ for all $x \in X$. Show that $\operatorname{Ran}(T)$ is complete, and hence closed.

   Solution: Suppose $(y_n)_n$ is a Cauchy sequence in $\operatorname{Ran}(T)$, write $y_n = Tx_n$, so that $\|x_n - x_m\| \leq \delta^{-1}\|y_n - y_m\| \to 0$. Therefore (x_n) is Cauchy, and so converges say to $x \in X$. Therefore by continuity of T, $y_n = Tx_n \to Tx \in \operatorname{Ran}(T)$. Thus $\operatorname{Ran}(T)$ is complete, and hence closed.[1]

2. Let X be a vector space and suppose that Y is a subspace of X.

   (a) By extending a Hamel basis for Y to X, construct a linear map $P\colon X \to X$ such that $P^2 = P$ and $\operatorname{Ran} P = Y$.

   (b) Deduce that Y is *algebraically complemented* in X, which is to say that there exists a further subspace Z of X such that every $x \in X$ can be expressed uniquely as $x = y + z$ with $y \in Y$ and $z \in Z$.

   (c) Is the subspace Z in part (b) uniquely determined by Y?

   [One can achieve the main point of this question: subspaces are algebraically complemented directly, by the same Hamel basis extension argument hinted at in (a).]

   Solution:

   (a) Let B be a Hamel basis for Y and B' a Hamel basis for X such that $B \subseteq B'$. Define P by setting $Px = x$ for all $x \in B$ and $Px = 0$ for all $x \in B' \setminus B$ and extending linearly. Then P has the required properties.

[1] Every complete subspace of a metric space is closed. Indeed, if $y_n \to y \in Y$ with $y_n \in \operatorname{Ran}(T)$, then (y_n) is Cauchy, so converges to some $z \in \operatorname{Ran}(T)$. By uniqueness of limits $y = z \in \operatorname{Ran}(T)$. But these details would not be needed in a part C answer.

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(b) Since $P^2 = P$ we know that $X = \operatorname{Ran} P \oplus \operatorname{Ker} P$, so we may take $Z = \operatorname{Ker} P$.

(c) No. Take for instance $X = \mathbb{F}^2$ and $Y = \operatorname{Span}\{(1,0)\}$. Let $Z_1 = \operatorname{Span}\{(0,1)\}$ and $Z_2 = \operatorname{Span}\{(1,1)\}$. Then $X = Y \oplus Z_k$ for $k = 1, 2$ but $Z_1 \neq Z_2$.

3. Let X be a vector space on which two norms $\|\cdot\|, |||\cdot|||$ are defined, and suppose that $\|x\| \leq C|||x|||$ for some constant $C > 0$ and all $x \in X$.

   (a) Suppose that X is complete with respect to one of these two norms. Show that it is complete with respect to the other if and only if the two norms are equivalent.

   (b) Give an example in which $(X, |||\cdot|||)$ is complete but $(X, \|\cdot\|)$ is not.

   [See Question B.1 for examples with $(X, \|\cdot\|)$ complete but $(X, |||\cdot|||)$ is not.]

   Solution:

   (a) If the norms are equivalent, and one norm is complete, then we can apply Question 1 to the identity map $(X, \|\cdot\|) \to (X, |||\cdot|||)$ (or the otherway round) to learn that the other norm is complete.

   For the converse we need to show that if both norms are complete then they are equivalent. This follows from the inverse mapping theorem applied to $I : (X, |||\cdot|||) \to (X, \|\cdot\|)$.

   (b) Let $X = \ell^1$ and let $\|\cdot\| = \|\cdot\|_\infty$ and $|||\cdot||| = \|\cdot\|_1$. Then $\|x\| \leq |||x|||$ for all $x \in X$ and $(X, |||\cdot|||)$ is complete but $(X, \|\cdot\|)$ is not.

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Section B

1. Let X be an infinite-dimensional normed space, and suppose that $\{x_\alpha : \alpha \in A\}$ is a Hamel basis for X and that $\|x_\alpha\| = 1$ for all $\alpha \in A$. Given a vector $x \in X$ which has the expansion $x = \sum_{\alpha\in A} \lambda_\alpha x_\alpha$ we let

$$|||x||| = \sum_{\alpha\in A} |\lambda_\alpha|.$$

   (a) Check that $|||\cdot|||$ defines a norm on X.

   (b) Now let X be a Banach space. Show that $(X, |||\cdot|||)$ is not separable.

   (c) Deduce that in the Closed Graph Theorem the assumption that the codomain be complete cannot be omitted.

   [Note that this provides loads of examples of Banach spaces X with norms $\|\cdot\|$ and $|||\cdot|||$ as in Question 3, in which $(X, \|\cdot\|)$ is complete but $(X, |||\cdot|||)$ is not.]

2. Let X be an infinite-dimensional Banach space with norm $\|\cdot\|$, and let $f\colon X \to \mathbb{F}$ be an unbounded linear functional. Given a vector $x_0 \in X$ such that $f(x_0) = 1$, consider the linear operator $T\colon X \to X$ defined by

$$Tx = x - 2f(x)x_0, \quad x \in X.$$

   Show that $T^2 = I$. Hence show that the map $|||\cdot|||\colon X \to [0,\infty)$ given, for $x \in X$, by $|||x||| = \|Tx\|$ defines a complete norm on X which is not equivalent to $\|\cdot\|$.

3. (a) Let X, Y and Z be vector spaces and suppose that $T\colon X \to Y$ and $S\colon X \to Z$ are linear maps. Show that there exists a linear map $\pi\colon Z \to Y$ such that $T = \pi \circ S$ if and only if $\operatorname{Ker} S \subseteq \operatorname{Ker} T$.

   (b) Hence or otherwise show that if $n \in \mathbb{N}$ and if $f_1, \dots, f_n$ and f are linear functionals on a vector space X, then $f \in \operatorname{Span}\{f_1, \dots, f_n\}$ if and only if

$$\bigcap_{k=1}^{n} \operatorname{Ker} f_k \subseteq \operatorname{Ker} f.$$

   (c) Let X be a normed space and let F be a finite dimensional subspace of X^*. Write $F_\circ$ for the preanhilator of F, i.e. $F_\circ = \{x \in X : f(x) = 0,\ f \in F\}$ and $(F_\circ)^\circ$ for the anhilator of $F_\circ$, i.e. $(F_\circ)^\circ = \{g \in X^* : g(x) = 0,\ x \in F_\circ\}$. Show directly that $F = (F_\circ)^\circ$.

   [We will use part (b) of this question a lot later in the course, so make sure you keep this result handy.]

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4. Let $Y, Z \subseteq \ell^2$ be given by

$$Y = \{(y_n) \in \ell^2 : y_{2n} = 0 \text{ for all } n \geq 1\},$$
$$Z = \{(z_n) \in \ell^2 : z_{2n-1} = nz_{2n} \text{ for all } n \geq 1\}.$$

   (a) Show that Y and Z are closed subspaces of ℓ^2 and that $Y \cap Z = \{0\}$.

   (b) Letting $X = Y \oplus Z$ denote the algebraic direct sum of Y and Z, prove that X is dense in ℓ^2 but that $X \neq \ell^2$, and deduce that X is not the topological direct sum of Y and Z.

   (c) Let $P\colon X \to X$ be the linear map given by $P(y + z) = y$ for all $y \in Y$, $z \in Z$. Show directly that P is unbounded.

5. Let X be a normed vector space and let Y and Z be subspaces of X such that $X = Y \oplus Z$ algebraically. Show that if Y is closed, then X is the topological direct sum of Y and Z if and only if the restriction $\pi|_Z\colon Z \to X/Y$ of the canonical quotient map $\pi\colon X \to X/Y$ is an isomorphism.

6. Let Y and Z be closed subspaces of a Banach space X with $Y \cap Z = \{0\}$. Equip the algebraic direct sum $Y \oplus Z$ with the ℓ^1-norm: $|||y + z||| = \|y\| + \|z\|$.

   (a) Show that $|||\cdot|||$ is complete on $Y \oplus Z$.

   (b) Show that the following are equivalent:

      (i) $|||\cdot|||$ is equivalent to the original norm on $Y \oplus Z$ (as a subspace of X);

      (ii) $Y \oplus Z$ is closed in X;

      (iii) Y is complemented by Z in $Y + Z$ (so $Y \oplus Z$ is a topological direct sum).

7. Let X and Y be Banach spaces and suppose that $T \in \mathcal{B}(X, Y)$ is such that $\operatorname{Ran} T$ has finite codimension in Y. Show that $\operatorname{Ran} T$ is closed.

   [Recall that $\operatorname{Ran} T$ being of finite co-dimensional means that $Y/\operatorname{Ran} T$ is finite dimensional. Start by checking that $\operatorname{Ran} T$ is algebraically complemented in Y by a finite dimensional subspace Z. You may then want to consider a map $X/\operatorname{Ker} T \to Y/Z$ where Z is a finite dimensional subspace which complements $\operatorname{Ran} T$.]

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[2] This exercise came from Fabian, where it didn't ask for Y to be complemented by Z, just complemented. This doesn't work, as in general a bounded projection $Y \oplus Z \to Y$ does not have to be of the form $y + z \mapsto y$. As a counter-example, see Q4: since Y is complemented in ℓ^2, it is complemented in any subspace; but the Z of that question is not a complement.

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Section C

The extensional material in Section C for this sheet looks at the interplay between the fundamental applications of Baire's category theorem in functional analysis. Most of the questions below are probably no harder than Section B, but they are a little off the main topic of the course, though as we will use these applications of Baire category theorem it certainly does not harm to know how they fit together.

1. Prove that the Closed Graph Theorem, the Inverse Mapping Theorem and the Open Mapping Theorem are all equivalent.

   [We saw that OMT⇒IMT⇒CGT in Appendix A, and potentially in your earlier courses. You should avoid using any form of the axiom of choice (or even a countable version of the axiom of choice). Countable choice plays a role in the proof of all of these theorems, in that it is needed for the Baire category theorem.]

   Solution: Throughout let X and Y be Banach spaces and let $T\colon X \to Y$ be linear. We show first that OMT $\iff$ IMT and then that IMT $\iff$ CGT. Assume the OMT and suppose that $T \in \mathcal{B}(X, Y)$ is a bijection. If $U \subseteq X$ is open then so is $T(U)$ and hence T^{-1} is continuous, so T is an isomorphism. Thus OMT $\Longrightarrow$ IMT. Now assume the IMT and suppose that T is surjective. Then the operator $T_0\colon X/\operatorname{Ker} T \to Y$ defined by $T_0(x + \operatorname{Ker} T) = Tx$, $x \in X$, is a continuous linear bijection between two Banach spaces and hence is an isomorphism. If $\pi\colon X \to X/\operatorname{Ker} T$ denotes the canonical quotient operator then for every open set $U \subseteq X$ we have $T(U) = T_0(\pi(U))$. But π is an open map, so $\pi(U)$ is open and hence so is $T(U)$, so T is an open map. Thus IMT $\Longrightarrow$ OMT. Next assume the IMT and endow $X \times Y$ with any of the p-norms, $1 \le p \le \infty$. If T is continuous then G_T is closed in $X \times Y$ by an elementary argument. Conversely, suppose that G_T is closed in $X \times Y$. Since $X \times Y$ is complete so is G_T. Consider the linear map $S\colon G_T \to X$ given by $S(x, Tx) = x$, $x \in X$. Then S is a continuous linear bijection between two Banach spaces and hence by the IMT the operator $S^{-1}\colon X \to G_T$ is also continuous. Hence T is bounded, so IMT $\Longrightarrow$ CGT. Now assume the CGT. If $T \in \mathcal{B}(X, Y)$ then G_T is closed. If T is a bijection then the graph of the algebraic inverse T^{-1} satisfies

   $$G_{T^{-1}} = \{(y, T^{-1}y) : y \in Y\} = \{(y, x) : (x, y) \in G_T\},$$

   which is closed because the map $(x, y) \mapsto (y, x)$ maps $X \times Y$ homeomorphically onto $Y \times X$. Hence T^{-1} is bounded, so CGT $\Longrightarrow$ IMT.

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2. Show, in the spirit of Question 1, that the principle of uniform boundedness is also equivalent to the open mapping theorem, inverse mapping theorem and closed graph theorem.

Solution: Assume the CGT. Suppose that $\mathcal{F}$ is a family of bounded linear operators $X \to Y$ such that $\sup_{T\in\mathcal{F}} \|T(x)\| < \infty$ for all $x \in X$. Form the Banch space $\ell^\infty(\mathcal{F}, Y) = \{(y_T)_{T\in\mathcal{F}} : \sup_{T\in\mathcal{F}} \|y_T\| < \infty\}$ with the norm $\|(y_T)_{T\in\mathcal{F}}\| = \sup_{T\in\mathcal{F}} \|y_T\|$. Now define a map $\Phi : X \to \ell^\infty(\mathcal{F}, Y)$ given by $\Phi(x) = (T(x))_{T\in\mathcal{F}}$. The hypothesis on $\mathcal{F}$ ensures that Φ does map into $\ell^\infty(\mathcal{F}, Y)$ and it is clearly linear. Suppose $x_n \to x$ in X and $\Phi(x_n) \to (y_T)_{T\in\mathcal{F}} \in \ell^\infty(\mathcal{F}, Y)$. Then, for each $T \in \mathcal{F}$, $T(x_n) \to y_T$. But as T is bounded, $T(x_n) \to T(x)$, so $y_T = T(x)$, i.e. $(y_T)_{T\in\mathcal{F}} = \Phi(x)$. Thus Φ has closed graph, so is bounded. Then $\|\Phi\| = \sup_{\|x\|\le 1} \sup_{T\in\mathcal{F}} \|T(x)\| = \sup_{T\in\mathcal{F}} \|T\|$, so $\sup_{T\in\mathcal{F}} \|T\| < \infty$.

Now assume the principle of uniform boundedness and let $T : X \to Y$ be a surjective map between Banach spaces. For each n define a new norm $\|y\|_n = \inf\{\|x\| + n\|z\| : x \in X,\ z \in Y,\ Tx + z = y\}$. It is easily checked that this is a norm (and we do not need to worry about whether it is complete) and let $Z = \{(y_n)_{n=1}^\infty : y_n \in Y,\ \sup \|y_n\|_n < \infty\}$, the ℓ^∞-direct sum of copies of Y with the norm $\|\cdot\|_n$. For each n, let $T_n : Y \to Z$ be the map embedding Y into the n-th summand of Z (i.e. $T_n(y) = (0, \dots, 0, y, 0, \dots)$, with y in the n-th position. This is linear, and $\|T_n\| \le n$, so each T_n is bounded. Also, for $y \in Y$, there exists $x \in X$ with $Tx = y$. Thus $\sup_n \|T_n(y)\| \le \|x\|$, so the principle of uniform boundedness (which does not require Z to be complete) shows that $C = \sup \|T_n\| < \infty$.

Suppose $y \in Y$ has $\|y\| < 1/C$. Then $\|y\|_n = \|T_n y\| \le C\|y\| < 1$. Therefore for each n there exists $x_n \in X$, $z_n \in Y$ with $Tx_n + z_n = y$ and $\|x_n\| + n\|z_n\| < 1$. Thus $\|z_n\| < 1/n \to 0$, so $Tx_n \to y$. Therefore $y \in \overline{T(B_X^\circ)}$. The successive approximation lemma then shows that $B_Y^\circ(1/C) \subset T(B_X^\circ)$, and we have deduced the open mapping theorem.

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