C4.1 Further Functional Analysis

Sheet 2 — MT 2025

For classes in week 5 or 6

This problem sheet is based on material up to and including Section 6 of the notes, together with appendix A.

Section A

1. Let X and Y be normed spaces and $T \in \mathcal{B}(X, Y)$

   (a) Show $(\text{ran } T)^\circ = \ker T^*$.

   (b) Use the Hahn-Banach theorem to show $(\text{ran } T^*)_\circ = \ker T$.

   Solution:

   (a)

$$(\text{ran } T)^\circ = \{f \in Y^* : f(Tx) = 0,\ x \in X\} = \{f \in Y^* : T^* f = 0\} = \ker T^*.$$

   (b) If $x \in \operatorname{Ker} T$, then for any $f \in Y^*$, $(T^* f)(x) = f(Tx) = 0$, so $x \in (\text{ran } T^*)_\circ$. For the converse, if $x \in (\text{ran } T^*)_\circ$, then for all $f \in Y^*$, $(T^* f)(x) = f(Tx) = 0$. Hahn-Banach then implies that $x \in \operatorname{Ker} T$ (as if not $Tx \neq 0$, and so there would exist $f \in Y^*$ with $0 \neq f(Tx) = (T^* f)(x)$).

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2. Let X be a normed space.

   (a) Let $C \subset X$ be convex set. Show that the closure, $\overline{C}$ is convex.

   (b) Given a subset $A \subset X$, show that

   $$\left\{ \sum_{i=1}^{n} \lambda_i a_i : n \in \mathbb{N},\ a_i \in A,\ \lambda_i \geq 0,\ \sum_i \lambda_i = 1 \right\}$$

   is the smallest convex subset of X containing A. This is known as the *convex hull* of A, and denoted $\mathrm{co}(A)$.

   (c) Given a subset $A \subset X$, show that $\overline{\mathrm{co}}(A)$ is the smallest closed convex subset of X containing A. This is known as the closed convex hull of A, denoted $\overline{\mathrm{co}}(A)$.

   (d) Use an example on the last sheet to provide closed convex sets $C_1, C_2 \subseteq X$ such that $\mathrm{co}(C_1 \cup C_2)$ is not closed.

   [You may find the closed convex hull construction useful in question B.4.]

Solution:

(a) Suppose $x, y \in \overline{C}$ and $0 < \lambda < 1$. Taking sequences $x_n \to x$ and $y_n \to y$ with $x_n, y_n \in C$, we have $\lambda x_n + (1-\lambda)y_n \in C$ and $\lambda x_n + (1-\lambda)y_n \to \lambda x + (1-\lambda)y$. Thus $\lambda x + (1-\lambda)y \in \overline{C}$ and so $\overline{C}$ is convex.

(b) It is easy to see that the given set is convex. Then show by induction that any convex set C has the property that $\sum_{i=1}^{n} \lambda_i c_i \in C$ whenever $c_1, \ldots, c_n \in C$ and $\lambda_i > 0$ have $\sum_{i=1}^{n} \lambda_i = 1$. Thus the displayed set is contained in any closed convex set containing A.

(c) In this case $\mathrm{co}(C_1 \cup C_2) = \{\lambda c_1 + (1-\lambda)c_2 : c_1 \in C_1,\ c_2 \in C_2,\ 0 \leq \lambda \leq 1\}$. Once one sees this closure follows exactly as in (a). To see this, note that the set on the right hand side is clearly contained in $\mathrm{co}(C_1 \cup C_2)$ from the expression of this set in (b); but it is also easily checked to be convex and contains $C_1 \cup C_2$.

(d) Sheet 1, B4 provides an example, with $C_1 = Y$ and $C_2 = Z$.

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Section B

1. Let X be a normed vector space and let Y be a subspace of X.

   (a) Suppose that Y is finite-dimensional. Show that Y is complemented in X, and that if Z is any closed subspace of X such that $X = Y \oplus Z$ algebraically, then X is in fact the topological direct sum of Y and Z.

   (b) What can you say if Y has finite codimension in X? [*Recall that the codimension of Y in X is the dimension of the quotient vector space X/Y.*]

2. (a) Let X be a Banach space and suppose that $\{x_n : n \geq 1\}$ is a bounded subset of X. Show that there exists a unique operator $T \in \mathcal{B}(\ell^1, X)$ such that $Te_n = x_n$ for all $n \geq 1$ and $\|T\| = \sup_{n\geq 1} \|x_n\|$.

   (b) Prove that if X is a separable Banach space then $X \cong \ell^1/Y$ for some closed subspace Y of ℓ^1.

   (c) Deduce that ℓ^1 contains closed subspaces which are uncomplemented.

   [*You may assume that any closed infinite-dimensional subspace of ℓ^1 has non-separable dual. We might prove this at the end of the course.*]

3. (a) Let X be an infinite dimensional real normed space, and $f : X \to \mathbb{R}$ a linear functional. Show that if there is an open ball $B^0_X(x_0, r)$ such that $f(x) > 0$ for $x \in B^0_X(x_0, r)$, then f is continuous. Deduce that if f is unbounded, then $\ker f$ is dense in X.

   (b) Use the previous result to show that any infinite dimensional normed space X can be decomposed into a union $A \cup B$ of disjoint convex sets, with both A and B dense in X.

4. (a) Let C be a convex absorbing subset of a normed space. Show

$$\{x \in X : p_C(x) < 1\} \subseteq C \subseteq \{x \in X : p_C(x) \leq 1\},$$

   with equality in the first inclusion when C is open, and equality in the second when C is closed.

   (b) Let C be a convex balanced subset of a normed space, which contains a neighbourhood of 0 and is bounded. Show that p_C gives an equivalent norm on X.

   (c) Let Y be a subspace of a normed space $(X, \cdot)$, and let $|||\cdot|||$ be an equivalent norm on Y. Show that $|||\cdot|||$ can be extended to an equivalent norm on X.

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5. Let X and Y be normed vector spaces and let $T \in \mathcal{B}(X,Y)$. Suppose there exists a constant $r > 0$ such that $\|T^* f\| \geq r\|f\|$ for all $f \in Y^*$.

   (a) Using the Hahn-Banach Separation Theorem, or otherwise, show that $B_Y(r)$ is contained in the closure of $T(B_X)$.

   (b) If X is complete, deduce that T is a quotient operator, and that T is an isometric quotient operator if T^* is an isometry.

   [This question is asking you to complete the missing bits from Theorem 5.16.]

6. Let $X = \ell^\infty$ and $Sx = (x_{n+1})$ for $x = (x_n) \in X$. Moreover, let $T = I - S$.

   (a) Show that $\operatorname{Ker} T = \{(\lambda, \lambda, \lambda, \dots) : \lambda \in \mathbb{F}\}$ and that $\operatorname{Ran} T \cap \operatorname{Ker} T = \{0\}$.

   (b) Let $Y = \operatorname{Ran} T \oplus \operatorname{Ker} T$ and let $P\colon Y \to Y$ be the projection onto $\operatorname{Ker} T$ along $\operatorname{Ran} T$. By considering the operators

   $$A_n = \frac{1}{n}\sum_{k=0}^{n-1} S^k, \quad n \geq 1,$$

   or otherwise, show that P is bounded and that $\|P\| = 1$.

   (c) Prove that there exists a functional $f \in X^*$ with $\|f\| = 1$ such that $f(Sx) = f(x)$ for all $x \in X$ and

   $$f(x) = \lim_{n\to\infty} x_n$$

   whenever $x = (x_n) \in c$.[1] Evaluate $f(x)$ when x is a periodic sequence.

7. Let X be a normed vector space and let Y be a subspace of X.

   (a) Writing $Y^{\circ\circ} = (Y^\circ)^\circ$ for the double annihilator of Y in X^{**}, show that there exists an isometric isomorphism $T\colon Y^{**} \to Y^{\circ\circ}$ such that $T \circ J_Y = J_X|_Y$. Deduce that Y is reflexive if and only if $Y^{\circ\circ} \subseteq J_X(Y)$.

   (b) Show that if X is reflexive and Y is closed, then both Y and X/Y are reflexive. [*Hint: Do the case of* Y *being reflexive first, and then consider* $(X/Y)^*$.]

   (c) Now suppose X is Banach and Y is a closed subspace such that both Y and X/Y are reflexive.

      (i) Fix $\phi \in X^{**}$. Let $\pi : X \to X/Y$ denote the canonical quotient operator, and show that there exists $x \in X$ such that $J_{X/Y}(x+Y) = \phi \circ \pi^*$.

      (ii) Show $\phi - J_X(x) \in Y^{\circ\circ}$, and use part (a) to show find some y for which $\phi = J_X(x+y)$.

[1] Recall that c is the subspace of ℓ^∞ consisting of convergent sequences.

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Section C

1. (a) Let X be a normed vector space and let $P \in \mathcal{B}(X^{***})$ be given by $P = J_{X^*}J_X^*$. Show that P is the projection onto $J_{X^*}(X^*)$ along $J_X(X)^\circ$ and that $\|P\| = 1$.

   (b) (i) Show that if $T \in \mathcal{B}(\ell^\infty)$ with $\|T\| = 1$ and $Te_n = e_n$, $n \geq 1$, then $T = I$.

   (ii) Deduce that there does not exist a projection of norm 1 from ℓ^∞ onto c_0.

   (iii) Prove that there is no normed vector space X such that $X^* \cong c_0$.

Solution:

(a) For $f \in X^*$ we have

$$\big(J_X^*(J_{X^*}f)\big)(x) = (J_X x)(f) = f(x), \quad x \in X,$$

so $J_X^* J_{X^*}$ is the identity operator on X^*. Hence $P^2 = P$. Note also that

$$J_{X^*}(X^*) = P(J_{X^*}(X^*)) \subseteq \operatorname{Ran} P \subseteq J_{X^*}(X^*),$$

so $\operatorname{Ran} P = J_{X^*}(X^*)$. Since J_{X^*} is injective, $\operatorname{Ker} P = \operatorname{Ker} J_X^* = J_X(X)^\circ$. Moreover, $\|P\| \leq \|J_{X^*}\|\|J_X^*\| = 1$. Since $P \neq 0$ we must have $\|P\| = 1$.

(b) (i) Let $x \in \ell^\infty$ and let $y = Tx$. Fix $n \geq 1$. Then $\|x + \lambda e_n\|_\infty = |x_n + \lambda|$ and $\|y + \lambda e_n\|_\infty = |y_n + \lambda|$ for all $\lambda \in \mathbb{F}$ with $|\lambda|$ sufficiently large. Thus

$$|y_n + \lambda| = \|y + \lambda e_n\|_\infty = \|T(x + \lambda e_n)\|_\infty \leq |x_n + \lambda|$$

for $\lambda \in \mathbb{F}$ as above. It follows from elementary geometric considerations that $x_n = y_n$. Since $n \geq 1$ was arbitrary, we have $x = y$ and hence $T = I$.

(ii) Suppose that $P \in \mathcal{B}(\ell^\infty)$ has norm 1 and is such that $P^2 = P$ and $\operatorname{Ran} P = c_0$. Then P fixes elements of c_0 and in particular $Pe_n = e_n$, $n \geq 1$. By the previous result we see that $P = I$ contradicting the fact that $\operatorname{Ran} P \neq \ell^\infty$.

(iii) Let $Y = c_0$ and suppose that $S\colon X^* \to Y$ is an isometric isomorphism. Let $\Phi\colon \ell^\infty \to Y^{**}$ be the usual isometric isomorphism and let $P \in \mathcal{B}(X^{***})$ be as in part (a). Note that $J_Y \circ S = S^{**} \circ J_{X^*}$ and that S^{**} is an isometric isomorphism. Consider the operator $T \in \mathcal{B}(\ell^\infty)$ given by

$$T = \Phi^{-1} J_Y S J_{X^*}^{-1} P (S^{**})^{-1} \Phi,$$

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where we write $J_{X^*}^{-1}$ for the inverse of the map J_{X^*} with codomain $J_{X^*}(X^*)$. Then $\|T\| = 1$ and, using the fact that $J_Y(y) = \Phi(y)$ for $y \in c_0$, we have

$$Ty = \big(\Phi^{-1} J_Y S J_{X^*}^{-1} P (S^{**})^{-1} J_Y\big)(y) = \big(\Phi^{-1} J_Y S J_{X^*}^{-1} P J_{X^*} S^{-1}\big)(y) = y.$$

By part (b)(i) we have that $T = I$. In particular, J_Y must be surjective, which is a contradiction because c_0 is non-reflexive.

2. Given a normed vector space X, we say that X is injective [2] if whenever Y is a subspace of a normed vector space Z and $T \in \mathcal{B}(Y, X)$ there exists an operator $S \in \mathcal{B}(Z, X)$ such that $\|S\| = \|T\|$ and $S|_Y = T$.

   (a) (i) Show that ℓ^∞ is injective

   (ii) By proving first that any operator $T \in \mathcal{B}(\ell^\infty, c_0)$ such that $Te_n = e_n$, $n \geq 1$, must have norm $\|T\| \geq 2$, or otherwise, show that c_0 is not injective.

   (iii) Is c_0 complemented in c, and if so what can you say about the norm of a complementing projection?

   (b) Suppose that X is an injective normed vector space, and Y is a subspace of a normed vector space Z such that Y is isomorphic to X. Prove that Y is complemented in Z.

Solution:

(a) (i) Suppose that Y is a subspace of Z and that $T \in \mathcal{B}(Y, \ell^\infty)$. For $n \geq 1$ let $p_n \in (\ell^\infty)^*$ by given by $p_n(x) = x_n$ and let $g_n \in Y^*$ be given by $g_n = T^* p_n$. Thus $Ty = (g_n(y))$, $y \in Y$, and hence

$$\|T\| = \sup_{y \in B_Y} \sup_{n \geq 1} |g_n(y)| = \sup_{n \geq 1} \sup_{y \in B_Y} |g_n(y)| = \sup_{n \geq 1} \|g_n\|.$$

By the Hahn-Banach Theorem there exist $f_n \in Z^*$, $n \geq 1$, such that $f_n|_Y = g_n$ and $\|f_n\| = \|g_n\|$ for all $n \geq 1$. Let $S \in \mathcal{B}(Z, \ell^\infty)$ be given by $Sz = (f_n(z))$, $z \in Z$. Then $S|_Y = T$ and moreover $\|S\| = \sup_{n \geq 1} \|f_n\| = \sup_{n \geq 1} \|g_n\| = \|T\|$, as required.

(ii) Suppose that $T \in \mathcal{B}(\ell^\infty, c_0)$ satisfies $Te_n = e_n$, $n \geq 1$, and that $\|T\| < 2$. Let $e = (1, 1, 1, \dots)$ and let $x = Te$. Then

$$|x_n - 2| \leq \|x - 2e_n\| = \|T(e - 2e_n)\| \leq \|T\|\|e - 2e_n\| \leq \|T\|, \quad n \geq 1,$$

[2] The terminology comes from category theory; X is an injective object in the category of normed spaces with contractive linear maps.

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and hence $|x_n| \geq 2 - \|T\| > 0$, $n \geq 1$, contradicting the fact that $x \in c_0$. So if $T \in \mathcal{B}(\ell^\infty, c_0)$ and $Te_n = e_n$, $n \geq 1$, then $\|T\| \geq 2$. In particular, taking $Y = c_0$ and $Z = \ell^\infty$ it follows that the identity operator on Y has no norm-preserving extension to Z, so c_0 is not injective.

(iii) The same proof above shows that any projection $P\colon c \to c_0$ must have $\|P\| \geq 2$. c_0 is complemented in c by a projection of norm exactly 2. Indeed, define $f : c \to \mathbb{F}$ by $f((x_n)) = \lim_{n\to\infty} x_n$, and then $P : c \to c_0$ by $P((x_n)_{n=1}^\infty) = (x_n - f(x))_{n=1}^\infty$. This is a projection onto c (as $P(e_n) = e_n$, and the span of the e_i is dense in c_0). Viewing P as an operator $c \to c$, it is the sum of id_c and $x \mapsto f(x)(1, 1, \dots)$, so has norm at most 2. Hence $\|P\| = 2$.

(b) If $T \in \mathcal{B}(Y, X)$ is an isomorphism then we may find $S \in \mathcal{B}(Z, X)$ such that $S|_Y = T$. Consider the operator $P \in \mathcal{B}(Z)$ given by $Pz = T^{-1}Sz$, $z \in Z$. Then $P^2 = P$ and $\operatorname{Ran} P = Y$, so by a result from lectures Y is complemented in Z.

3. Let Y and Z be closed subspaces of a Banach space X and suppose that $X^* = Y^\circ \oplus Z^\circ$ as a topological direct sum. Show that $X = Y \oplus Z$ as a topological direct sum.

Solution: Not so much as a solution, as a link to some hints. This was on the 2020 exam paper as Q3(b), broken up there in to parts to get you going (the question had not appeared on a problem sheet in 2020). The last part though is still pretty tricky - see the solutions to the 2020 exam, which are available for current students through the institute website.

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