# B8.5 Graph Theory

## Sheet 0 — MT25

*Not for classes*

The following problems (mostly mentioned in the lectures/notes) are primarily intended for students who did not do Part A Graph Theory, to help you get yourself up to speed. Other students may find them useful too. They will not be discussed in classes; solutions will be posted on the course website.

You don't need to do all the questions, but it may be helpful. I suggest comparing your answers to the model solutions one (or a few) at a time; having seen the model solution, try to write the next answer in a similar style/level of detail.

1. Let $x$ and $y$ be vertices of a graph $G$. Show that $G$ contains an (i.e., at least one) $x$–$y$ walk if and only if $G$ contains an $x$–$y$ path.

2. Let $G = (V, E)$ be a graph, and define a relation $\sim$ on $V$ by $x \sim y$ if $x$ and $y$ are connected in $G$, i.e., if there is an $x$–$y$ path/walk in $G$ (possibly of length 0). Show, giving full details, that $\sim$ is an equivalence relation.

3. [A little tedious; omit if you like.] Check that any graph $G$ is the disjoint union of its components (maximal connected subgraphs). It may help to first show that the components correspond to equivalence classes of the relation $\sim$ in the previous question.

4. Show that TFAE (The Following Are Equivalent): (a) $T$ is a tree, (b) $T$ is a minimal (w.r.t. edges) connected graph, (c) $T$ is a maximal (w.r.t. edges) acyclic graph.

5. Modify the argument in lectures to show that any tree with at least two vertices has at least two leaves.

6. Let $T$ be a tree with $|T| \geqslant 2$, and let $P$ be a longest path in $T$. Prove, giving full details, that the ends of $P$ are leaves. Deduce that $T$ has at least two leaves.

7. Show that any two vertices of a tree $T$ are joined by a *unique* path in $T$.

8. Let $(d_1, \ldots, d_n)$ be a sequence of integers with $n \geqslant 2$. Show that there is a tree on $[n]$ with $d(i) = d_i$ for each $i$ if and only if $d_i \geqslant 1$ for all $i$ and $\sum_{i=1}^{n} d_i = 2n - 2$.

9. Show that deleting any edge from a tree $T$ leaves a graph with exactly two components. Show that deleting a vertex $v$ leaves $d(v)$ components. [Hint: you could do this directly, or try a short cut using what we know about numbers of edges in trees.]


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