Topics in Fluid Mechanics, C5.7

1.1 Navier Stokes Equations, recap

Let $\underline{u}(\underline{x}, t)$ be the velocity at location $\underline{x}$, time t.

Then

density

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \underline{u} = 0 \quad , \quad \text{mass conservation}$$

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p + \mu \nabla^2 \underline{u} + \underline{F} \quad , \quad \text{momentum conservation}$$

pressure, viscosity, body force, eg. $\underline{F} = \rho \underline{g}$

where $\frac{D}{Dt}(\cdot) = \left[\frac{\partial}{\partial t} + \underline{u} \cdot \nabla\right](\cdot)$. ← Derivative following the fluid.

Non-dimensionalise

$$\underline{u}' = U \underline{u} \quad , \quad t' = \frac{L}{U} t \quad , \quad \underline{x}' = L \underline{x}$$

$$p' = p^* p \quad , \quad \underline{F}' = p^*/_L \, \underline{F}$$

$$(Re) \frac{D\underline{u}'}{Dt'} = \left(\frac{p^* L}{\mu U}\right)\left(-\nabla' p' + \underline{F}'\right) + \nabla'^2 \underline{u}'$$

Reynolds number $= \dfrac{\rho U^2 / L}{\mu U / L^2} = \dfrac{\rho U L}{\mu}$

either work for $Re \ll 1$.

$Re \ll 1$, viscous dominated. Let $p^* = \mu U / L$ for balance

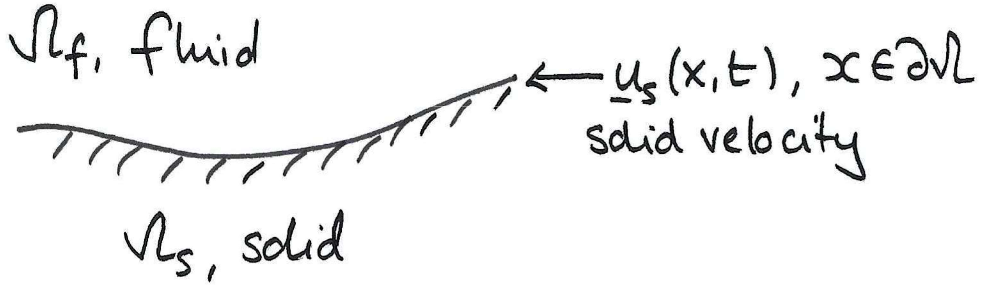
$Re \gg 1$, inertia dominated. Let $p^* = (Re) \dfrac{\mu U}{L}$ for balance.

Drop Primes henceforth

Boundary conditions

Boundary with a solid, with known motion

Typically zero, i.e. stationary.

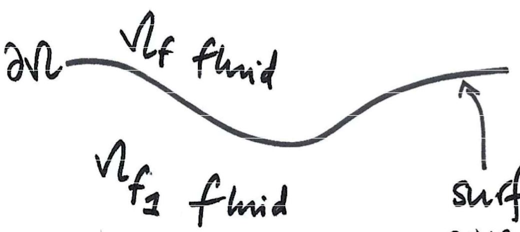


Then $\underline{u}(x,t) = \underline{u}_s(x,t)$, $x \in \partial\Omega$.

Tangential components, no slip BC

Normal Component, kinematic BC

Boundary with a fluid



surface often given via $z = f(x,y,t)$.

Continuity of velocity at $\partial\Omega$

$$\underline{u}(x,t) = \underline{u}_{f_1}(\underline{x},t), \quad x \in \partial\Omega$$

Then kinematic condition often enforced via

$$\underline{u}\cdot\underline{e}_z\Big|_{\Omega} \equiv w\big|_{\Omega} = \frac{Df}{Dt} = \partial f/\partial t + u\,\partial f/\partial x$$

Fluid velocity = velocity of surface following fluid i.e. no gaps, same as kinematic condition

we have double the number of unknowns at the interface Hence ...

Stress boundary conditions also required ... will return to this later. Will introduce surface tension for example

When $Re \ll 1$, Stokes Equations

$$0 = -\nabla p + \underline{F} + \nabla^2 \underline{u}$$

$Re \gg 1$, Eulers Equations $\dfrac{D\underline{u}}{Dt} = -\nabla p + \underline{F}$

Boundary layer is a thin regime near surface where $\nabla^2 \underline{u}$ term important... do not consider here.

Cannot satify no-slip conditions ... leads to boundary layers, thin layers where viscosity cannot be neglected

Boussinesq Approximation: changes in density negligible except when coupled to $\underline{g}$ in a buoyancy body force

$$\therefore \frac{D\rho}{Dt} + \rho \nabla \cdot \underline{u} = \rho \nabla \cdot \underline{u} = 0$$ and we have

$\nabla \cdot \underline{u} = 0$, fluid incompressibility

See online notes for more details on why this is normally an excellent approximation

Two dimensional flows: the streamfunction (ψ).

Planar flows

Ask Why is this useful

$\underline{u}(x, y) = (u, v, 0)$ and we let $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$

This ensures $\nabla \cdot \underline{u} = 0$.

Analogous streamfunctions for axisymmetric flows are in the on-line notes

1.2 Other conservation laws

- Concentration of a dissolved solute

$$\frac{DC}{Dt} = \nabla \cdot (\kappa \nabla C)$$

diffusion coefficient (often constant)

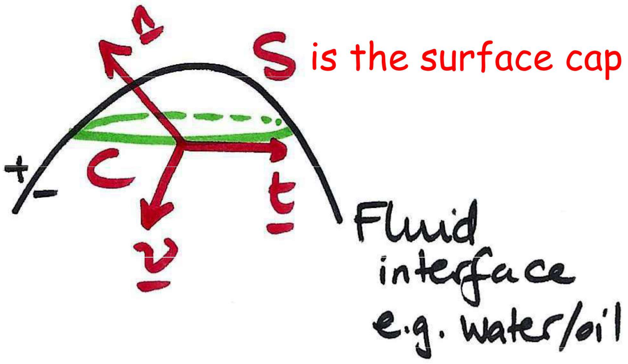
- Thermal energy

$$\frac{DT}{Dt} = \nabla \cdot (\kappa_T \nabla T)$$

neglecting viscous dissipation and any other heat sources (and taking $\nabla \cdot \underline{u} = 0$).

2.1 Surface Tension

Stress conditions at a fluid-fluid interface



Surface tension: force at interface in direction $\underline{\nu}$, of magnitude γ per unit length.

Extend S to the interval $(-\epsilon, \epsilon)$ in the normal direction $\underline{n}$ to form the volume V.

Force balance

$\underline{t}^+$ the force per unit area the upper fluid exerts on S

$$\underbrace{\int_V \rho \frac{D\underline{u}}{Dt}\, dV = \int_V \underbrace{\underline{f}}_{\text{body force}}\, dV}_{\to 0 \text{ as } \epsilon \to 0} + \int_S (\underline{t}^+ + \underline{t}^-)\, dS + \underbrace{\int_C \gamma \underline{\nu}\, ds}_{\text{Surface Tension force}}$$

$\boxed{\underline{t}^- \text{ the force per unit area the lower fluid exerts on } S}$

ds: arclength

$\therefore$ Let $\epsilon \to 0$,

$$0 = \int_S (\underline{t}^+ + \underline{t}^-)\, dS + \int_C \gamma \underline{\nu}\, ds$$

Assuming Navier Stokes Eqns and Newtonian fluid

The stress tensor is

$$\underline{\underline{T}} = -p\underline{\underline{I}} + \mu\left[\nabla\underline{u} + (\nabla\underline{u})^T\right]$$

Aside, in component form

$$T_{ij} = -p\delta_{ij} + \mu\left(\partial u_i/\partial x_j + \partial u_j/\partial x_i\right)$$

Hence $\quad \underline{t}^+ = \underline{n}\cdot\underline{\underline{T}}^+, \quad \underline{t}^- = -\underline{n}\cdot\underline{\underline{T}}^-$

Define γ, $\underline{n}$ off-surface by an invariant extension in direction $\underline{n}$

Then (Q1, Problem Sheet 1)

$$\int_C \gamma\,\underline{\nu}\,ds = \int_S \left[\nabla\gamma - (\nabla\cdot\underline{n})\underline{n}\gamma\right] dS$$

With $\quad \nabla_s := \nabla - \underline{n}\dfrac{\partial}{\partial n}, \quad \nabla\gamma = \nabla_s\gamma$

as γ has no variation in direction $\underline{n}$ by definition

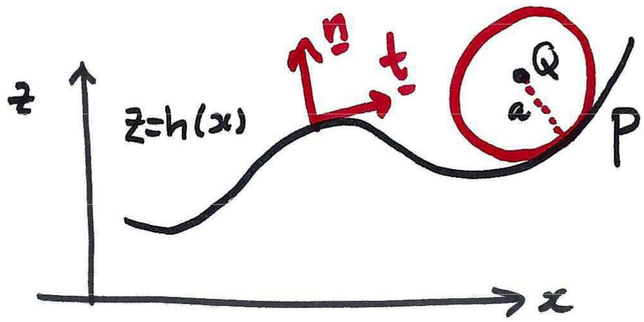
$$\therefore \quad 0 = \int_S \left(\underline{n}\cdot\underline{\underline{T}}^+ - \underline{n}\cdot\underline{\underline{T}}^- + \nabla_s\gamma - \gamma\nabla\cdot\underline{n}\,\underline{n}\right) dS, \quad \text{all } S.$$

$$\therefore \quad \boxed{\underbrace{\underline{n}\cdot\underline{\underline{T}}^+ - \underline{n}\cdot\underline{\underline{T}}^-}_{\text{Jump in normal stress}} = \underbrace{\gamma(\nabla\cdot\underline{n})\underline{n} - \nabla_s\gamma}_{\text{Surface tension}}} \quad x \in S.$$

Surface tension, $\nabla_s\gamma \sim$ Maragoni dynamics due to surface tension gradients

Vector Equation, with 3 components

Curvature (2D)



In 2D the magnitude of the curvature at P is the reciprocal of the best fit circle radius, a.

With the surface $z = h(x)$, the unit normal and tangent are

$$\underline{n} = \frac{(-h', 1)}{(1+h'^2)^{1/2}} \qquad \underline{t} = \frac{(1, h')}{(1+h'^2)^{1/2}}$$

Recall $\underline{n}$ defined off-surface by invariant normal extension

Then

$$\frac{1}{a} = \frac{d}{dx}\left(\frac{h'}{(1+h'^2)^{1/2}}\right) = -\nabla\cdot\underline{n} = \frac{h''}{(1+h'^2)^{3/2}}$$

[as shown in Appendix of on-line notes]

More generally (see Appendix), in 2D or 3D, the curvature, κ, is given by

$$\kappa = -\nabla\cdot\underline{n}$$

Hence in 3D, for a surface $z = h(x,y)$,

$$\kappa = -\nabla\cdot\underline{n} \qquad \text{with} \qquad \underline{n} = \frac{\nabla(z-h(x,y))}{|\nabla(z-h(x,y))|}$$

so that

$$\kappa = \frac{h_{xx}(1+h_y^2) + h_{yy}(1+h_x^2) - 2h_xh_yh_{xy}}{(1+h_x^2+h_y^2)^{3/2}}.$$

Capillary Statics

For static configurations, $\underline{u} = \underline{0}$, $\underline{\underline{T}} = -p\underline{\underline{I}}$, $\underline{n} \cdot \underline{\underline{T}} = -p\underline{n}$

Normal stress BC reduces to $\quad p^+ - p^- = \gamma \kappa$

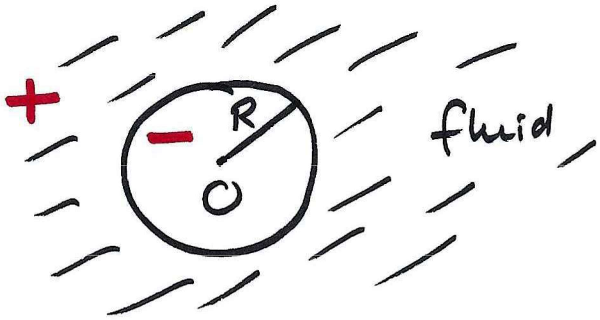
Curvature

Tangential stress BC

$$0 = -\nabla_s \gamma .$$

The Marangoni term $\nabla_s \gamma$ must be zero for consistency as tangential stress is zero.

Stationary Bubble



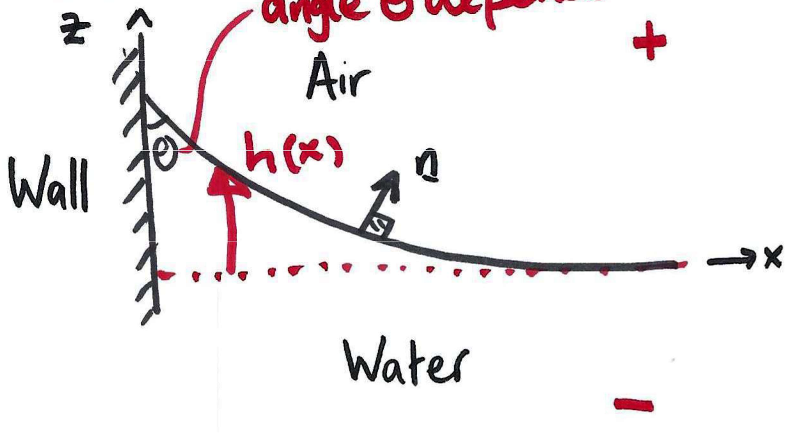
$$\text{Curvature} = \kappa = -\nabla \cdot \underline{n}\big|_{r=R}$$
$$= -\nabla \cdot \left(\underline{r}/r\right)\big|_{r=R}$$
$$= -\frac{1}{r^2}\frac{d}{dr}\left(r^2 \cdot 1\right)\Big|_{r=R} = -2/R .$$

$$\therefore \quad p^- - p^+ = 2\gamma/R$$

Smaller bubbles have higher internal pressure. Thus they are louder when they burst at a free surface ∴ Champagne fizz louder than beer.

Static meniscus

angle θ depends on water/air/wall properties { hydrophilic vs hydrophobic

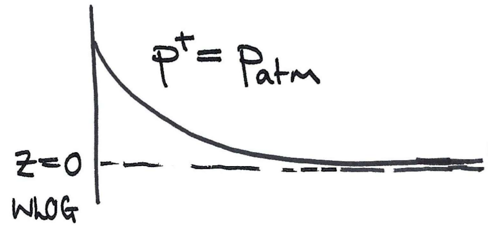


$$P_{air} = p^+ = P_{atm}, \text{ constant}$$

atmospheric pressure

Navier-Stokes ($\underline{u}=0$).

$$0 = -\nabla p^- + \rho \underline{g} = -\nabla p^- - \rho g \underline{e}_z \therefore \frac{\partial p^-}{\partial x} \overset{①}{=} 0,\ \frac{\partial p^-}{\partial z} \overset{②}{=} -\rho g$$



Here curvature of surface $\simeq 0$

$$p^+ - p^- = \gamma K \simeq 0$$

$$\lim_{x\to\infty} p^-(x,0) = p^{atm}$$

$$\left.\begin{aligned} \therefore p^-(x,0) &\overset{\text{by }①}{=} p^{atm}, \text{ all } x \\ \therefore p^-(x,h) &\overset{②}{=} p^{atm} - \rho g h \end{aligned}\right\} \therefore \boxed{\rho g h = \gamma K}$$

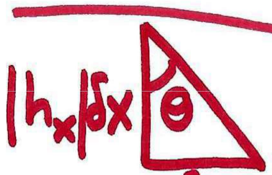
Laplace-Young Eqn

Recall $K = \dfrac{h_{xx}}{(1+h_x^2)^{3/2}}$ $\therefore$ $h = \left(\dfrac{\gamma}{\rho g}\right) \dfrac{h_{xx}}{(1+h_x^2)^{3/2}}$

gives shape of meniscus.

BCs $h \to 0$ as $x \to \infty$

$h_x \to -\cot\theta$ as $x \to 0$, for contact angle to be θ.



$\therefore \tan\theta = \dfrac{\delta x}{|h_x|\delta x} = \dfrac{1}{|h_x|}$

$h_x<0, \cot\theta>0$

Fixing signs, $h_x = -\cot\theta$

For $\theta \simeq \pi/2$, $|h_x| \ll 1$

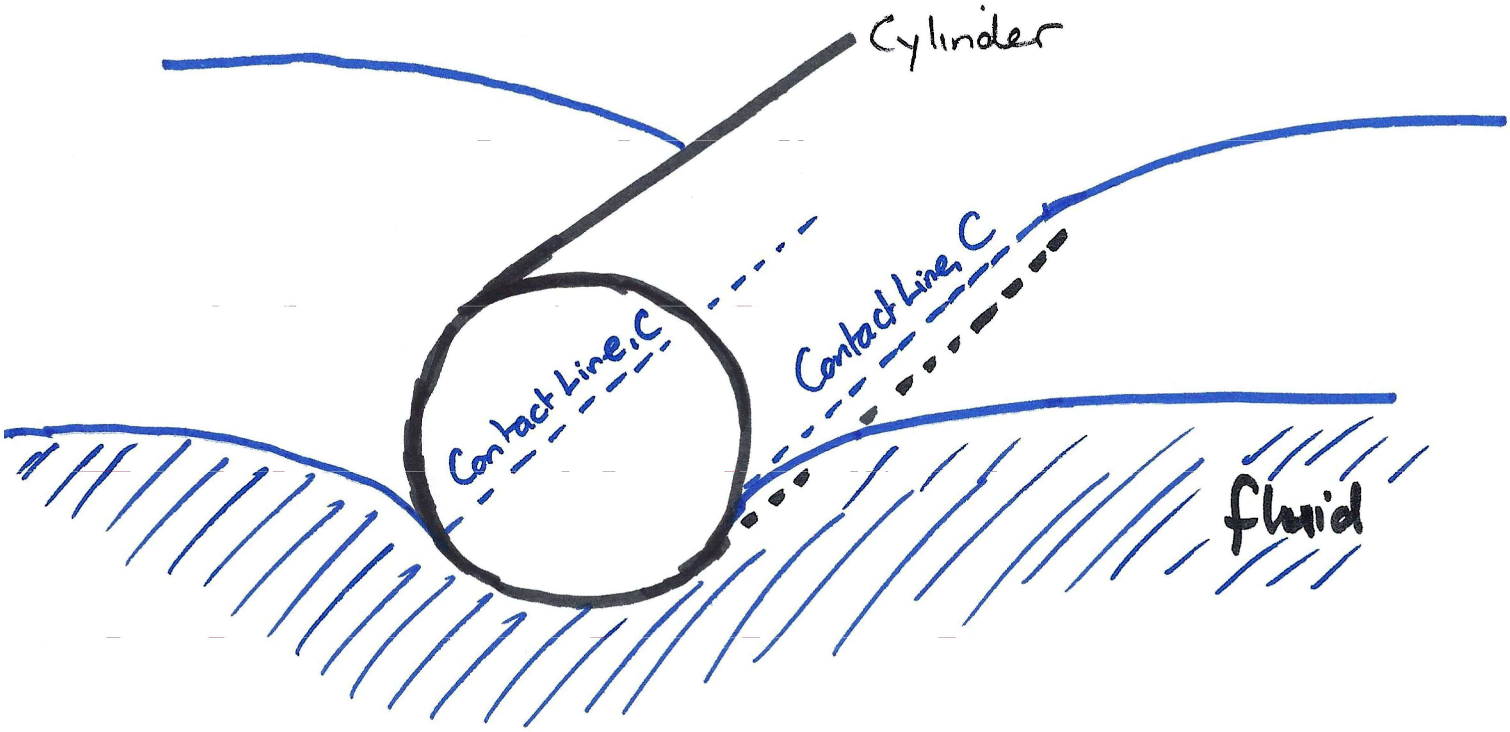
$$\boxed{\begin{aligned} &h = \left(\frac{\gamma}{\rho g}\right) h_{xx} \\ &h \to 0 \text{ as } x \to \infty \\ &h_x \to -\cot\theta \text{ as } x \to 0. \end{aligned}}$$

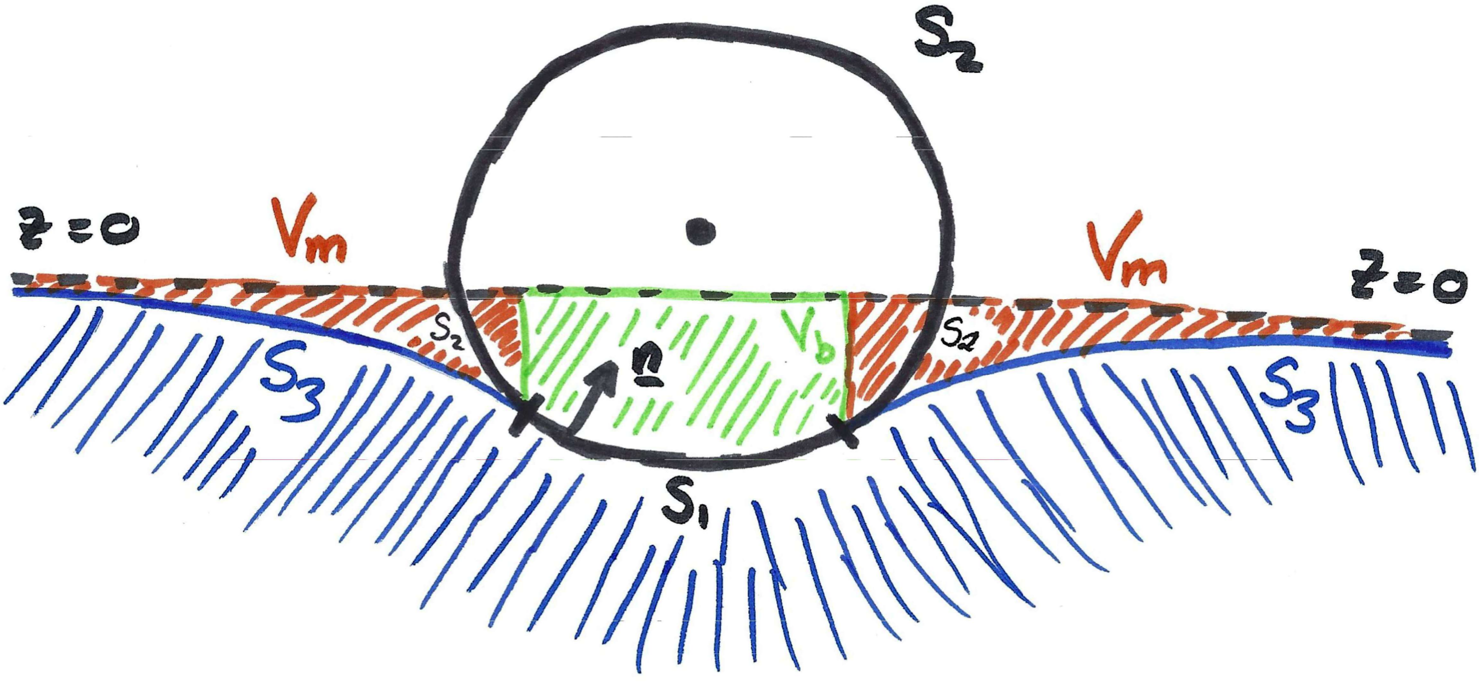
Solving $h(x) = l_c \cot\theta \, e^{-x/l_c}$, $\underbrace{l_c = \sqrt{\frac{\gamma}{\rho g}}}_{\text{capillary length}}$

For air-water $l_c \simeq 3\text{mm}$

so one can just about see the meniscus.

Floating Bodies: Supported by buoyancy & surface tension





V_m : Volume displaced by meniscus
V_b : Volume additionally displaced by body

Surface tension between fluid and air, γ, and fluid density, ρ, are constant.

As previously, the fluid pressure is

(NS Eqns) $\quad p = -\rho g z + P_{atm}$ since $\underline{u} = 0$.

Force on body from fluid is (noting $\underline{n}$ points into body)

$$\underline{F} = \underbrace{\int_{S_1} p\underline{n}\, dS}_{P_{atm}\int_{S_1}\underline{n}\, dS - \rho g\int_{S_1} z\underline{n}\, dS} + P_{atm}\int_{S_2}\underline{n}\, dS + \underbrace{\int_C \gamma\underline{\nu}\, ds}_{\underbrace{\int_{S_1}\gamma\kappa\underline{n}\, dS}_{\nabla\gamma = 0 \text{ as } \underline{u}=0}}$$

0 as $\int_{S_1 \cup S_2}\underline{n}\, dS = 0$ by div theorem

$$\therefore \underline{F} = \cancel{P_{atm}\int_{S_1\cup S_2}\underline{n}\, dS} - \rho g\int_{S_1} z\underline{n}\, dS + \int_{S_1}\gamma\kappa\underline{n}\, dS$$

γ not defined on S1
Allowed as γ constant so just work with Stokes' theorem and

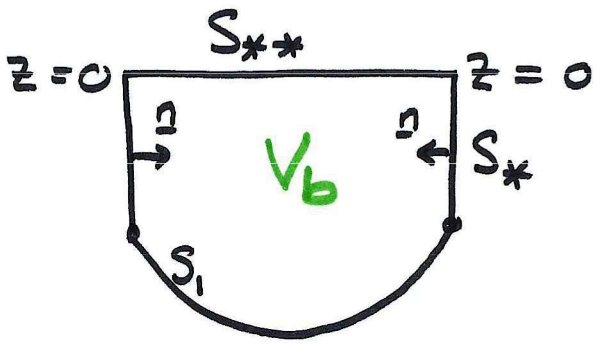
$$\int_C \underline{\nu}\, ds$$

F+M**g**=0 as no net force

$$\therefore \quad \underline{F} = \underbrace{-\rho g\int_{S_1} z\underline{n}\, dS + \int_{S_1}\gamma\kappa\underline{n}\, dS}_{\text{force due to fluid}} = -M\underline{g}, \text{ force due to gravity.}$$

g =-ge_z for minus sign

$$\therefore Mg = \underbrace{\underline{e}_z \cdot \left[-\rho g\int_{S_1} z\underline{n}\, dS\right]}_{F_b,\text{ buoyancy force}} + \underbrace{\underline{e}_z \cdot \left[\int_{S_1}\gamma\kappa\underline{n}\, dS\right]}_{F_c,\text{ surface tension force}}$$



$$\underline{e}_z \cdot \int_{S_{**}} z \underline{n}\, dS = \underline{e}_z \cdot \int_{S_*} z \underline{n}\, dS = 0$$

Let $S_1^* = S_1 \cup S_* \cup S_{**}$

$$F_b = -\rho g\, \underline{e}_z \cdot \int_{S_1} z \underline{n}\, dS = -\rho g\, \underline{e}_z \cdot \int_{S_1^*} z \underline{n}\, dS$$

$$= +\rho g \int_{V_b} dV = \left\{ \begin{array}{l} \text{Weight of fluid} \\ \text{displaced from } V_b \end{array} \right\}$$

divergence theorem (sign as $\underline{n}$ points inwards)

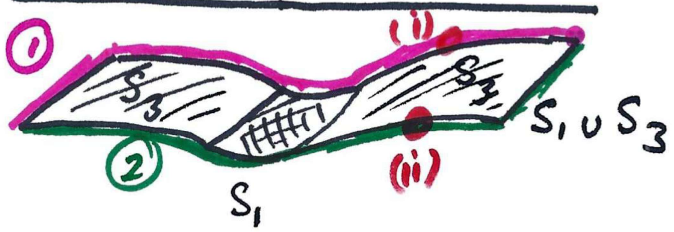
Q1, Problem sheet 1

$$\int_C \gamma \underline{\nu}\, ds = \int_S [\cancel{\nabla \gamma} + \gamma \kappa \underline{n}]\, dS$$

0 as static

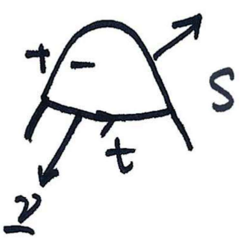
curvature

Surface Tension Force



$\underline{\nu} \perp \underline{t}, \underline{n}$

$\underline{\nu} = \underline{t} \wedge \underline{n}$



Going from point (i) on ① to point (ii) on ② $\Big\}$ $\underline{t}, \underline{n} \to -\underline{t}, \underline{n}$ $\therefore \underline{\nu} \to -\underline{\nu}$

$$\therefore \underbrace{\int_{S_1 \cup S_3}}_{\text{per unit length}} \kappa \underline{n}\, dS = \underbrace{\int_{\partial(S_1 \cup S_3)}}_{\text{per unit length}} \underline{\nu}\, ds = 0$$

by cancellation between ① and ②

$\therefore$ Noting γ constant $\displaystyle\int_{S_1} \gamma \kappa \underline{n}\, dS = -\int_{S_3} \gamma \kappa \underline{n}\, dS$

$$\therefore F_c = \underline{e}_z \cdot \int_{S_1} \gamma \kappa \underline{n}\, dS = -\underline{e}_z \cdot \int_{S_3} \gamma \kappa \underline{n}\, dS$$

On S_3 surface tension and buoyancy balance.

Similar reasoning to $F_b = \rho g \int_{V_b} dV$

$\therefore$ Laplace Young $\rho g z = \gamma \kappa$

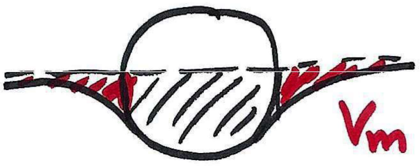
$$F_c = -\underline{e}_z \cdot \int_{S_3} \rho g \underline{n} z \, dS = \rho g \int_{V_m} dV$$

$$= \left\{ \begin{array}{l} \text{Weight of fluid} \\ \text{displaced by meniscus} \end{array} \right\}$$

$\therefore$ Force on body = Weight of displaced fluid

Archimedes' Principle, to include impact of surface tension.

<u>Overall Scales</u>



$V_b \sim \text{depth} \cdot L^2$

$V_m \sim \text{depth} \cdot L \cdot l_c$

where L is the body lengthscale

meniscus lengthscale $l_c \sim \sqrt{\frac{\gamma}{\rho g}}$

$$\therefore \frac{F_b}{F_c} \sim \frac{V_b}{V_m} \sim L\sqrt{\frac{\rho g}{\gamma}}$$

$\therefore$ Once $L \ll \sqrt{\frac{\gamma}{\rho g}}$ floating bodies primarily supported by surface tension.

Marangoni Flows

These are flows where surface tension gradient effects dominate.

Examples (see on-line notes) Tears of wine, surfactant flows, ...

Example Thermo-capillary bubble

- A bubble in a temperature gradient can move due to Marangoni effects ~ surface tension changes with temperature

Assumptions

- Bubble remains spherical, radius a, constant.
- Temperature in both fluid and bubble given by

$$T = \underbrace{T_\infty + T'}_{\text{constant}} z$$

in frame moving with bubble

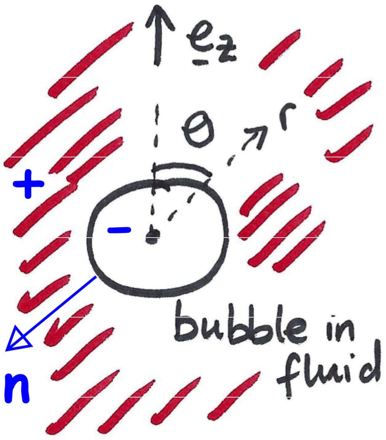
- Surface tension linear in T, hence

$$\gamma = \underbrace{\gamma_0 + \gamma'}_{\text{constant}} z = \gamma_0 + \gamma' a \cos\theta$$

- Length scales sufficiently small to ensure

$$Re = \frac{\rho |U| a}{\mu} \ll 1.$$

a is radius of bubble,
U is the speed of the bubble



Bubble radius a

$\underline{u} \sim -U\underline{e}_z$ at infinity, as working in bubble frame

$$0 = -\nabla p + \mu \nabla^2 \underline{u} + \rho \underline{g} \quad , \quad \text{Stokes flow}$$

$$\underline{u} = u_r \underline{e}_r + u_\theta \underline{e}_\theta \quad , \quad \text{Spherical polars}$$

$$\therefore \quad u_r = \frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta} \, , \quad u_\theta = -\frac{1}{r\sin\theta} \frac{\partial \psi}{\partial r} \Bigg\} \text{This ensures } \nabla \cdot \underline{u} = 0$$

for streamfunction ψ.

as $\underline{e}_r \cdot \underline{e}_z = \cos\theta$ as $\underline{e}_z \cdot \underline{e}_\theta = -\sin\theta$

$$u_r \sim -U\cos\theta \, , \quad u_\theta \sim U\sin\theta \quad \text{as } r \to \infty$$

$\therefore$ Seek streamfunction of form

$$\psi = f(r)\sin^2\theta$$

✓ as $\frac{1}{\sin\theta}\frac{\partial \psi}{\partial \theta} \sim \cos\theta$, $-\psi/\sin\theta \sim \sin\theta$

Standard methods for viscous flow [e.g. §3.3 of Ockendon & Ockendon] give

$$0 = -\nabla \wedge \nabla p + \mu \nabla^2 \nabla \wedge \underline{u} + \nabla \wedge (\rho \underline{g})$$

$$= \mu \nabla^2 (\nabla \wedge \underline{u}).$$

Substitute u_r, u_θ one finds (algebra)

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2} \right)^2 f(r) = 0.$$

ODE homogeneous in powers of r $\therefore$ Try $f \sim r^m$.

Thus $$\psi = \left[Ar^4 + Br^2 + Cr + D/r \right] \sin^2\theta.$$

BCs

- $u_r(r=a) \overset{①}{=} \left.\dfrac{1}{a^2\sin\theta}\dfrac{\partial\psi}{\partial\theta}\right|_{r=a} = 0$ Bubble constant radius
- $\psi \overset{②}{\to} -\dfrac{Ur^2}{2}\sin^2\theta$ as $r\to\infty$ to match flow at infinity

Noting bubble gas viscosity negligible compared to fluid viscosity, the stress balance BC gives

$$\underline{e_r}\cdot\underline{\underline{T}}^{fluid}\cdot\underline{e_\theta} = -\underline{e_\theta}\cdot\nabla\gamma = -\frac{1}{a}\frac{\partial\gamma}{\partial\theta}$$

Proportional to viscosity & bubble gas viscosity term drops

as

$$\therefore \quad \underbrace{\mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right]_{r=a}} \overset{③}{=} \gamma'\sin\theta$$

$\gamma=\gamma_0+a\gamma'\cos\theta$

From expression for stress

$\underline{e_r}\cdot\left[(\nabla u)+(\nabla u)^T\right]\cdot\underline{e_\theta}$

see eg Appendix of Bachelor, Introduction to Fluid Dynamics.

② gives $A=0,\ B=-U/2$

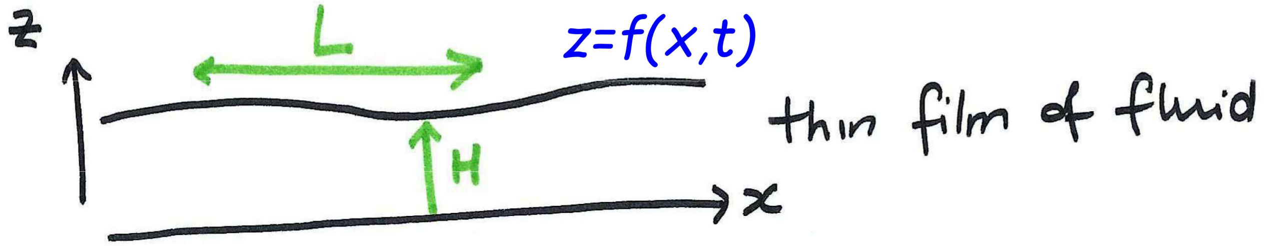
① gives $Ua^2/2 = Ca + D/a$

③ gives $D = -\dfrac{1}{6}\dfrac{\gamma' a^4}{\mu}$ (lots of cancellation)

$$\therefore\ C = Ua/2 + \frac{1}{6}\frac{\gamma' a^2}{\mu},$$

and we thus know ψ, and hence $\underline{u}$.

2.2 Thin films and the lubrication approximation



$$\delta = \frac{\text{Thickness scale}}{\text{lengthscale of horizontal variation}} = \frac{H}{L} \ll 1.$$

Navier-Stokes

$$\rho(\underline{u}_t + \underline{u}\cdot\nabla\underline{u}) = -\nabla p + \rho\underline{g} + \mu\nabla^2\underline{u}$$

$$\nabla\cdot\underline{u} = 0$$

2D Cartesian

Let $\underline{u} = u\underline{e}_x + w\underline{e}_z$, with velocity scales U, W.

$$0 = \nabla\cdot\underline{u} = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}, \quad \frac{\partial}{\partial x} \sim \frac{1}{L}, \quad \frac{\partial}{\partial z} \sim \frac{1}{H}$$

$$\therefore \quad W \sim H/L\, U \sim \delta U$$

For x-components of the momentum equation

$$\therefore \quad \rho\underline{u}\cdot\nabla u \sim \rho\frac{U^2}{L}, \quad \nabla p \sim P/L, \quad \mu\nabla^2 u \sim \frac{\mu U}{H^2}$$

largest term $\mu\frac{\partial^2 u}{\partial z^2}$

$\therefore$ Can neglect inertia if $\frac{\rho U^2}{L} \ll \frac{\mu U}{H^2}$

$$\text{i.e.} \left(\frac{\rho}{\mu}UH\right)\frac{H}{L} = \delta Re \ll 1$$

δRe is the reduced Reynolds number

Also, pressure is of the scale

$$P \sim \mu \frac{UL}{H^2}$$

The leading order equation is thus

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial z^2}$$

z - components

$$\nabla p \sim P/H \sim \mu \frac{UL}{H^3} \quad , \rho \underline{u} . \nabla w \sim \rho \frac{UW}{L} \sim \delta \frac{\rho U^2}{L}$$

$$\sim \delta^3 Re \, \nabla p$$

$$\mu \nabla^2 w \sim \mu \frac{1}{H^2} W \sim \mu \frac{\delta U}{H^2} \sim \delta^2 \nabla p .$$

$\therefore$ Pressure term dominates viscosity and inertia

$$\therefore \quad \frac{\partial p}{\partial z} = \rho \underline{g} \cdot \underline{e}_z \quad \text{at leading order}$$

and $|w| \sim O(\delta)|u|$, neglect vertical velocity at leading order

except in $\nabla \cdot \underline{u} = 0$

BCs

Fixed boundary, $\underline{u} = \underline{0}$.

Free boundary. Typically Air-Liquid interface.

- Viscosity of air negligible.

$$\underline{\underline{T}}^{air} = -P_{atm} \underline{\underline{I}}$$

- Boundary given by $z = f(x,t)$

Kinematic BC $\quad w = f_t + u f_x$

For stress balance, $\underline{\underline{T}} = \begin{pmatrix} -p + 2\mu u_x & \mu(u_z + w_x) \\ \mu(u_z + w_x) & -p + 2\mu w_z \end{pmatrix}$

Note

$$\frac{|\mu u_x|}{|p|} \sim \frac{\mu U/L}{\mu U L/H^2} \sim \delta^2 \sim \frac{|\mu w_z|}{|p|}$$

$$\frac{|w_x|}{|u_z|} \sim \frac{\delta U/L}{U/H} \sim \delta^2 \qquad \frac{|\mu u_z|}{|p|} \sim \delta$$

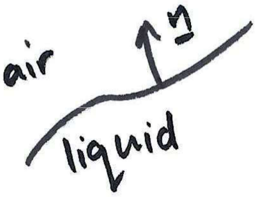
$$\therefore \quad \underline{\underline{T}} = \begin{pmatrix} -p(1 + O(\delta^2)) & \mu u_z(1 + O(\delta^2)) \\ \mu u_z(1 + O(\delta^2)) & -p(1 + O(\delta^2)) \end{pmatrix}$$

Similarly $|f_x| \sim H/L \sim \delta$ and

$$\underline{n} = (-f_x, 1)(1 + O(\delta^2)), \quad \underline{t} = (1, f_x)(1 + O(\delta^2))$$

$$\kappa = f_{xx}(1 + O(\delta^2))$$

With surface tension constant,



$$\underline{n}\cdot\underline{\underline{T}}^{air} - \underline{n}\cdot\underline{\underline{T}} = -\gamma \kappa \underline{n}$$

Absorbing p_{atm} into p, we have up to $O(\delta^2)$

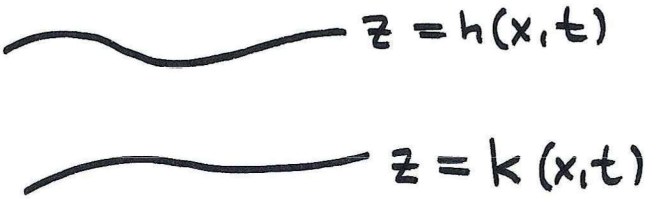
$\underline{n}\cdot\underline{\underline{T}}\cdot\underline{n}$ will pick out the zz term i.e. pressure $\sim$

$$-p = \gamma f_{xx} \quad \text{(normal component)}$$

$$\frac{\partial u}{\partial z} = 0 \quad \text{(tangential component).}$$

$\underline{n}\cdot\underline{\underline{T}}\cdot\underline{t}$ will pick out cross term, i.e. $\partial u/\partial z$

Conservation of mass / incompressibility



If $z = k(x,t)$ fixed surface, then $\partial k/\partial t = 0$, $u = w = 0$ on the surface so that

$$0 = w = \partial k/\partial t + u\,\partial k/\partial x$$ and kinematic condition still holds

Thus,

$$\frac{\partial}{\partial x}\int_{\bar{z}=k(x,t)}^{\bar{z}=h(x,t)} u(x,\bar{z},t)\,d\bar{z} = \int_k^h \frac{\partial u}{\partial x}\,d\bar{z} + u(x,h,t)h_x - u(x,k,t)k_x$$

$$= -\int_k^h \frac{\partial w}{\partial \bar{z}}\,d\bar{z} + u(x,h,t)h_x - u(x,k,t)k_x$$

$$= w(x,k,t) - w(x,h,t) + u(x,h,t)h_x - u(x,k,t)k_x$$

$$= \frac{\partial k}{\partial t} - \frac{\partial h}{\partial t}\,,$$ by kinematic condition.

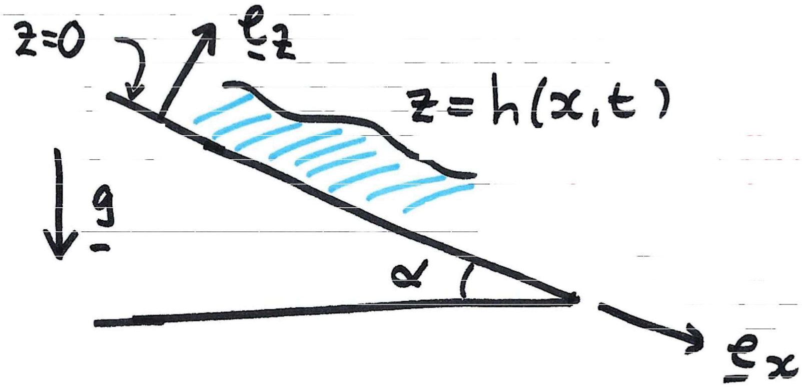
With $\mathcal{H} := h-k$, film thickness, the average velocity is $\bar{u} = \frac{1}{\mathcal{H}}\int_k^h u(x,\bar{z},t)\,d\bar{z}$

and thus

$$\frac{\partial \mathcal{H}}{\partial t} + \frac{\partial}{\partial x}\left(\mathcal{H}\bar{u}\right) = 0$$

$\therefore$ We need to find $\bar{u}$ in terms of $\mathcal{H}$ which is context specific.

Free surface flow down an inclined plane



$$\underline{g} = g(\sin\alpha, -\cos\alpha)$$

z - component, momentum

$$\frac{\partial p}{\partial z} = -g\rho\cos\alpha \qquad \text{with} \quad -p = \gamma h_{xx} \ \text{ on } z = h.$$

$$\therefore \quad p = -\gamma h_{xx} + \rho g (h-z)\cos\alpha.$$

x - component, momentum

$$\mu \frac{\partial^2 u}{\partial z^2} = \frac{\partial p}{\partial x} - \rho g \sin\alpha$$

$$\therefore \mu \frac{\partial^2 u}{\partial z^2} = -\gamma h_{xxx} + \rho g \cos\alpha \, h_x - \rho g \sin\alpha$$

For α non-zero and a sufficiently shallow film,

$$|h_x| \ll \tan\alpha$$

We rescale using a velocity scale U and lengthscale R

$$\frac{\partial^2 u}{\partial z^2} = -\left[\frac{\gamma}{\mu U}\right] h_{xxx} - \left(\frac{\rho g R^2}{\mu U}\right)\sin\alpha$$

$$= -\frac{1}{Ca} h_{xxx} - \frac{1}{Ca}\underbrace{\left(\frac{\rho g}{\gamma} R^2\right)}_{\text{Bo, Bond Number}}\sin\alpha$$

Capillary Number

Still retain horizontal and vertical ratios

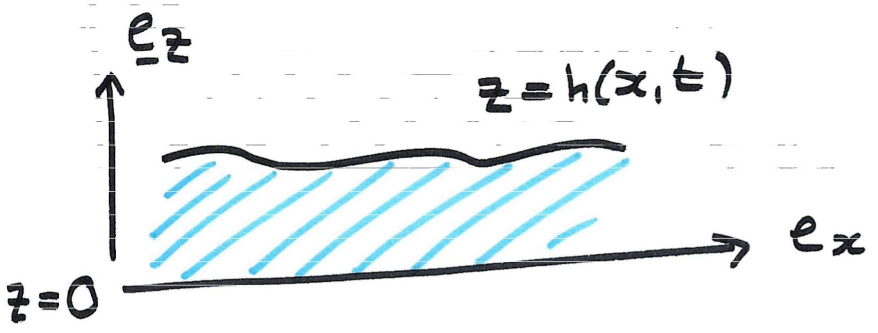
$$\therefore \quad u = \frac{-z(z-2h)}{2Ca}\left(h_{xxx} + Bo\sin\alpha\right)$$

$$\bar{u} = \frac{1}{h}\int_0^h u\,dz = \frac{h^2}{3Ca}\left(h_{xxx} + Bo\sin\alpha\right)$$

$$\therefore \quad 0 = h_t + \frac{\partial}{\partial x}(h\bar{u}) = \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\frac{h^3}{3Ca}\left(h_{xxx} + Bo\sin\alpha\right)\right)$$

gives evolution of free surface.

Free surface flow on a horizontal plane



We no longer have $|h_x| \ll \tan\alpha$. Thus,

$$\mu \frac{\partial^2 u}{\partial z^2} = -\gamma h_{xxx} + \rho g h_x$$

$\cos\alpha = 1$ here.

After rescaling

$$\therefore \quad u = -\frac{z(z-2h)}{2Ca}\left(h_{xxx} - Bo\, h_x\right)$$

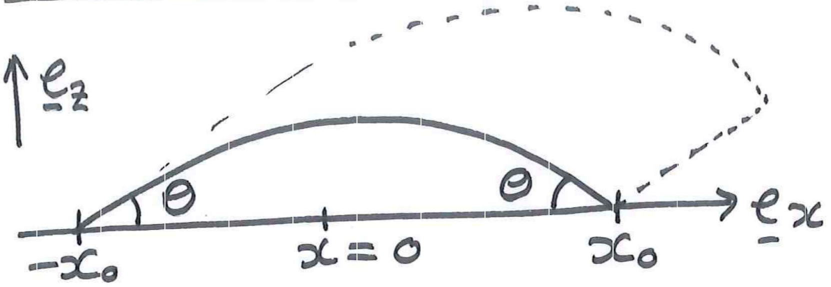
$$Bo = \frac{\rho g}{\gamma} R^2$$

$$\therefore \quad \bar{u} = \frac{h^2}{3Ca}\left(h_{xxx} - Bo\, h_x\right)$$

and hence

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\frac{h^3}{3Ca}\left(h_{xxx} - Bo\, h_x\right)\right) = 0$$

Finite fluid drop on a horizontal plate



2D drop, $\theta \ll 1$

We assume

- Drop is static
- drop is symmetric about $x=0$
- $h(x_0) = h(-x_0) = 0$
- $h_x(x_0) = -\tan\theta \simeq -\theta$, $h_x(-x_0) \simeq \theta$.
- $\displaystyle\int_{-x_0}^{x_0} h\,dx = 2\int_0^{x_0} h\,dx = A$.

Find x_0 given A, θ

Since $h_t = 0$,

$$\frac{h^3}{3Ca}\left(h_{xxx} - B_o h_x\right) = \text{Constant} = 0$$

Since the drop is of finite extent, at some point we must have h=0.

$$\therefore\ h_{xxx} - Bo\, h_x = 0$$

$$\therefore\ h = a + \underbrace{\cancel{b}}_{\text{by parity}}\!{}^{0}\sinh\left(x/l_c\right) + c\cosh\left(x/l_c\right),$$

$$l_c = 1/\sqrt{Bo}.$$

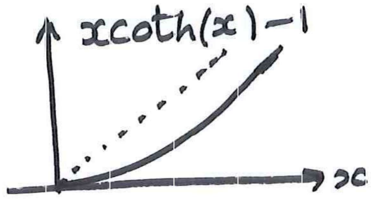
$h_x(x_0) = -\theta$ gives

$$c = \frac{-\theta l_c}{\sinh(x_0/l_c)}$$

$h(x_0) = 0$ gives $\quad a = -c\cosh\left(\frac{x_0}{l_c}\right) = \theta l_c \coth\left(\frac{x_0}{l_c}\right)$

$h_x(-x_0) = \theta$ holds by parity.

Also,
$$\frac{A}{2} = \int_0^{x_0} h \, dx$$
$$= \int_0^{x_0} a + c\cosh(x/l_c) \, dx$$
$$= \theta l_c^2 \left[\frac{x_0}{l_c} \coth(x_0/l_c) - 1 \right]$$



$\therefore$ For a given area A, there is a unique drop width, x_0.

<u>Radial drop</u> Static, $Bo \ll 1$. On-line notes for more detail.

$p = -\kappa \simeq \nabla^2 h$

$|h_x|, |h_y| \ll 1$

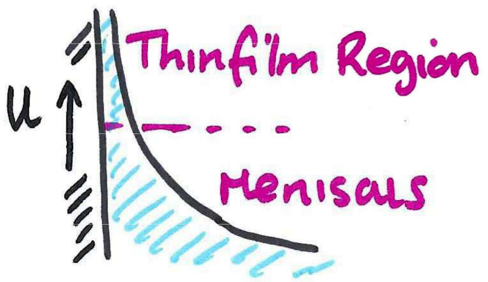
$$u_{zz} = -\frac{1}{Ca}\frac{\partial}{\partial r}(\nabla^2 h)$$

$$\therefore \quad h\bar{u} = \frac{-h^3}{3Ca}\frac{\partial}{\partial r}(\nabla^2 h)$$

$\partial h / \partial t = 0$ as static $\therefore \quad \frac{\partial}{\partial r}(\nabla^2 h) = 0$, as previously.

$\therefore \quad h_{rrr} - \frac{1}{r^2} h_r + \frac{1}{r} h_{rr} = 0$ with solution
$$h = A(r_0^2 - r^2).$$

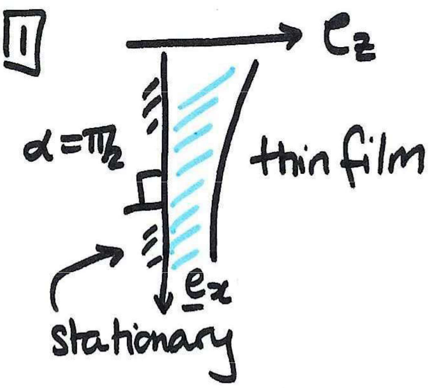
2.3 The Landau-Levich Problem, Dip Coating



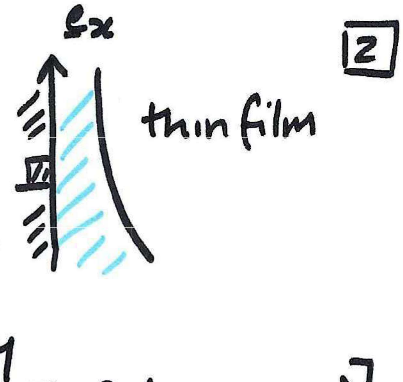
Industrial coating: draw a sheet at constant speed U from a fluid bath.

Non-dimensionalise such that $U=1$.

To use previous work, for thin film region

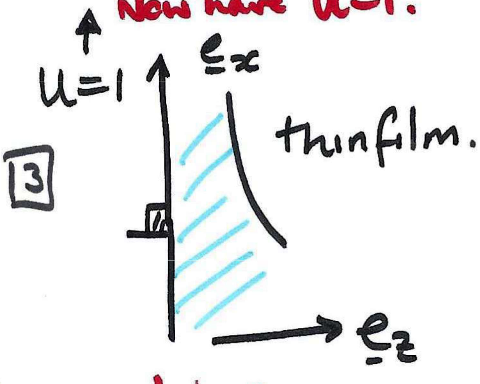


$x \to -x$



$$h_t + \frac{\partial}{\partial x}\left[\frac{h^3}{3Ca}\left(h_{xxx}+B_o\right)\right] = 0 \longrightarrow h_t + \frac{\partial}{\partial x}\left[\frac{h^3}{3Ca}\left(h_{xxx}-B_o\right)\right] = 0.$$

Now have $U=1$.



Jump into moving frame,

$$\bar{x} = x - t, \quad \bar{t} = t, \quad \bar{h}(\bar{x},\bar{t}) = h(x,t).$$

Same equation (as still an inertial frame)

$$\therefore \quad \bar{h}_{\bar{t}} + \frac{\partial}{\partial \bar{x}}\left[\frac{\bar{h}^3}{3Ca}\left(\bar{h}_{\bar{x}\bar{x}\bar{x}} - B_o\right)\right] = 0$$

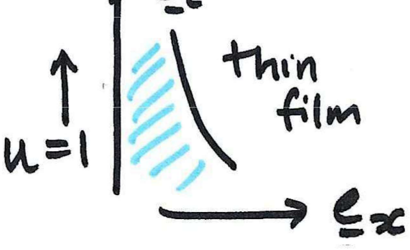
by rewriting in terms of x,t

Transform back to lab frame: $h_t + h_x + \dfrac{\partial}{\partial x}\left[\dfrac{h^3}{3Ca}\left(h_{xxx}-B_o\right)\right] = 0.$

---

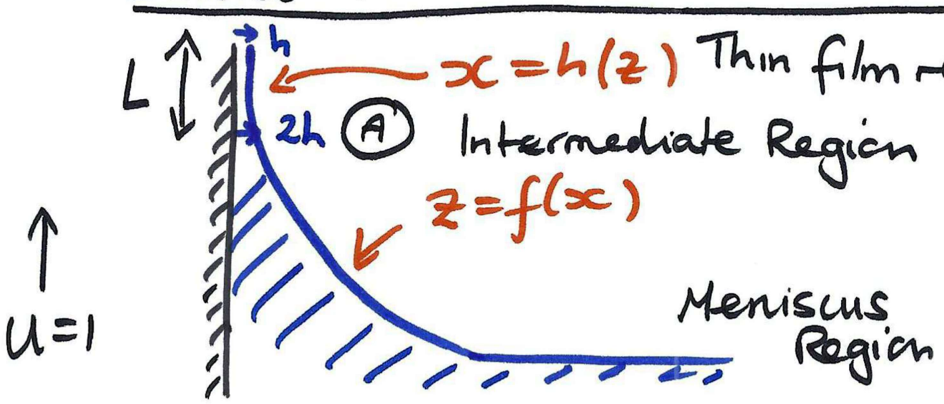
4 $(x,z) \to (z,x)$

Equations for thin film region



$$h_t + h_z + \frac{\partial}{\partial z}\left[\frac{h^3}{3Ca}\left(h_{zzz}-B_o\right)\right] = 0.$$

Scales for Landau-Levich Problem $\frac{\mu U}{\gamma} \ll 1$



Thin Film Equations

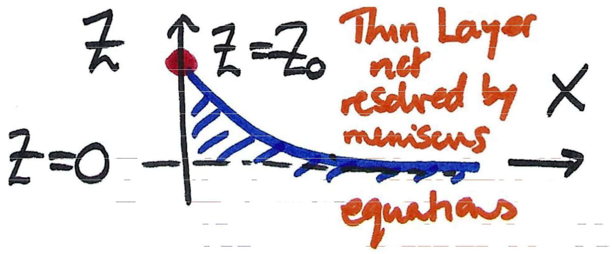
$$\cancel{h_t}^{\,0 \text{ as steady}} + h_z + \partial/\partial z \left[\frac{h^3}{3Ca}\left(h_{zzz} - Bo\right)\right] = 0$$

unsteady — plate velocity — $\sim \mu U/\gamma$ — surface tension — gravity

Leaving thin film region, h becomes large; h^3 dominate ... suggests meniscus region is a balance between surface tension & gravity

Choose R in Bond Number $Bo = \frac{\rho g}{\gamma} R^2$ to be $R = \sqrt{\frac{\gamma}{\rho g}}$

Then $Bo = \rho g/\gamma \; \gamma/\rho g = 1$.



Use l_c to non-dim

f, κ now non-dim

Meniscus

$$\rho g f = \gamma \kappa$$

$$\rho g l_c f = \gamma \kappa / l_c$$

$$\therefore \quad \kappa = (\rho g/\gamma) l_c^2 f = f$$

$$X = H(Z)$$

non-dimensionalised wrt l_c

$$\boxed{\begin{array}{c} \dfrac{H_{ZZ}}{(1+H_Z^2)^{3/2}} = Z \\ H_Z \to -\infty,\ H \to 0 \text{ as } Z \to 0 \\ H \to 0,\ H_Z \to 0 \text{ as } Z \to Z_0 \\ Z_0 \text{ unknown} \end{array}}$$

Meniscus is an outer solution in the framework of matched asymptotics & the thin film an inner solution

Solve for H : $\dfrac{H_Z}{(1+H_Z^2)^{1/2}} = Z^2/2 + \underbrace{\text{const}}_{H_Z \to 0 \text{ as } Z \to Z_0} = \dfrac{Z^2 - Z_0^2}{2}$

$$\therefore H_z = \frac{z^2 - z_0^2}{\left(4 - (z^2 - z_0^2)^2\right)^{1/2}}$$

Solve for H_z; sign choice via $H_z < 0$.

$$\text{As } z \to 0, \quad H_z \simeq \frac{-z_0^2}{(4 - z_0^4)^{1/2}} \to -\infty \quad \underbrace{\therefore z_0 = \sqrt{2}}_{z_0 > 0}$$

assumed; not coating in mercury

$$\therefore H(z_0) = 0, \quad H_z(z_0) = 0$$

$$H_{zz}(z_0) = z_0 \left(1 + [H_z(z_0)]^2\right)^{3/2} = \sqrt{2}.$$

$$\therefore H \simeq \frac{(z - z_0)^2}{\sqrt{2}} = \frac{(z - \sqrt{2})^2}{\sqrt{2}} \text{ as } z \to z_0 = \sqrt{2}$$

Can solve for H, but this will suffice.

Back to thin film equations

With $R=\sqrt{\frac{\gamma}{\rho g}}$, $Bo=1$,

$$h_t + \left(\frac{h^3}{3Ca}\left(h_{zzz}-1\right)\right)_z = 0$$

Let $z = z_0 + \epsilon \bar{Z}$ to blow up near z_0

$$H = \frac{(Z-Z_0)^2}{\sqrt{2}} + \dots = \frac{\epsilon^2 \bar{Z}^2}{\sqrt{2}} + \dots$$

which has been rescaled by $l_c \equiv R$

meniscus scaling

thin film scaling

$\therefore$ Can directly equate as they are equal.

$\therefore$ Let $h = H = \epsilon^2 \bar{h}$

$$z = Z = z_0 + \epsilon \bar{Z}$$

thin film variables

meniscus variables

$$\therefore \quad 0 = \epsilon \bar{h}_{\bar{z}} + \left(\frac{\epsilon^6 \bar{h}^3}{3Ca} \left(\frac{1}{\epsilon} \bar{h}_{\bar{z}\bar{z}\bar{z}} - 1 \right) \right)_{\bar{z}} \frac{1}{\epsilon}$$

$$0 = \bar{h}_{\bar{z}} + \left(\frac{\epsilon^3 \bar{h}^3}{3Ca} \left(\bar{h}_{\bar{z}\bar{z}\bar{z}} - \epsilon \right) \right)_{\bar{z}}$$

Let $\epsilon = (Ca)^{1/3}$ to give $0 = \bar{h}_{\bar{z}} + \left(\frac{\bar{h}^3}{3} \left(\bar{h}_{\bar{z}\bar{z}\bar{z}} - \epsilon \right) \right)_{\bar{z}}$

Leading order

$$0 = \bar{h}_{\bar{z}} + \partial/\partial \bar{z} \left(\bar{h}^3/3 \, \bar{h}_{\bar{z}\bar{z}\bar{z}} \right)$$

$\bar{h} \to \bar{h}_0$ asymptotic thickness as $\bar{z} \to \infty$

$\bar{h} \sim \frac{1}{\sqrt{2}} \bar{z}^2$ as $\underbrace{H \sim \frac{1}{\sqrt{2}} (z - z_0)^2 \text{ near } z = z_0}_{\text{on meniscal scale}}$

$\therefore \bar{h} \sim \frac{1}{\sqrt{2}} \bar{z}^2$ on a much longer lengthscale

$\therefore \bar{h} \to +1/\sqrt{2}\, \bar{z}^2$ for $\bar{z} < 0$, $|\bar{z}| \gg 1$

$\therefore$ Take $\bar{h} \to \bar{z}^2/\sqrt{2}$ as $\bar{z} \to -\infty$.

<u>Integrate</u> $\quad \bar{h} + \bar{h}^3/3 \, h_{zzz} = \text{const} = \bar{h}_0$

Let $\bar{h} = \bar{h}_0 g \quad \bar{z} = \bar{h}_0 \zeta$ Then

$$g + \frac{g^3}{3} g_{\zeta\zeta\zeta} = 1 \qquad \begin{aligned} &g \to 1 \text{ as } \zeta \to \infty \\ &g_\zeta \to 0 \text{ as } \zeta \to \infty \text{ (flat)} \end{aligned}$$

Solve numerically to find g and find

$$g \sim 0.67\,\zeta^2 \quad \text{as } \zeta \to -\infty$$

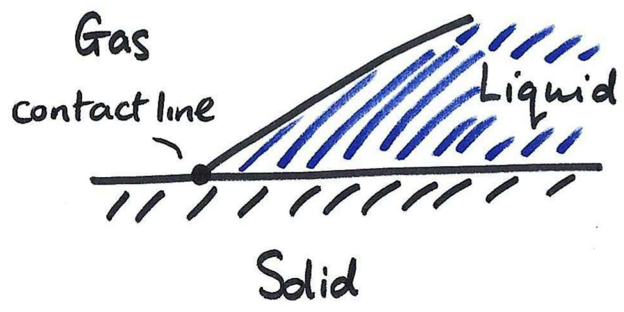
but $g \sim \dfrac{\bar{h}_0 \zeta^2}{\sqrt{2}}$ as $\zeta \to -\infty$

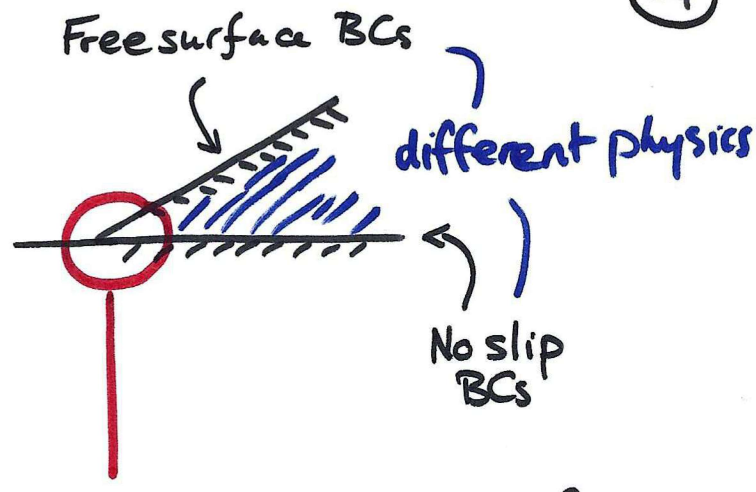
$$\therefore\ \bar{h}_0 = 0.67\sqrt{2} \simeq 0.948$$

$$\therefore\ (h_0)_{phys} = R\epsilon^2 \bar{h}_0 = R(Ca)^{2/3}\bar{h}_0$$

$$= \sqrt{\frac{\gamma}{\rho g}}\,(Ca)^{2/3}\,.\,(0.948)$$

Moving Contact Lines

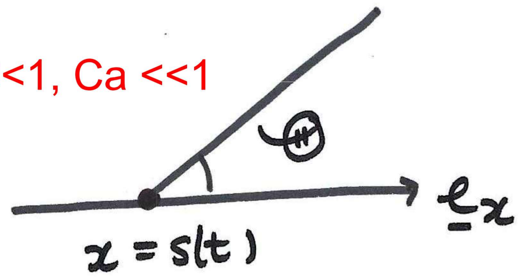




Close to the contact line, influence of free-surface physics and liquid-solid interface physics present... choice of BC ?

Try BC that previously worked for a static drop, once contact line is moving:

Bo <<1, Ca <<1



Scales such that $\boxed{Bo \ll 1}$ $\boxed{Ca \ll 1}$

See online notes for Bo significant, Ca not small.

At $x = s(t)$, $h = 0$, $h_x = \Theta$. Also no mass loss at interface.

Near contact point

$$h_t + \frac{\partial}{\partial x}\left[\frac{h^3}{3Ca}\left(h_{xxx}\right)\right] = 0 \qquad \boxed{Bo \ll 1}$$

Let $x = s(t) + \xi$, $h = \Theta\xi + f$, $h \geq 0$, $|f| \ll \Theta\xi$.

$$h_t - \dot{s}h_\xi + \frac{\partial}{\partial \xi}\left[\frac{h^3}{3Ca}h_{\xi\xi\xi}\right] = 0$$

$$f_t - \dot{s} f_\xi - \dot{s}\Theta + \frac{\partial}{\partial \xi}\left[\frac{\Theta^3 \xi^3}{3Ca} f_{\xi\xi\xi}\right] = 0$$

Scaling $f = Ca\, g$ consistent with $f \ll 1$, we have at leading order

$$\frac{\partial}{\partial \xi}\left[\frac{\Theta^3 \xi^3}{3} g_{\xi\xi\xi}\right] = \dot{s}\Theta$$

$$\therefore \quad g_{\xi\xi\xi} = \frac{3}{\Theta^2}\frac{1}{\xi^2}\dot{s} + \frac{\text{constant}}{\xi^3}$$

$$g = \frac{3\dot{s}}{\Theta^2}\left(\xi \ln \xi - \xi\right) + \text{constant} \cdot \ln \xi$$

+ A ξ^2 + B ξ + C

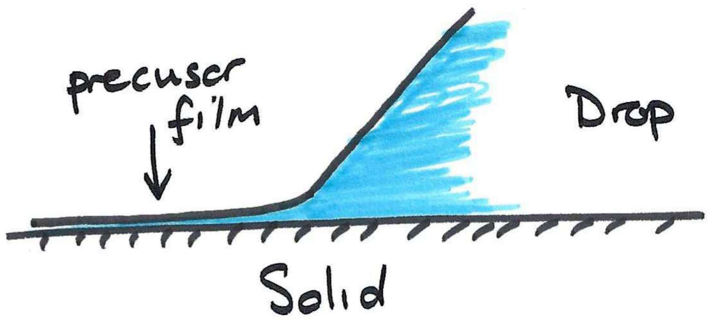
Note Problem only if $\dot{s} \neq 0$.

$\therefore \quad |Ca\, g| = |f| \not\ll \Theta \xi$ for ξ sufficiently small

No consistent solution on approaching contact line.

Potential resolutions

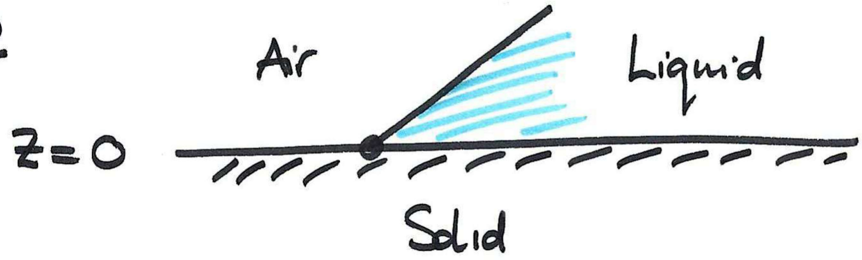
Precursor film



Solid has a precursor film e.g. if drop edge oscillating back and forth.

No contact line $\therefore$ No problem

Slip



slip length.

Rather than $u=0$ on $z=0$, impose $u=\lambda\dfrac{\partial u}{\partial z}$.

The thin film equation becomes ($Bo \ll 1$, $Ca \ll 1$)

See example sheets

$$h_t + \left[\frac{1}{Ca}\left(\frac{h^3}{3}+\lambda h^2\right)h_{xxx}\right]_x = 0.$$

With $h = \Theta\xi + f = \Theta\xi + Ca\,g$ as previously,

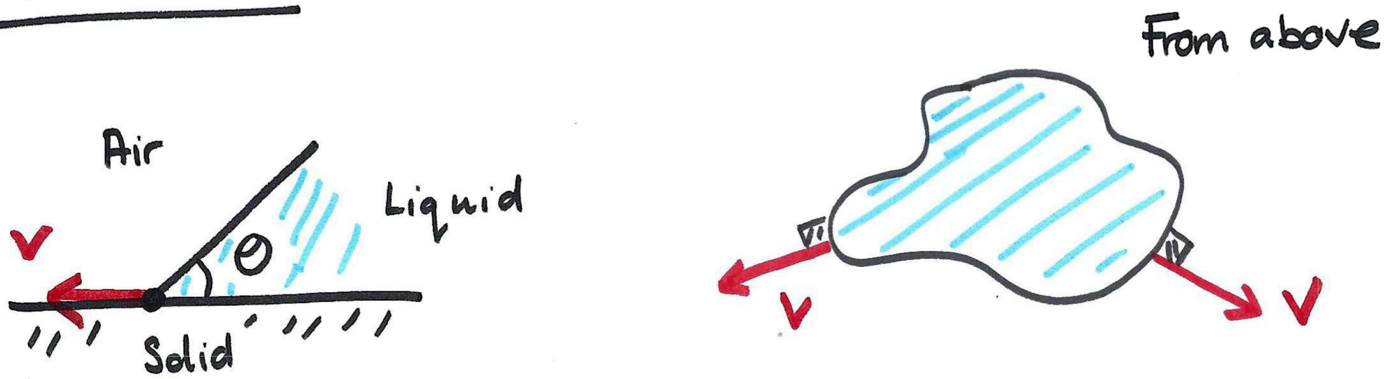
$$\lambda\Theta^2\xi^2 g_{\xi\xi\xi} \simeq \dot{s}\,\Theta\xi$$ on balancing dominant terms

$$\therefore \quad g \simeq \frac{\dot{s}}{\Theta\lambda}\,\xi^2\ln\xi$$

$\therefore \; |Ca\,g| = |f| \ll \Theta\xi$ for ξ sufficiently small, as required.

Tanner's Law

Empirical, based on observation



Normal velocity, $v = K(\theta^3 - \Theta^3)$

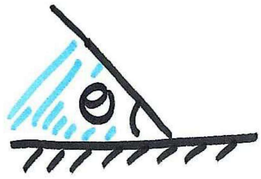
actual contact angle

Static contact angle, i.e. angle when drop stationary

For the thin film approximation

$$v = K\left(h_x^3 - \Theta^3 \right)$$

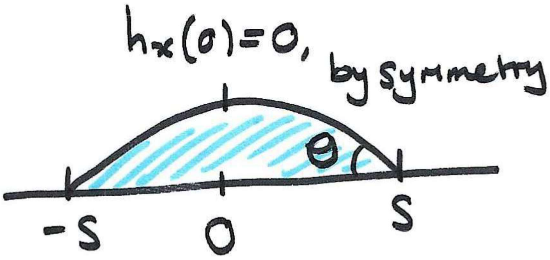
↳ for figure.

$-h_x^3$ for 

Quasi-static evolution with $Bo \ll 1$, $Ca \ll 1$

For a symmetric drop spreading via Tanner's law, where drop transients relaxed, so drop dynamics slow, i.e. $h_t \ll O\left(\frac{1}{Ca}\right)$, i.e. quasi-static.

Then $$\left[\frac{h^3}{3Ca} h_{xxx}\right]_x = 0$$



$h(s) = 0, \quad h_x(s) = -\Theta$

$h_x(0) = 0$

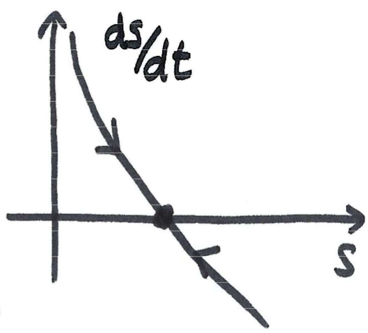
At s, $\dot{s} = -h_x^3 - \Theta^3$, Tanner's law, after non-dim, use K as velocity scale.

$$A = \int_{-s}^{s} h\,dx = 2\int_0^s h\,dx,$$ mass conservation

$\therefore h^3 h_{xxx} = \text{Constant} = 0$

$$\therefore \quad h = \frac{\theta}{2s}(s^2 - x^2), \qquad A = 2\int_0^s \frac{\theta}{2s}(s^2 - x^2)\,dx$$
$$= \tfrac{2}{3}\theta s^2.$$

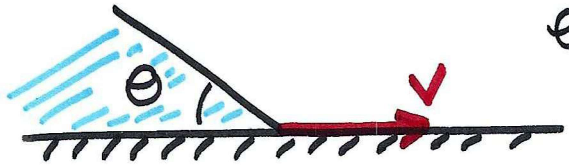
$$\therefore \quad \frac{ds}{dt} = \theta^3 - \theta_e^3 = \frac{27}{8}\frac{A^3}{s^6} - \theta_e^3$$



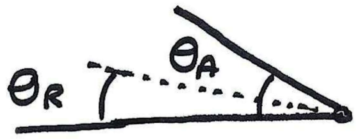
$$\therefore \quad s \to \left(\tfrac{3}{2}\right)^{1/2} \sqrt{\frac{A}{\theta_e}} \quad \text{as } t \to \infty$$

Advancing and receding contact angles

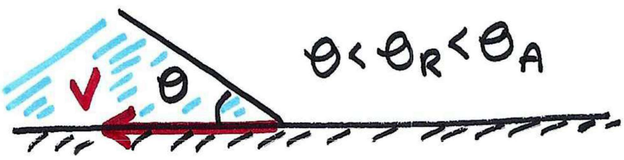
Advancing



$\theta > \theta_A > \theta_R$



Receding



$\theta < \theta_R < \theta_A$

In a generalisation of Tanner's law it is often observed that

$$V = \begin{cases} K_A(\theta^3 - \theta_A^3) & \theta > \theta_A \\ 0 & \theta_R < \theta < \theta_A \\ -K_R(\theta_R^3 - \theta^3) & \theta < \theta_R \end{cases}$$

Evaporating drops and coffee stains

Symmetric drop

- Slowly evaporating drop
- Drop is thin, $|h_x| \ll 1$.
- $Bo \ll 1$, gravity negligible
- $Ca \ll 1$
- Quasi-static $(h_t \simeq 0)$.
- Contact line does not move, s constant.

The drop is slowly evaporating:

$$\frac{d}{dt}\int_{-s}^{s} h\,dx = 2\frac{d}{dt}\int_0^s h\,dx = -2E\int_0^s (1+h_x^2)^{1/2}\,dx$$

E: evaporation rate per unit surface area, assumed constant and ord(1)

$$\therefore\ 2\frac{d}{dt}\int_0^s h\,dx = -2Es,$$ to leading order.

Fluid flow

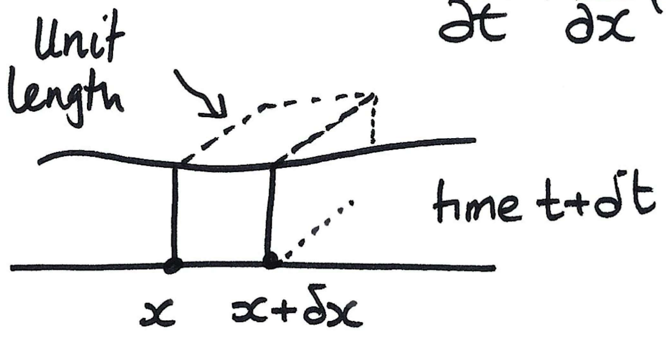
$h(s)=0,\ h_x(s) = -\theta,\ h_x(0)=0$ ✓ as previously

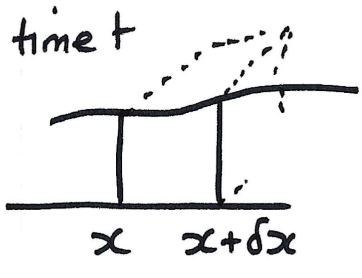
$$h_t + (h\bar{u})_x = h_t + \left(\frac{h^3}{3Ca}h_{xxx}\right)_x = -E$$

$\bar{u}$: average velocity

$-E$: sink due to evaporation.

Show $\dfrac{\partial h}{\partial t} + \dfrac{\partial}{\partial x}(\bar{u}h) = -E$





$$\delta x\, h(x,t+\delta t) - \delta x\, h(x,t)$$

$$= \underbrace{-E\,\delta x\,\delta t}_{\substack{\text{evaporation}\\ \text{per unit length}\\ \text{per unit time}}} + \underbrace{\int_0^{h(x,t)} \delta t\, u(x,\bar{z},t)\,d\bar{z}}_{\text{Amount entering/leaving on left}}$$

$$- \underbrace{\int_0^{h(x+\delta x,t)} \delta t\, u(x+\delta x,\bar{z},t)\,d\bar{z}}_{\text{Amount entering/leaving on right}}$$

$$\therefore\ -E\,\delta x\,\delta t = \delta x\,\delta t\,\frac{\partial h}{\partial t} + \delta t\,\delta x\,\frac{\partial}{\partial x}\int_0^{h(x,t)} u(x,\bar{z},t)\,d\bar{z}$$

+ higher orders

$$\therefore\ \boxed{-E = \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(h\bar{u}\right)}$$

<u>Recall $Ca \ll 1$</u> Let $q = h^2/_3\, h_{xxx}$ $\therefore$ $\bar{u} = q/Ca$

Let $h = h_0 + Ca\, h_1 + \cdots$ $\quad q = q_0 + Ca\, q_1 + \cdots$

Leading order $(h_0 q_0)_x = \left(h_0^3/_3\, h_{0xxx}\right)_x = 0$

As previously $h_{0xxx} = 0$ $\quad\therefore\ h_0 = \theta/_{2s}\,(s^2 - x^2)$

As previously $A = 2\int_0^s h\,dx$

$$\simeq 2\int_0^s h_0\,dx = 2\theta s^2/3$$

We also have $2\,d/dt\int_0^s h\,dx = \dot{A} = -2Es$

s constant

$$\therefore \quad h \simeq h_0 = \frac{3A(t)}{4s^3}(s^2 - x^2) \qquad h_{0t} = -\frac{3E}{2s^2}(s^2-x^2)$$

$h_0 h_{0xxx} = 0$

<u>Next order</u> $\quad \frac{\partial h_0}{\partial t} + \frac{\partial}{\partial x}(h_0 q_1 + h_1 q_0) = -E$

$-h_{0t}$

$\bar{u} = 0 = q/Ca$
at $x = 0$
by symmetry

$$\therefore \quad h_0 q_1 = \int_0^x -E + \frac{3E}{2s^2}(s^2 - x^2)\,dx$$

$$\therefore \quad q_1 = \frac{xE}{2s^2}(s^2 - x^2)\frac{1}{h_0} = \frac{xE(s^2-x^2)}{2s^2}\frac{1}{\frac{\theta}{2s}(s^2-x^2)}$$

$$= \frac{xE}{s\theta} = \frac{2xEs}{3A}$$

$$\therefore \quad \bar{u} \simeq \frac{1}{Ca}(Ca\,q_1) = \frac{2xEs}{3A}$$

$\therefore$ Average velocity is from centre to contact line.

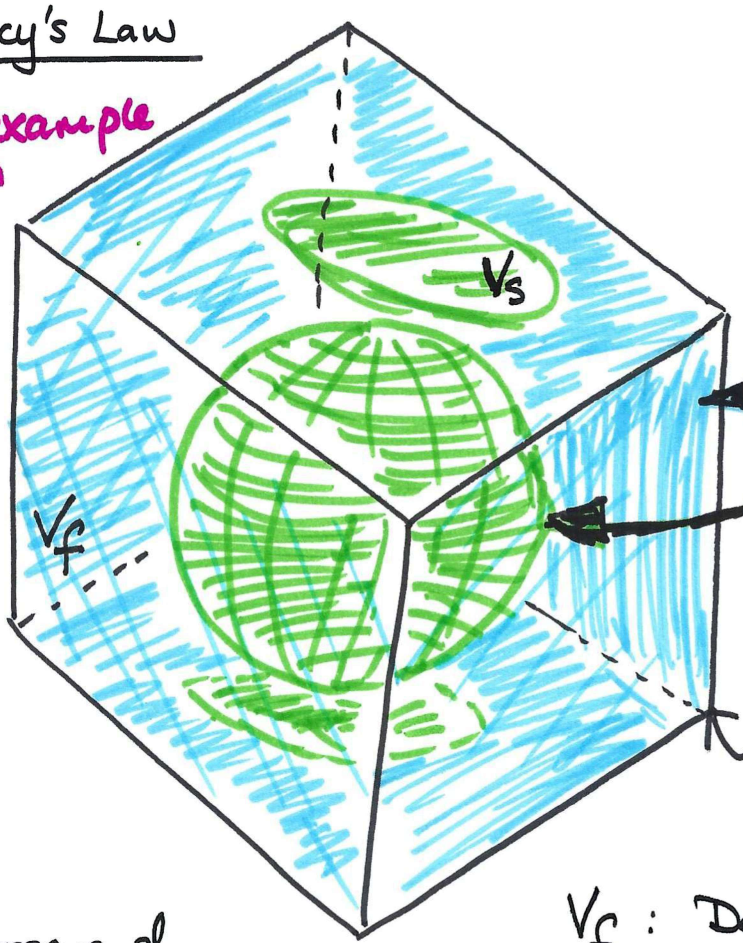
$\therefore$ For a drop of coffee, dissolved coffee carried to rim of drop, where it is deposited as liquid evaporates

$\therefore$ Coffee rings.

3 Porous Media & Thermal Convection

3.1 Darcy's Law

Typical example Flow in Porous Rock



Porous medium:

2 phases,
1 fluid,
1 rigid solid

Geometry: Repeating periodic, cuboidal unit, V.

V_s : Domain of solid within the unit

V_f : Domain of fluid within the unit.

Objective: Show that

For small Reynolds number, and fixed density ρ, the volume averaged flow, $\underline{u}$, satisfies

will define in detail below, but velocity field averaged over repeating unit.

$$\underline{u} = -k/\mu \nabla p - \frac{k}{\mu} \rho g \underline{e}_z . \qquad \text{Darcy's law}$$

permeability

viscosity

volume averaged pressure.

We also have $\quad 0 = \nabla \cdot \underline{u} = -k/\mu \, \nabla^2 p.$

We use homogenisation, starting from

simplifying equations by averaging

$$\rho \underline{u}_t + \rho \underline{u}.\nabla \underline{u} = -\nabla(\overbrace{p+\rho g z}^{p^*}) + \mu \nabla^2 \underline{u}$$

with $\underline{u}$ redefined to be the microscale velocity field (no averaging).

Similarly p is redefined analogously.

Boundary conditions $\underline{u} = 0$ on ∂V_s.

On cuboidal faces $\notin \partial V_s$, we have periodic boundary conditions.

Scales $\quad r_p =$ Pore lengthscale $\sim [\text{Volume}(V)]^{1/3}$

$L =$ lengthscale of pressure gradient driving flow

$\epsilon = r_p/L \ll 1$, typically

Large scale pressure gradient scale $\gg$ pore size as in porous rock.

Rescale p^* by P. For a balance

$$\frac{P}{L} \sim \mu U / r_p^2$$

With this pressure scaling, non-dimensionalisation with lengthscale r_p, timescale r_p/U, gives

$$Re(\underline{u}_t + \underline{u}.\nabla \underline{u}) = -\frac{1}{\epsilon}\nabla p^* + \nabla^2 \underline{u},$$

$$Re = \frac{\rho U r_p}{\mu}.$$

We have two lengthscales, r_p and L.

$\underline{x}$ is on the porescale. Let $\underbrace{\underline{X} = \epsilon \underline{x}}$, macroscale.

$\left.\begin{array}{r}\Delta x \sim 1\\ \text{then } \Delta X \ll 1\end{array}\right\} \therefore X$ larger scale

Write
$$\underline{u} = \underline{u}(\underline{x}, \underline{X}), \qquad p^* = p^*(\underline{x}, \underline{X})$$

$$\nabla_i \to \underbrace{\frac{\partial}{\partial x_i}}_{\text{microscale variation}} + \underbrace{\frac{\partial X_j}{\partial x_i}\frac{\partial}{\partial X_j}}_{\text{macroscale variation}} = \frac{\partial}{\partial x_i} + \epsilon \frac{\partial}{\partial X_i}$$

Separating these out by their scale wrt ϵ.

With
$$\underline{u} = \underline{u}^0 + \epsilon \underline{u}^1 + \cdots, \quad p^* = p^{*0} + \epsilon p^{*1} + \cdots$$

and
$$Re \sim O(\epsilon)$$

$$\therefore \quad \frac{\partial p^{*0}}{\partial x_i} = 0 \qquad \text{i.e.} \quad p^{*0} = p^{*0}(\underline{X})$$

$$\underbrace{\frac{\partial p^{*0}}{\partial X_i} \overset{①}{=} -\frac{\partial p^{*1}}{\partial x_i} + \frac{\partial^2 u_i^0}{\partial x_p \partial x_p} \quad \text{with} \quad \frac{\partial u_i^0}{\partial x_i} \overset{②}{=} 0 \text{ from } \nabla\cdot\underline{u} = 0}_{\text{Stokes Equation, forced by } \frac{\partial p^{*0}}{\partial X_i}}$$

$\therefore$ By linearity

$$u_i^0 = -\underbrace{K_{ij}(x, X)}_{\text{linear in forcing}} \frac{\partial p^{0*}}{\partial X_j}$$

With $$\langle f \rangle := \frac{1}{|V|}\int_{V_f} f \, dV_{\underline{x}}$$

Volume element in microscale variables i.e. $dx_1 dx_2 dx_3$

$$\langle u_i^0 \rangle = -\langle K_{ij} \rangle \frac{\partial p^{0*}}{\partial X_j}$$

<u>Incompressibility</u> $\nabla \cdot \underline{u} = 0$

$$\therefore \quad \frac{\partial u_i^0}{\partial x_i} = 0 \qquad \frac{\partial u_i^0}{\partial X_i} + \frac{\partial u_i^1}{\partial x_i} = 0$$

$$\therefore \quad \nabla \cdot \langle \underline{u}^0 \rangle = \frac{\partial}{\partial X_i}\left[\frac{1}{|V|}\int_{V_f} u_i^0(\underline{x}, \underline{X}) \, dV_{\underline{x}}\right]$$

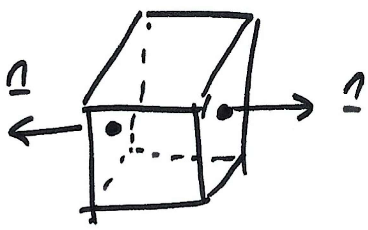
$$= \frac{1}{|V|}\int_{V_f} \frac{\partial u_i^0}{\partial X_i} \, dV_{\underline{x}}$$

$$\therefore \quad \nabla \cdot \langle \underline{u}^0 \rangle = -\frac{1}{|V|} \int_{V_f} \frac{\partial u_i^1}{\partial x_i} \, dV_{\underline{x}}$$, by incompressibility

$$= -\frac{1}{|V|} \left[\underbrace{\int_{\partial V_f \cap \partial V_s} n_i u_i^1 \, dS}_{1} + \overbrace{\int_{\partial V_f \cap \overline{\partial V_s}} n_i u_i^1 \, dS}^{2} \right]$$

Boundaries of cell which are not solid phase, i.e. boundaries of cell V.

1 $= 0$ as $\underline{u} = 0$ on $\partial V_f \cap \partial V_s$



2 $= 0$ by periodicity. On boundary $\underline{u}$ same at opposite point, by periodicity, but $\underline{n}$ changes sign. Hence no contribution.

$$\therefore \quad \nabla \cdot \langle \underline{u}^0 \rangle = 0$$

For $\langle K_{ij}(\underline{x}, \underline{X}) \rangle$ isotropic, we have $\langle K_{ij}(\underline{x}, \underline{X}) \rangle = \delta_{ij} K(\underline{X})$, for some $K(\underline{X})$.

Redefining $\underline{u} = \langle \underline{u}^0 \rangle$ and $p^* = p^{*0}(\underline{X})$ we have

$$\underline{u} = -K(\underline{X}) \nabla p^*$$ where $\nabla \equiv \nabla_{\underline{X}}$, the macroscale gradient.

Redimensionalisng

$$\underline{u} = -\frac{k}{\mu} \nabla p^* = -\frac{k}{\mu} \nabla (p + \rho g z).$$ Darcy's Law

Also $\nabla \cdot \underline{u} = 0$ and hence $0 = -\frac{k}{\mu} \nabla^2 p$

Finally,

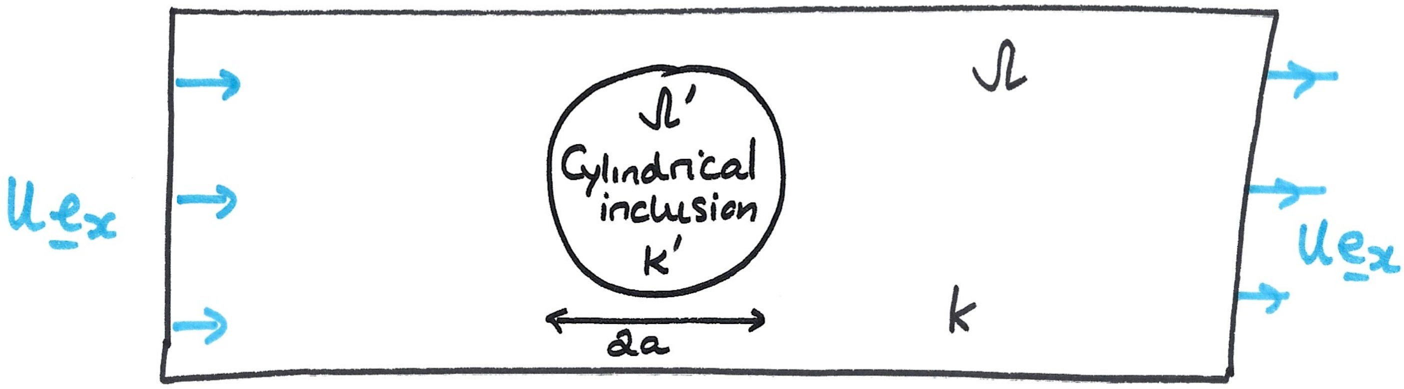
$$\underbrace{\underline{u}}_{u_{\text{macroscale}}} = \langle \underbrace{\underline{u}^{0}}_{\text{microscale}} \rangle = \frac{1}{|V|} \int_{V_f} \underline{u}^{(0)} \, dV_{\underline{x}}$$

If $\underline{u}^{(0)}$ constant $\underline{u}_{\text{macroscale}} = \frac{|V_f|}{|V|} \underline{u}^{(0)} = \varphi \underline{u}^{(0)}$

$$= \varphi \, \underline{u}_{\text{microscale}}.$$

Useful later.

Flow focussing



Consider a porous medium, permeability k, with a cylindrical inclusion, Ω', permeability k', radius a.

We assume

- k, k' constant
- No gravity
- $\underline{u} = U\underline{e}_x$ far from inclusion
- Problem is steady
- Darcy's law holds

$$\therefore \quad \underline{u} = -k/\mu \nabla p = -\nabla\left(\frac{kp}{\mu}\right), \qquad 0 = \nabla\cdot\underline{u} = -\nabla^2\left(\frac{kp}{\mu}\right).$$

At the boundary $\partial\Omega'$, pressure p and normal flux

$\underline{n}\cdot\underline{u} = -\underline{n}\cdot\nabla\left(kp/\mu\right)$ are continuous.

r, θ polars

Let $\varphi = -kp/\mu$. $\quad \underline{u} = \nabla\varphi \to U\underline{e}_x = U\cos\theta\,\underline{e}_r - U\sin\theta\,\underline{e}_\theta$

$$\therefore \quad \varphi \sim \cos\theta \quad \text{as } r \to \infty$$

$$\therefore \quad \varphi = \left(Ar + B/r\right)\cos\theta$$

since $\nabla^2\varphi = 0$, and WLOG origin at centre of inclusion.

$\boxed{A = U \text{ as } \underline{u} = \nabla\varphi \to U\underline{e}_x \text{ at } \infty}$

$$\therefore \quad \varphi = \begin{cases} (Ur + B/r)\cos\theta & r > a \\ (A_< r + B_</r)\cos\theta & r < a \end{cases}$$

$B_< = 0$ as $\varphi(r=0)$ finite.

$$0 = \left[p \right]_{a^-}^{a^+} = -\frac{\mu}{k}\varphi(a^+) + \frac{\mu}{k'}\varphi(a^-)$$

$$\therefore \quad \frac{1}{k}\left(Ua + B/a\right) = \frac{A_<}{k'} a$$

$$0 = \left[\underline{n} \cdot \nabla\varphi \right]_{a^-}^{a^+} = \left[\frac{\partial\varphi}{\partial r} \right]_{a^-}^{a^+} = \cos\theta \left[\left(U - B/a^2\right) - \left(A_<\right) \right]$$

$$\therefore \quad U - B/a^2 = A_< .$$

Solving for the unknowns $A_<$, B gives

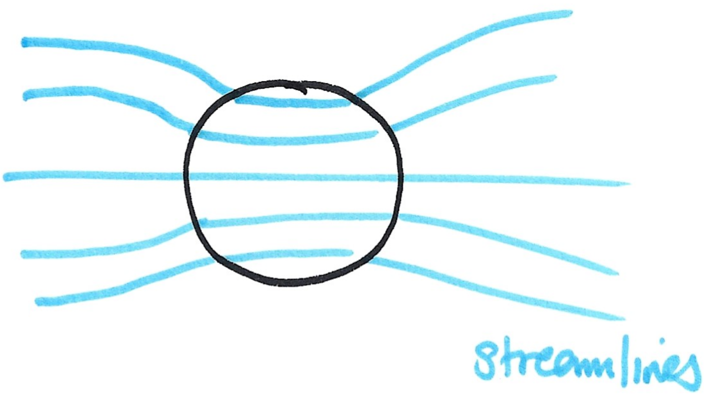
$$\varphi = \begin{Bmatrix} -kp/\mu & r > a \\ -k'p/\mu & r < a \end{Bmatrix} = U\cos\theta \begin{cases} r + \frac{(k-k')a^2}{(k+k')r}, & r > a \\ \frac{2k'}{k+k'} r, & r < a \end{cases}$$

As $k' \to 0$, $\quad \varphi \to U\cos\theta\,(r^2 + a^2/r)$, $\; r > a$,

the solution for flow around a cylinder.

For $k' \gg k$,

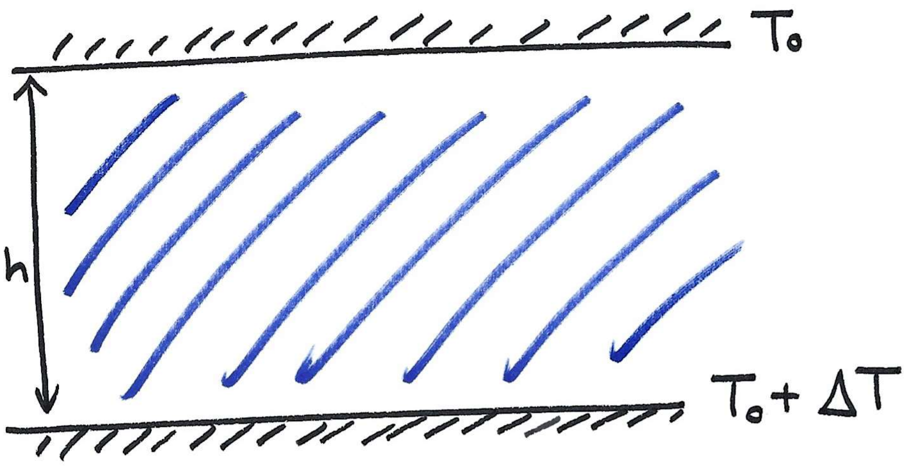
flow focussed by inclusion.



3.2 Thermal Convection

- We assume
  - the Boussinesq approx (variations in density only important for buoyancy)
  - $\rho = \rho_0 (1 - \beta (T - T_0))$ for the fluid density
    $\beta > 0$
  - specific heat capacities of solid and fluid in porous medium the same. Similarly for density

- We have



Porous layer, depth h, heated from below.

ΔT small enough to allow equilibrium, $\underline{u} = 0$, $\partial/\partial t = 0$.

$\therefore$ Heat equation $\frac{DT}{Dt} = \partial T/\partial t + \underline{u} \cdot \nabla T = \kappa_T \nabla^2 T$ reduces

to $\nabla^2 T_s = 0$, T_s for steady state.

$\therefore \quad T_s = T_0 + \Delta T (1 - z/h)$.

$\therefore \quad \rho_s = \rho_0 \left[1 - \beta\Delta T(1 - z/h)\right]$, steady state.

As $\quad 0 = -\dfrac{\partial p}{\partial z} - \rho g \quad$ at steady state

Vertical Darcy's law at steady state

$$P_s = P_0 - \rho_0 g\left[z - \beta\Delta T(z - z^2/2h)\right].$$

Linear instability. Perturb the steady state:

$$\underline{u} = \underline{0} + \underline{u}' \qquad T = T_s + T' \qquad p = p_s + p' \qquad \rho = \rho_s + \rho'$$

This requires the Boussinesq approx

We thus have $\nabla\cdot\underline{u}' = 0$ and from D'Arcy's law

i.e. $\quad \underline{u} = -\dfrac{k}{\mu}\nabla p - \dfrac{k}{\mu}\rho g \underline{e}_z$

we have $\quad \underline{u}' = -\dfrac{k}{\mu}\nabla p' - \dfrac{k}{\mu}\rho' g \underline{e}_z = -\dfrac{k}{\mu}\left(\nabla p' - \beta\rho_0 g T'\underline{e}_z\right)$

as $\rho' = -\beta\rho_0 T'$.

Heat Eqn $\quad \dfrac{\partial T'}{\partial t} + \underline{u}'\cdot\nabla T_s + \cancel{\underline{0}\cdot\nabla T'} = \kappa_T\nabla^2 T'$

T_s only has a z-dependence

$$\therefore \quad \frac{\partial T'}{\partial t} - w'\Delta T/h = \kappa_T \nabla^2 T$$

With $\quad \underline{x} = h\underline{X}, \quad \underline{u}' = \dfrac{\kappa_T}{h}\underline{U}, \quad t = \dfrac{h^2}{\kappa_T}\tau, \; P' = \dfrac{\mu\kappa_T}{k}P$

$T' = \Delta T\,\Theta$,

$$\nabla\cdot\underline{U} \overset{①}{=} 0, \qquad \underline{U} \overset{②}{=} -\nabla P + Ra\,\Theta\,\underline{e}_z$$

$$\frac{\partial\Theta}{\partial\tau} - W \overset{③}{=} \nabla^2\Theta, \quad \text{with } Ra = \frac{\beta\rho_0 g\Delta T k h}{\mu\kappa_T}$$

Rayleigh Number

From 2

$$\nabla \wedge (\nabla \wedge \underline{u}) = Ra \, \nabla \wedge (\nabla \wedge \Theta \underline{e}_z)$$ eliminates pressure

as $\nabla \cdot \underline{u} = 0$

$$-\nabla^2 \underline{u} = Ra \left[\Theta_{xz} \underline{e}_x + \Theta_{yz} \underline{e}_y - \nabla_H^2 \Theta \underline{e}_z \right]$$

dot with $\underline{e}_z$ where $\nabla_H^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

$$\therefore \quad \nabla^2 W = Ra \, \nabla_H^2 \Theta$$

<u>BCs</u> $W = 0 = \Theta$ at $z = 0, 1$ as perturbations zero there.

$$\therefore \text{ Try } \quad W = \underbrace{\sin(m\pi z)}_{\text{zero at } z=0,1} e^{\sigma t} e^{ipx} e^{iqy}, \quad m \in \{1, 2, \dots\}$$

$$\Theta = \Theta_m \sin(m\pi z) e^{\sigma t} e^{ipx} e^{iqy}.$$

Forms a Fourier expansion of W, Θ

$$\therefore \quad \sigma \Theta_m - 1 = -(m^2\pi^2 + \underbrace{p^2 + q^2}_{\alpha^2}) \Theta_m$$

$$-(m^2\pi^2 + \alpha^2) = -Ra \, \alpha^2 \Theta_m$$

$$\therefore \quad \Theta_m = \frac{m^2\pi^2 + \alpha^2}{Ra \, \alpha^2},$$

growth rate: $\sigma = \frac{1}{\Theta_m} - (m^2\pi^2 + \alpha^2) = \frac{Ra \, \alpha^2}{m^2\pi^2 + \alpha^2} - (m^2\pi^2 + \alpha^2)$

Instability $\sigma > 0$. $\sigma \downarrow$ as $m \uparrow$ $\therefore$ $m = 1$ most unstable Fourier mode.

For $m=1$, $\quad \sigma = \dfrac{Ra\,\alpha^2}{\pi^2+\alpha^2} - (\pi^2+\alpha^2)$

For α fixed, unstable if $Ra > Ra_c(\alpha)$ where

$$0 = Ra_c(\alpha)\left[\frac{\alpha^2}{\pi^2+\alpha^2}\right] - (\pi^2+\alpha^2)$$

$\therefore$ Varying over all α, unstable if

$$Ra > \min_{\alpha} Ra_c(\alpha) = \min_{\alpha}\left[\frac{(\pi^2+\alpha^2)^2}{\alpha^2}\right] = \underline{\underline{4\pi^2}}$$

Double - diffusive convection

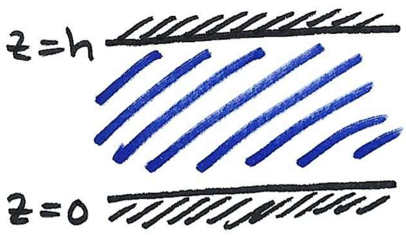
* Buoyancy where density depends on two quantities, e.g. salt and temperature

We consider

$$\rho = \rho_0\left[1 - \beta_T(T-T_0) + \beta_c(C-C_0)\right]$$

salt concentration, higher concentration, fluid denser.

Salt balance $\quad \dfrac{\partial C}{\partial t} + \underline{u}\cdot\nabla C = K_c\nabla^2 C$.



$C=C_0$, $T=T_0$.

Porous Layer

$C=C_0+\Delta C$

$T=T_0+\Delta T$

Static solutions

$$T_s = T_0 + \Delta T(1 - z/h)$$

$$C_s = C_0 + \Delta C(1 - z/h)$$

Hence $\rho_s = \rho_0 \left[1 + \{ \beta_c \Delta C - \beta_T \Delta T \} (1 - z/h) \right]$

with $0 = -\dfrac{\partial p}{\partial z} - \rho g$

<u>Linear stability</u> $\underline{u} = 0 + \underline{u}'$

$$\begin{pmatrix} T \\ C \\ p \end{pmatrix} = \begin{pmatrix} T_s \\ C_s \\ p_s \end{pmatrix} + \begin{pmatrix} T' \\ C' \\ p' \end{pmatrix}$$

so that $\nabla \cdot \underline{u}' = 0$ $\quad \underline{u}' = -\dfrac{k}{\mu} \left[\nabla p' + \rho_0 g (\beta_c C' - \beta_T T') \underline{e_z} \right]$.

Analogous to previously $\dfrac{\partial}{\partial t} \begin{pmatrix} T' \\ C' \end{pmatrix} - \dfrac{w'}{h} \begin{pmatrix} \Delta T \\ \Delta C \end{pmatrix} = \begin{pmatrix} \kappa_T \nabla^2 T' \\ \kappa_c \nabla^2 C' \end{pmatrix}$.

Non-dimensionalise, as previously, with $C' = \Delta C \cdot \chi$.

$\nabla \cdot \underline{u} = 0$, $\quad \underline{u} = -\nabla P + \left(Ra\, \Theta - \dfrac{Ra_s}{Le} \chi \right) \underline{e_z}$

where $\underset{\sim}{Ra_s} = \dfrac{\beta_c \rho_0 g \Delta C k h}{\mu \kappa_c}$, $\underset{\sim}{Le} = \kappa_T / \kappa_c$

Solutal Rayleigh number

Lewis Number

Note $Ra_s = Ra \cdot Le \cdot N$

As previously

$$\frac{\partial \Theta}{\partial \tau} - W \overset{\boxed{1}}{=} \nabla^2 \Theta, \quad Le\left(\frac{\partial \chi}{\partial \tau} - W\right) \overset{\boxed{2}}{=} \nabla^2 \chi.$$

Taking the curl twice of $\underline{u} = -\nabla P + \left(Ra\,\Theta - \frac{Ra_s \chi}{Le}\right)\underline{e}_z$,

using $\nabla \cdot \underline{u} = 0$, gives

$$\nabla^2 W \overset{\boxed{3}}{=} Ra\left(\nabla_H^2 \Theta - N \nabla_H^2 \chi\right)$$

$$\nabla_H^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2, \quad N = \frac{\beta_C \Delta C}{\beta_T \Delta T}.$$

Also, $W = \Theta = \chi = 0$ at $z = 0, 1$.

Let $f_m := \sin(m\pi z)\exp(\sigma t)\exp(ipX + iqY)$.

As before, let $W = f_m$ (to find Fourier mode)

$\Theta = \theta_m f_m, \quad \chi = \chi_m f_m.$

From $\boxed{1}$ $\quad \sigma\theta_m - 1 = -g_m \theta_m, \quad g_m := m^2\pi^2 + p^2 + q^2 = m^2\pi^2 + \alpha^2.$

$$\therefore \theta_m = \frac{1}{\sigma + g_m}$$

From $\boxed{2}$ $\quad Le(\sigma\chi_m - 1) = -g_m \chi_m$

$$\therefore \chi_m = \frac{Le}{\sigma Le + g_m}$$

From $\boxed{3}$ $\quad -g_m = Ra(-\alpha^2 \theta_m + N\alpha^2 \chi_m)$

$$= -\frac{Ra\,\alpha^2}{\sigma + g_m} + \frac{\boxed{Ra\,N\,Le}\,\alpha^2}{\sigma Le + g_m}$$

Ra_s

$$\therefore\ g_m(\sigma+g_m)(\sigma Le+g_m) = Ra\,\alpha^2(\sigma Le+g_m) - Ra_s\,\alpha^2(\sigma+g_m)$$

$$\underbrace{Le\, g_m}_{A}\sigma^2 + B\sigma + \underbrace{C}_{g_m^3 - Ra\, g_m\alpha^2 + Ra_s g_m\alpha^2} = 0$$

Note $A = Le\, g_m = Le(m^2\pi^2+\alpha^2) > 0.$

$$2A\,\sigma_\pm = -B \pm (B^2-4AC)^{1/2}.$$

Guaranteed instability if $C<0$, as then $\sigma_+ > 0$.

$$\therefore\ g_m\left(g_m^2 - (Ra-Ra_s)\alpha^2\right) < 0$$

$$\therefore\ (Ra-Ra_s) > g_m^2/\alpha^2 = \frac{(\pi^2m^2+\alpha^2)^2}{\alpha^2}.$$

Once $(Ra-Ra_s) > \min\limits_{\alpha}\ \min\limits_{m\in\{1,2,\dots\}} \dfrac{(\pi^2m^2+\alpha^2)^2}{\alpha^2}$

$$= \min_\alpha \frac{(\pi^2+\alpha^2)^2}{\alpha^2} = 4\pi^2, \text{ as before}$$

guaranteed an instability.

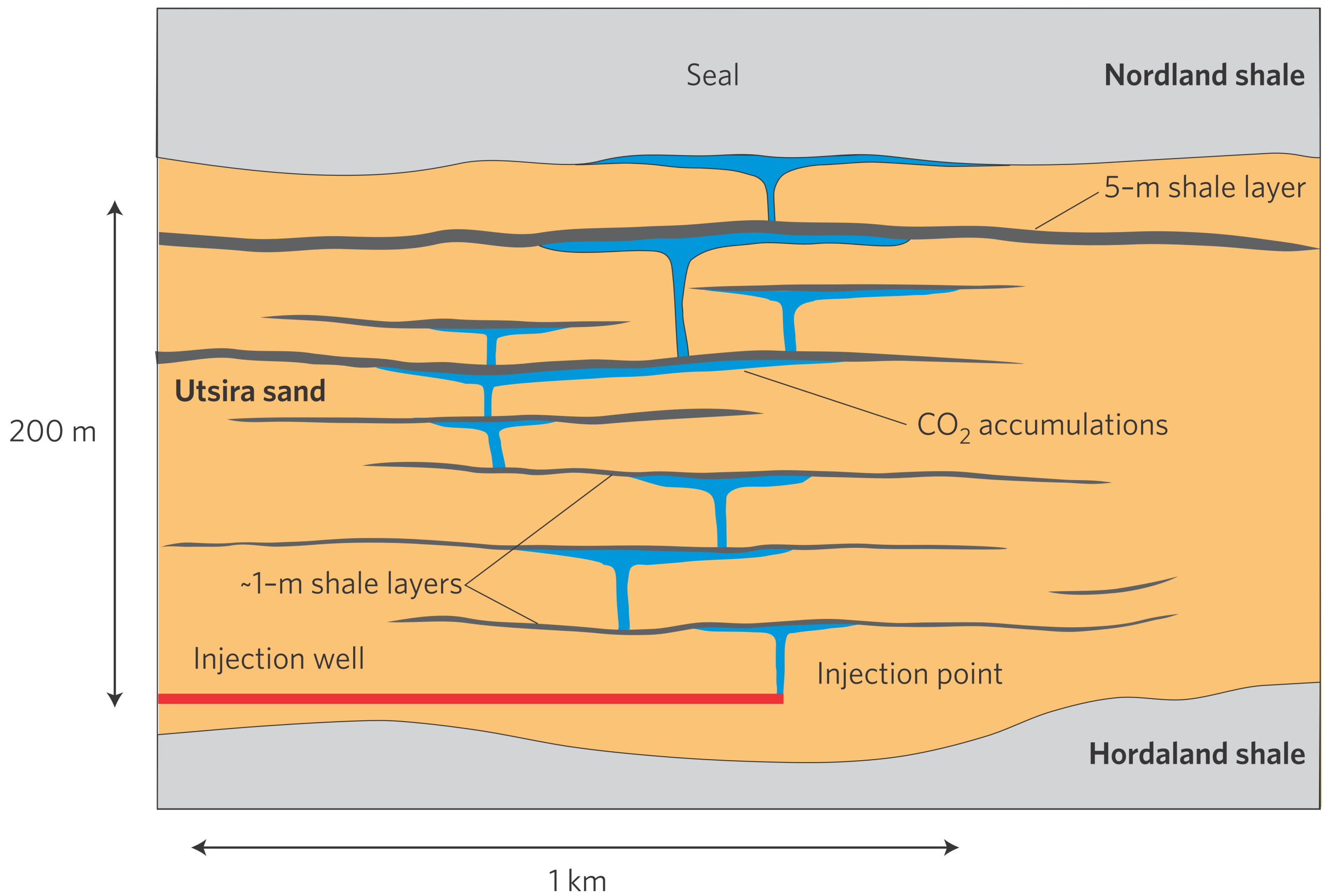
Other instabilities possible depending on parameter values.

See example sheet for details

Geological carbon storage

Mike J. Bickle

Storage of the carbon dioxide that is produced by burning fossil fuels is one way to avoid the damaging consequences of climate change. A range of observations suggests that geological carbon storage is much less risky than unabated carbon emissions to the atmosphere.

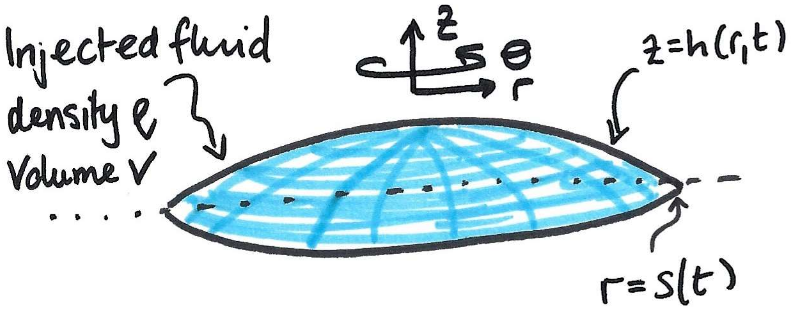


Horizontal gravity driven waves

* Finite volume of fluid, V, released above an impermeable boundary within a porous medium.

Porous medium analogue of spreading drop (no surface tension). Related to carbon storage.

Assumptions and scalings



- Axisymmetric flow
- Vertical scale H
- Horizontal scale R
- $\delta = H/R \ll 1$

- Velocity field $\underline{u} = u\underline{e_r} + w\underline{e_z}$, $u \sim U$, $w \sim H$.
- Conservation of mass $\quad U/R \sim W/H \quad \therefore W \sim \delta U$
- D'Arcy (horizontal) $\quad \dfrac{\partial p}{\partial r} = -\dfrac{\mu}{k} u \quad \therefore P \sim \mu/k \, RU.$

pressure scaling

- D'Arcy (vertical)

$$\frac{\mu W/k}{\partial p/\partial z} \sim \frac{\mu W/k}{P/H} \sim \frac{\mu/k \, U\delta}{\mu/k \, UR/H} \sim \delta^2$$

$$\therefore \quad \frac{\partial p}{\partial z} = -\rho g \quad \text{at leading order.}$$

Evolution of the flow profile

$$p(h,r) - p(z,r) = \int_z^h \frac{\partial p}{\partial z_*}\,dz_* = \int_z^h -\rho g\,dz_* = -\rho g(h-z)$$

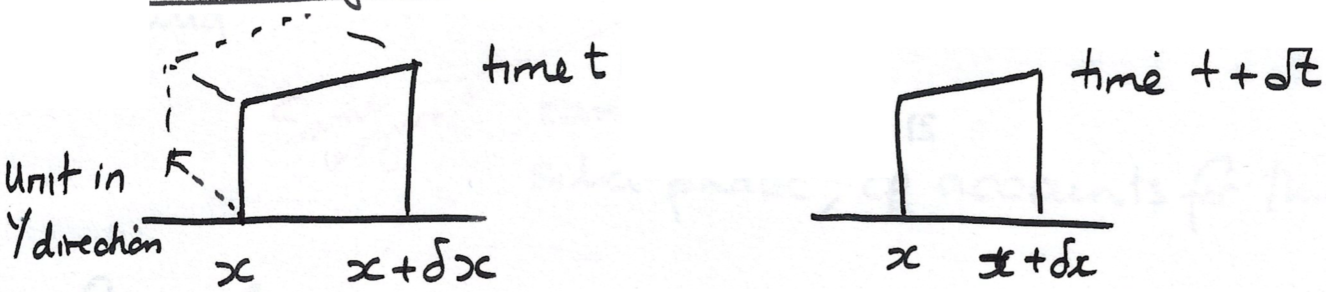
Take $p(h,r) \simeq p_0$ (const, a simplifying approximation)

$$p(z,r) = p_0 + \rho g(h-z).$$

Radial Darcy's Law

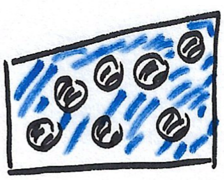
$$u_{\text{macroscale}} = u = -\frac{k}{\mu}\frac{\partial p}{\partial r} = -\frac{k}{\mu}\rho g \frac{\partial h}{\partial r}$$, independent of z.

Conservation of mass



$$\left.\begin{array}{l}\text{Volume of fluid}\\ \text{entering } (x, x+\delta x)\\ \text{in time } (t, t+\delta t)\end{array}\right\} = \frac{V_f}{|V|}\left[h(x,t+\delta t)\delta x - h(x,t)\delta x\right]$$

$$= \varphi\left[\frac{\partial h}{\partial t}\,\delta x\,\delta t\right]$$



$$\left.\begin{matrix}\text{Volume of fluid}\\ \text{entering}\end{matrix}\right\} = \delta t \int_0^{h(x,t)} u_{macro}(x,t,\bar{z})\,d\bar{z} - \delta t \int_0^{h(x+\delta x,t)} u_{macro}(x+\delta x,t,\bar{z})\,d\bar{z}$$

as previously

$$= -\delta t\,\delta x \frac{\partial}{\partial x}\int_0^{h} u_{macro}(x,t,\bar{z})\,d\bar{z}$$

$$\therefore \quad \varphi \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(h \left\{ \frac{1}{h}\underbrace{\int_0^{h} u_{macro}(x,t,\bar{z})\,d\bar{z}}_{-k/\mu\, \rho g\, \partial h/\partial x} \right\} \right) = 0$$

independent of z

Polars

$$\therefore \quad \varphi \frac{\partial h}{\partial t} = -\frac{k\rho g}{\mu}\frac{1}{r}\frac{\partial}{\partial r}\left(rh\,\partial h/\partial r\right)$$

Volume of fluid

$$V = 2\pi\varphi \int_0^{s(t)} rh\,dr$$

some of volume is solid phase, φ accounts for this.

Rescale with lengthscale $\left(V/\varphi\right)^{1/3}$

timescale $\left(\varphi^2 V\right)^{1/3}\dfrac{\mu}{k\rho g}$

Then

$$\frac{\partial H}{\partial T} = \frac{1}{R}\frac{\partial}{\partial R}\left(RH\frac{\partial H}{\partial R}\right)$$

$$2\pi \int_0^{S(T)} RH \, dR = 1$$

$$s(t) = \left(V/\varphi\right)^{1/3} S(T), \quad H(S(T),T) = 0.$$

We seek long time behaviour, hence look for a similarity solution.

Consider $\quad R = \eta T^{\alpha} \qquad H(R,T) = T^{-\beta} X(\eta)$

Similarity variable, WLOG scaling with R as taking more R derivatives

$$\therefore \quad \frac{\partial X}{\partial T} = \frac{\partial \eta}{\partial T}\frac{dX}{d\eta} = -\frac{\alpha}{T}\eta\frac{dX}{d\eta}, \quad \frac{\partial X}{\partial R} = \frac{\partial \eta}{\partial R}\frac{dX}{d\eta} = \frac{1}{T^{\alpha}}\frac{dX}{d\eta}.$$

Volume constraint

$$\frac{1}{2\pi} = \int_0^{S(T)} RH \, dR = \int_0^{\eta_s} \eta T^{\alpha} \frac{1}{T^{\beta}} X(\eta) T^{\alpha} \, d\eta$$

with $S(T) = \eta_s T^{\alpha}$.

RHS must be independent of time, T $\therefore$ $2\alpha - \beta = 0$, $\underline{\underline{2\alpha = \beta}}$.

$$\therefore \quad R = \eta T^{\alpha}, \quad H = T^{-2\alpha} X$$

$$\frac{\partial H}{\partial T} = -\frac{2\alpha}{T^{1+2\alpha}} X - \frac{\alpha}{T^{1+2\alpha}} \eta \frac{dX}{d\eta}$$

$$R H \frac{\partial H}{\partial R} = (\eta T^{\alpha})\left(\frac{1}{T^{2\alpha}} \chi\right)\left(\frac{1}{T^{2\alpha}} \frac{1}{T^{\alpha}} \frac{d\chi}{d\eta}\right) = \frac{1}{T^{4\alpha}} \eta \chi \frac{d\chi}{d\eta}$$

$$\frac{\frac{\partial}{\partial R}\left(R H \frac{\partial H}{\partial R}\right)}{R} = \frac{\frac{1}{T^{4\alpha}} \frac{1}{T^{\alpha}} \frac{d}{d\eta}\left(\eta \chi \frac{d\chi}{d\eta}\right)}{\eta T^{\alpha}}$$

$\therefore$

$$-\left[2\alpha \chi + \alpha \eta \chi'\right] \frac{1}{T^{2\alpha+1}} = \frac{1}{T^{6\alpha}} \frac{1}{\eta} (\eta \chi \chi')'$$

$\therefore \quad 2\alpha + 1 = 6\alpha \quad \therefore \boxed{\alpha = 1/4, \ \beta = 1/2}$

$$\therefore \quad -\frac{1}{4}\Big[\underbrace{2\eta \chi + \eta^2 \chi'}_{(\eta^2 \chi)'}\Big] = (\eta \chi \chi')'$$

$$\therefore \quad -1/4\, \eta^2 \chi = \eta \chi \chi' + \text{Constant}$$

when $\eta = \eta_s, \ \chi = 0 \ \therefore$ Constant is zero

$$\therefore \quad \chi' = -\eta/4$$

$$\therefore \quad \chi = 1/8 (\eta_s^2 - \eta^2)$$

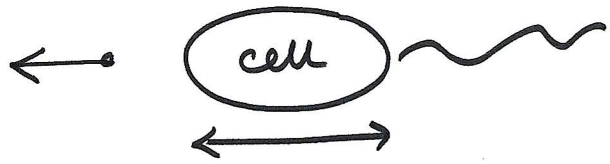
$$1 = 2\pi \int_0^{\eta_s} \eta \chi \, d\eta = \pi/16\, \eta_s^4 \quad \therefore \quad \eta_s = 2/\pi^{1/4}$$

$\therefore S(T) = \left(2/\pi^{1/4}\right) T^{1/4}$ gives extent of the fluid at large time.

4. Biofluid Flows: cilia and flagella

- Cilia and flagella are slender filaments on a cell, moved by molecular motors.
- Cilia actuate fluid, e.g. in the lung, and can also induce cell swimming.
- Flagella move cells.

Scales



$U \sim 100 \mu m\, s^{-1}$ $\quad L \sim 10 \mu m$

$\therefore$ $Re \sim 10^{-3}$, smaller for bacteria, for water.

$\therefore$ Stokes flow, $\quad \nabla \cdot \underline{u} = 0, \quad 0 = -\nabla p + \mu \nabla^2 \underline{u}$.

Force and Torque Free Swimming

Newton's 2nd for the cell, if it is swimming

$$M \ddot{\underline{X}} = \int_{cell} \underline{\underline{\sigma}} \cdot \underline{n} \, dS$$

M: cell mass $\quad$ $\underline{X}$ = cell centre of mass $\quad$ $\underline{\underline{\sigma}}$: Fluid stress tensor

$\rho_{cell} \doteq \rho_{water}$

$$R_s \sim \frac{|M \ddot{\underline{X}}|}{\left| \int_{cell} \underline{\underline{\sigma}} \cdot \underline{n} \, dS \right|} \sim \frac{\rho_{cell} L^3 \cdot L/T^2}{\mu \, U/L \, L^2}$$

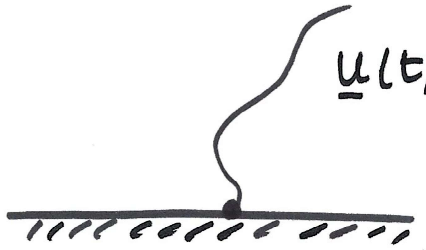
$$\sim \rho/\mu \, UL = Re \ll 1 \quad \therefore \int_{cell} \underline{\underline{\sigma}} \cdot \underline{n} \, dS \doteq 0$$

No net force.

No net torque by a similar argument.

Boundary conditions

Cilium



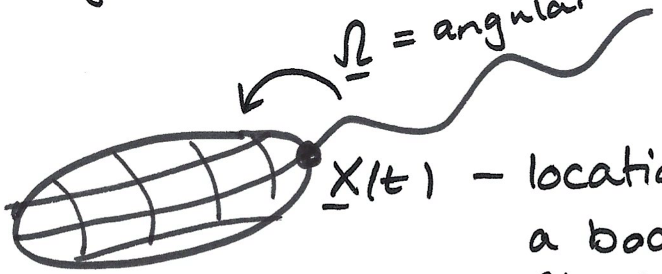
$\underline{U}(t,s)$, velocity of cilium at time t, arclength s.

$\therefore\ \underline{u} = \underline{U}(t,s)$ on cilium

& $\underline{u} = \underline{0}$ on substrate.

assumes cilium very thin so that its surface velocity is that of its centreline.

Flagellum & Swimming



$\underline{x}(t,p,q)$ parameterises cell surface

$$\underline{u} = \dot{\underline{X}}(t).$$

Then on the cell surface

$$\underline{u} = \underline{U}(t) + (\underline{x}(t,p,q) - \underline{X}(t)) \wedge \underline{\Omega}(t)$$

$$+ \begin{cases} \underline{u}_{\text{flagellum}}(t,p,q) & \text{on flagellum} \\ \underline{0} & \text{on cell body} \end{cases}$$

often given by $\underline{u}_{\text{flagellum}}(t,\underbrace{s}_{\text{arclength}})$

However the cell velocity, $\underline{U}(t)$, and angular velocity, $\underline{\Omega}(t)$, are a priori unknown.

These 6 degrees of freedom require 6 constraints: No net force and no net torque:

$$\int_{\text{Swimmer}} \underline{\underline{\sigma}} \cdot \underline{n} \, dS = \underline{0} = \int_{\text{Swimmer}} (\underline{x}(t,p,q) - \underline{X}(t)) \wedge \underline{\underline{\sigma}} \cdot \underline{n} \, dS$$

Purcell's scallop theorem not lectured

Purcell's Scallop Theorem

A time reversible swimming stroke (or flagellar beat pattern) generates no net motion.

Proof Suppose the flagellar beat is time reversible. Then there is $T > 0$ for which

$$\underline{u}_{\text{flagellum}}(t, p, q) = -\underline{u}_{\text{flagellum}}(2T - t, p, q) \qquad t \in [0, T].$$

i.e. flagellum moves for $t \in [0,T]$ and then reverses this motion for $t \in [T, 2T]$.

Let the cell swimming problem for $t \in [0,T]$ be solved by

$$\nabla p(t, \underline{x}), \ \underline{u}(t, \underline{x}), \ \underline{U}(t), \ \underline{\Omega}(t).$$

By linearity (and the absence of time derivatives) the solution for $t \in [T, 2T]$ given by

$$\underbrace{-\nabla p(2T-t, \underline{x})}_{\nabla p(t, \underline{x})}, \quad \underbrace{-\underline{u}(2T-t, \underline{x})}_{\underline{u}(t, \underline{x})}, \quad \underbrace{-\underline{U}(2T-t)}_{\underline{U}(t)}, \quad \underbrace{-\underline{\Omega}(2T-t)}_{\underline{\Omega}(t)}$$

these are also → $\nabla p(t, x)$ $\underline{u}(t, \underline{x})$ $\underline{U}(t)$ $\underline{\Omega}(t)$

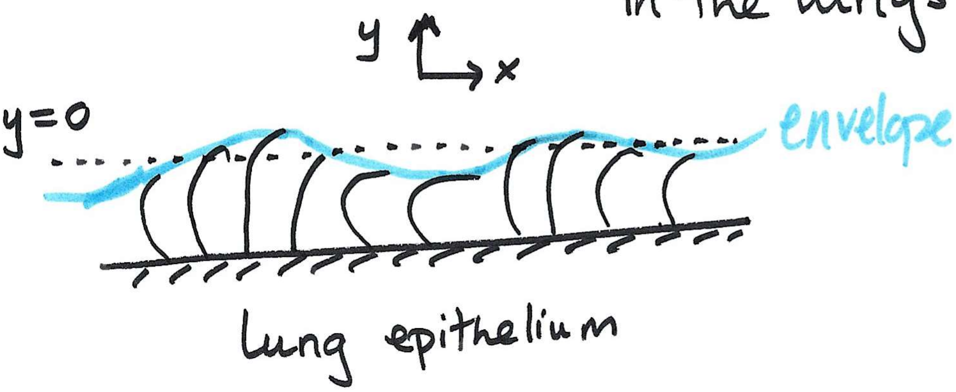
$\therefore \; -\underline{U}(2T-t) = \underline{U}(t)$, by uniqueness of Stokes solutions (eg Pozrikidis, Ch 1).

$$\therefore \int_0^T \underline{U}(t)\,dt + \int_T^{2T} \underline{U}(t_1)\,dt_1 = \int_0^T \underline{U}(t)\,dt + \int_0^T \underline{U}(2T-t)\,dt = \underline{0}$$

$t_1 = 2T - t$

Ciliary Pumping

Ciliated cells move fluid, eg. in the lungs.



We consider the envelope formed by the cilia, given by

$$x_e = x + \epsilon a \cos(x - t)$$
$$y_e = \epsilon b \sin(x - t)$$

for a travelling wave of ciliary beating

Note $y=0$ is the mid-plane of the envelope

On the envelope, we have the velocity is

$$u_e = \partial x_e/\partial t = \epsilon a \sin(x-t)$$
$$v_e = \partial y_e/\partial t = -\epsilon b \cos(x-t)$$

$\therefore$

$$\boxed{\begin{array}{l} \nabla\cdot\underline{u} = 0, \quad \underline{0} = -\nabla p + \nabla^2 \underline{u} \\ \text{with } (u,v) = (u_e, v_e) \text{ on } (x,y) = (x_e, y_e) \\ (u,v) \to (U, 0) \text{ as } y \to \infty, \ \underline{U \text{ to be found}} \end{array}}$$

✓ non dimensionalised already

<u>Solution</u> $\nabla\cdot\underline{u} = 0$ and 2D

$\therefore$ Let $\underline{u} = (u, v)$, $u = \psi_y$, $v = -\psi_x$, $\psi = \psi(x,y)$

where ψ is a streamfunction

$$\therefore \ 0 = \nabla\wedge(\nabla^2 \underline{u}) = \nabla^2(\nabla\wedge\underline{u})$$

$$\nabla\wedge\underline{u} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \partial_x & \partial_y & \partial_z \\ \psi_y & -\psi_x & 0 \end{vmatrix}$$

$$= -(\psi_{xx} + \psi_{yy})\underline{e}_z = -\nabla^2\psi\, \underline{e}_z$$

$$\therefore \ \boxed{\nabla^4\psi = 0}$$

Seek a perturbative solution

If $\epsilon = 0$, no flow. Hence we expand

$$\psi = \epsilon \psi_1 + \epsilon^2 \psi_2 + \cdots, \qquad U = \epsilon U_1 + \epsilon^2 U_2 + \cdots$$

At ord(ϵ)

$$\nabla^4 \psi_1 = 0$$

Also from

$$\psi_y(x_e, y_e) = u_e = \epsilon a \sin(x-t)$$
$$\psi_x(x_e, y_e) = -v_e = \epsilon b \cos(x-t)$$

$$\psi_{1y}(x, 0) = a\sin(x-t)$$

ϵ contributions to x_e, y_e will generate $O(\epsilon^2)$ terms

$$\psi_{1x}(x,0) = b\cos(x-t)$$

$$\lim_{y\to\infty} \psi_{1y}(x,y) = U_1, \quad \lim_{y\to\infty} \psi_{1x}(x,y) = 0.$$

Let $\psi_1 = f(y)\sin(x-t)$.

$$0 = \nabla^4 \psi_1 = \nabla^2 \left[(f'' - f)\sin(x-t) \right] = \left[(f'''' - 2f'' + f)\sin(x-t) \right]$$

$$\therefore \quad f = Ae^{-y} + Be^{y} + Cye^{-y} + Dye^{y}$$

No blow up in ψ_{1x}, ψ_{1y} as $y \to \infty$ gives $B = D = 0$.

$$\therefore \quad \psi_1 = (A + Cy)e^{-y}\sin(x-t).$$

$$\psi_{1x}(x,0) = A\cos(x-t) \quad \therefore A = b.$$

$$\psi_{1y}(x,y) = (-b + C - Cy)e^{-y}\sin(x-t)$$

$$\psi_{1y}(x,0) = (-b+c)\sin(x-t) \quad \therefore\ C = a+b$$

and $\psi_1 = (b + (a+b)y)e^{-y}\sin(x, t)$.

$$\therefore \quad U_1 = \lim_{y\to\infty} \psi_{1y}(x,y) = 0$$ due to the e^{-y} term.

<u>Need to go to next order</u>

<u>BCs</u> $\psi_y(x_e, y_e) = \epsilon a\sin(x-t)$, $\psi_x(x_e, y_e) = \epsilon b\cos(x-t)$

$x_e = x + \epsilon a\cos(x-t)$ $\quad y_e = \epsilon b\sin(x-t)$

$$\underbrace{\epsilon a\cos(x-t)\psi_{1xy}(x,0) + \epsilon b\sin(x-t)\psi_{1yy}(x,0)}_{\text{Second order terms from expanding } x_e,\, y_e \text{ within } \psi_1.} + \psi_{2y}(x,0) = 0$$

$$\epsilon a\cos(x-t)\psi_{1xx}(x,0) + \epsilon b\sin(x-t)\psi_{1xy}(x,0) + \psi_{2x}(x,0) = 0$$

$\therefore$
$$\psi_{2y}(x,0) = -a^2\cos^2(x-t) + b(b+2a)\sin^2(x-t)$$
$$= \tfrac{1}{2}(b^2+2ab-a^2) - \tfrac{1}{2}(a+b)^2\cos(2(x-t))$$

Subs in ψ_1 known.

Similarly

$$\psi_2(x,0) = ab\sin(x-t)\cos(x-t) - ab\cos(x-t)\sin(x-t)$$
$$= 0$$

<u>Note</u> the term $-\tfrac{1}{2}(a+b)\cos(2(x-t))$ will not contribute to U_2 by direct analogy to the calculation for U.

Thus consider only the term $\frac{1}{2}(b^2+2ab-a^2)$.

Therefore we solve

$$\nabla^4 \eta = 0$$

$$\left.\frac{\partial\eta}{\partial y}\right|_{y=0} = \tfrac{1}{2}(b^2+2ab-a^2) \qquad \left.\frac{\partial\eta}{\partial x}\right|_{y=0} = 0,$$

$(\eta_y, -\eta_x) \to (U_2, 0)$ as $y \to \infty$ for U_2 unknown.

No x-dependence. Hence let $\eta = \eta(y)$

$$\therefore \eta'''' = 0, \qquad \eta = Ay^3 + By^2 + Cy + D.$$

Can change η by a constant without changing flow $\therefore$ WLOG $D=0$. η_y finite at $y=\infty$ $\therefore A=B=0$

From $\eta_y|_{y=0} = \frac{1}{2}(b^2+2ab-a^2)$ we thus have

$$\eta = \tfrac{1}{2}(b^2+2ab-a^2)y$$

and hence $\epsilon^2 U_2 = \epsilon^2 \lim_{y\to\infty} \eta_y = \epsilon^2 \tfrac{1}{2}(b^2+2ab-a^2)$

is the non-dimensional pumping speed in the far field of the cilia.

Stokes' Equations : Observations and Solutions

Stokes' Equations are linear... we can build up solutions in terms of linear superpositions of point force solutions, dipoles, quadrapoles etc as in electostatics and the method of images.

The Stokeslet - the solution for a point force

With a point force $\underline{m}$ at $\underline{x}_0$, the Stokes equations are:

$$-\nabla p + \mu \nabla^2 \underline{u} + \underline{m}\,\delta(\underline{x}-\underline{x}_0) = 0 \qquad (1)$$

$$\nabla \cdot \underline{u} = 0$$

With $\left.\begin{array}{l}\hat{\underline{x}} = \underline{x} - \underline{x}_0 \\ r = |\underline{x} - \underline{x}_0|\end{array}\right\}$ $\quad u_i = \dfrac{1}{8\pi\mu}\Big[\underbrace{\dfrac{\delta_{ij}}{r} + \dfrac{\hat{x}_i \hat{x}_j}{r^3}}_{G_{ij},\text{ Greens function.}}\Big] m_j$

Derivation (one of many)

Translational invariance : WLOG set $\underline{x}_0 = 0$.

Linearity $\quad u_i = \dfrac{1}{\mu} K_{ij} m_j$, $\quad K_{ij}$ independent of μ and $\underline{m}$ & thus depends only on $\underline{x}$.

Dimensions $\quad [K_{ij}] = \dfrac{1}{\text{length}}$

δ-function $\quad \delta(\underline{x}) = -\dfrac{1}{4\pi}\nabla^2\left(\dfrac{1}{r}\right)$

$$\boxed{-\nabla p + \mu\nabla^2\underline{u} + \underline{m}\,\delta(\underline{x}-\underline{x}_0) = 0 \quad (1)}$$

Divergence of (1) gives

$$-\nabla^2 p + m\cdot\nabla\left(-\frac{1}{4\pi}\nabla^2\left(\frac{1}{r}\right)\right) = 0$$

Noting $\nabla p \to 0$ as $|x| \to \infty$

$$p = \frac{-1}{4\pi}\, m\cdot\nabla\left(\frac{1}{r}\right)$$

Substitute back into (1)

$$\mu\nabla^2\underline{u} = -\frac{1}{4\pi}\nabla\left(m\cdot\nabla\left(\frac{1}{r}\right)\right) + \frac{1}{4\pi}\underline{m}\,\nabla^2\left(\frac{1}{r}\right)$$

$$\therefore\ \mu\nabla^2 u_i = \frac{m_j}{4\pi}\left[\delta_{ij}\nabla^2 - \frac{\partial^2}{\partial x_i\partial x_j}\right]\left(\frac{1}{r}\right) \qquad (2)$$

$$\therefore\ \text{Let}\quad u_i = \frac{1}{4\pi\mu}\, m_j\left[\delta_{ij}\nabla^2 - \frac{\partial^2}{\partial x_i\partial x_j}\right]G(r)$$

where $\nabla^2 G = 1/r$.

Clearly solves (2) and satisfies $\nabla\cdot\underline{u} = 0$

$$\therefore\ \frac{1}{r}\frac{\partial^2}{\partial r^2}(rG) = 1/r \quad \text{for spherical symmetric solution}$$

$$\therefore\ G = r/2 + A + B/r\ ; \quad A, B \text{ constants.}$$

WLOG $A=0$, as shifting G by a constant has no effect.

Recall K_{ij} depends on $\underline{x}$ only and $[K_{ij}] = \frac{1}{\text{length}}$

$\therefore [G] = \text{length}$, $\underbrace{[B] = (\text{length})^2}$.

No parameters to form a constant with dimensions of length.
Using $\underline{x}\cdot\underline{x}$ will convert $B/r \to B*r$, already considered.

$\therefore$ By dimensions $B=0$.

e.g.

$$\frac{\partial r}{\partial x_i} = \frac{\partial}{\partial x_i}\sqrt{x_i^2+x_j^2+x_k^2} = \frac{x_i}{r}$$

$$\frac{\partial^2 r}{\partial x_i \partial x_j} = \frac{\delta_{ij}}{r} + x_i \cdot \frac{x_j(-1)}{r^3} = \delta_{ij}/r - \frac{x_i x_j}{r^3}$$

$\therefore G = r/2$

$$u_i = \frac{1}{4\pi\mu} m_j \left(\delta_{ij}\nabla^2 - \frac{\partial^2}{\partial x_i \partial x_j}\right)\frac{r}{2}$$

$$= \frac{1}{8\pi\mu}\left[\delta_{ij}/r + \frac{x_i x_j}{r^3}\right] m_j.$$

Let $\underline{x} \to \hat{\underline{x}} = \underline{x} - \underline{x}_0$, $r \to |\underline{x} - \underline{x}_0|$ for the general solution.

Differentiating wrt $\underline{x}_0$ generates many more solutions.

Potential solution

$p =$ Constant, WLOG zero

$$u_i = -\frac{\partial}{\partial x_i}\frac{1}{|\underline{x}-\underline{x}_0|} = -\frac{(\underline{x}-\underline{x}_0)}{r^3}$$

Excluding $\underline{x} = \underline{x}_0$ from the domain $\nabla \cdot \underline{u} = 0$ as

$$\underline{\nabla} \cdot \underline{u} \propto \nabla^2\left(\frac{1}{r}\right),$$

which is the delta function to within a multiplicative constant.

Similarly $\nabla^2 u_i = 0$,

as $\nabla^2 u_i \propto \frac{\partial}{\partial x_i}\nabla^2\left(\frac{1}{r}\right)$.

Point source dipole, potential dipole

Differentiating w.r.t. $\underline{x}_0$ generates more solutions.

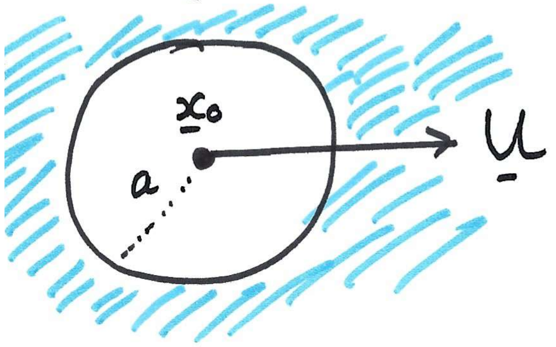
Let $$D_{ij} = \frac{\partial}{\partial x_{0,j}}\left(\frac{\underline{x}-\underline{x}_0}{r^3}\right)_i = -\delta_{ij}/r^3 + \frac{3\hat{x}_i\hat{x}_j}{r^5}.$$

q_j Constant,

Then $u_i = D_{ij}q_j$ solves Stokes' equations

and is a potential dipole solution.

Translating Sphere



① $-\nabla p + \mu \nabla^2 \underline{u} = 0$

② $\underline{u} = \underline{U}$ on $|\underline{x} - \underline{x}_0| = a$

③ $\underline{u} \to 0$ as $|\underline{x}| \to \infty$.

Consider $u_i = \underbrace{G_{ij}}_{\text{Stokeslet}} g_j + \underbrace{D_{ij}}_{\text{Potential dipole}} q_j$

∴ Satisfies ①, ③. Impose ②. When $r = a$

$$U_i = u_i = \left(\delta_{ij}/a + \frac{\hat{x}_i \hat{x}_j}{a^3}\right) g_j + \left(-\delta_{ij}/a^3 + \frac{3\hat{x}_i \hat{x}_j}{a^5}\right) q_j$$

$$\therefore \quad U_i = \left(\frac{1}{a} g_i - \frac{1}{a^3} q_i\right) + \hat{x}_i \hat{x}_j \left(\frac{1}{a^3} g_j + \frac{3}{a^5} q_j\right)$$

$$\underline{q} = -a^2/3 \, \underline{g} \qquad \underline{g} = \frac{3}{4} a \underline{U} = \frac{1}{8\pi\mu}\left[6\pi\mu a \underline{U}\right].$$

giving the solution.

Drag Potential dipole does not contribute (Sum of two opposite effects)

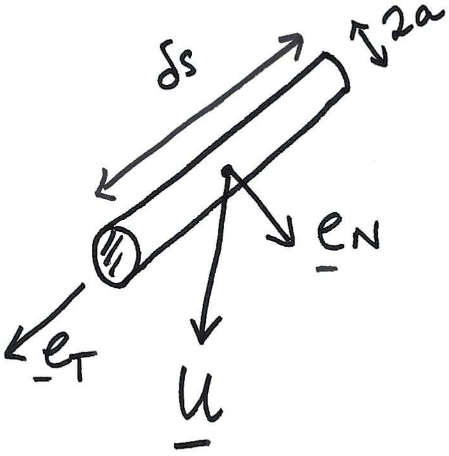
$$\underline{u} = \underbrace{\frac{1}{8\pi\mu} \underline{\underline{G}} \left[6\pi a \mu \underline{U}\right]} + \text{Potential Dipole}$$

Solves $0 = -\nabla p + \mu \nabla^2 \underline{u} + 6\pi\mu a \underline{U} \delta(\underline{x} - \underline{x}_0) = \nabla \cdot \underline{\underline{\sigma}} + 6\pi\mu a \underline{U} \delta(\underline{x} - \underline{x}_0)$

$$\text{Drag} = \int_{\text{Sphere}} \sigma_{ij} n_j \, dS = \int_{\text{Sphere}} \partial_j \sigma_{ij} \, dV = \int_{\text{Sphere}} -6\pi\mu a U_i \, \delta(\underline{x} - \underline{x}_0) \, dV$$

$$= -6\pi\mu a \underline{U}$$

Resistive Force Theory



Consider a slender filament with a circular cross section, radius a

For an element of the filament, of length δs and speed $\underline{u}$, its drag per unit length is

drag ∴ − sign.

$$\underline{f} \overset{\circledast}{=} \underbrace{\underline{\underline{M}}\,\underline{u}}_{\text{by linearity}} \underbrace{=}_{\text{by symmetry of filament}} -C_T\underbrace{(\underline{e}_T\cdot\underline{u})\underline{e}_T}_{\text{projects onto tangential direction}} - C_N\underbrace{(I-\underline{e}_T\otimes\underline{e}_T)\underline{u}}_{\text{rest of }\underline{u}\text{, normal to filament.}}$$

Note: Any theory based on ⊛ is a resistive force theory

$C_N \simeq 2C_T$ observed. ✓ About twice as much force required to pull a rod normal to its length compared to along its length.

Objective

Find C_T, C_N, for a filament radius a, length L.

We align the filament along the x-axis.

Subject the filament to an external force per unit length $\underline{f}^{ext}$.

To proceed split filament into $N = \frac{L}{2a} \gg 1$ segments, of length $2a$.

Neglect of elements interacting with their neighbours initially

<u>Linearity</u>:

$$u_i^\alpha = \frac{L}{N} \underbrace{A_{ij}} f_j^{ext}$$

speed at element α

all elements same ∴ No α dependence

To consider hydrodynamical interactions, approximate each element with a Stokeslet, strength $\frac{L}{N}\underline{f}^{ext}$, at $\underline{x}_0^\alpha = \left(\frac{L}{2N} + (\alpha - 1)\frac{L}{N}, 0, 0\right)$, $\alpha \in \{1, \dots, N\}$.

Flow due to α-element is

$$u_i^\alpha(x) = \frac{1}{8\pi\mu} G_{ij}(\underline{x} - \underline{x}_0^\alpha) \frac{L}{N} f_j^{ext}$$

[Approximate filament segment with a sphere of diameter $2a$, and neglect potential dipole term, which decays much more rapidly.]

Some theories include potential dipole terms and are more accurate but more complex.

For $\underline{x}^\beta$ <u>on the surface</u> of the β-element, we have

$$\overbrace{u_i^\beta}^{\text{element label}}(\underline{x}^\beta) = \underbrace{A_{ij}\left(f_j^{ext}\frac{L}{N}\right)}_{\substack{\text{due to} \\ \beta\text{-element}}} + \sum_{\substack{\alpha \\ \alpha\neq\beta}} \underbrace{\frac{1}{8\pi\mu}\left[\frac{\delta_{ij}}{r}+\frac{r_i r_j}{r^3}\right]\left(f_j^{ext}\frac{L}{N}\right)}_{\substack{\text{due to} \\ \text{other elements}}}$$

where $\underbrace{\underline{r} = \underline{x}^\beta}_{\text{on surface}} - \underbrace{\underline{x}_0^\alpha}_{\text{on centreline}}, \quad r = |\underline{r}|.$

For $N \gg 1$, approximate sum by an integral, excluding β element.

$$u_i^\beta(\underline{x}^\beta) \approx A_{ij}^{ext}\left(f_j^{ext}\frac{L}{N}\right) + \frac{1}{8\pi\mu}\int_0^{s^\beta} ds\left(\frac{\delta_{ij}}{r}+\frac{r_i r_j}{r^3}\right)f_j^{ext}$$

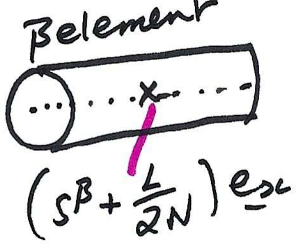
$\frac{L}{N} = \delta s \to ds$

$$+ \frac{1}{8\pi\mu}\int_{s^\beta+2a}^{L} ds\left(\delta_{ij}/r + \frac{r_i r_j}{r^3}\right)f_j^{ext}$$

where $s^\beta = (\beta-1)L/N$, start of β-element, and $x_0^\alpha \to s\underline{e}_x$, arclength along x-axis.

x^β fixed on surface on β-element; we approximate

$$\underline{r} = \underline{x}^\beta - \underline{x}_0 \simeq \underbrace{\left(s^\beta + \frac{L}{2N}\right)\underline{e}_x}_{\text{Centre of }\beta\text{ element, on centreline}} - s\underline{e}_x$$



$s^{\beta} = (\beta - 1)L/N$. Recalling $L/2N = a$,

$$\therefore\ u_i^{\beta} \simeq \left(A_{ij} f_j^{ext}\right)\frac{L}{N} + \frac{1}{8\pi\mu}\left[\delta_{ij} + \delta_{i1}\delta_{j1}\right] f_j^{ext}\left[\int_0^{s^{\beta}} \frac{ds}{\underbrace{|s^{\beta} - s + a|}} + \int_{s^{\beta}+2a}^{L} \frac{ds}{\underbrace{|s^{\beta} - s + a|}}\right]$$

denominators bound away from zero

$\therefore$ Integrals well defined

(Integrals can be done with elementary techniques

assuming away from ends since ...

$$\therefore\ u_i^{\beta} \simeq A_{ij} f_j^{ext}\frac{L}{N} + \frac{1}{8\pi\mu}\left[\delta_{ij} + \delta_{i1}\delta_{j1}\right] f_j^{ext} \cdot \log\left(\frac{(s^{\beta}+a)(L-(s^{\beta}+a))}{a^2}\right)$$

Away from rod ends so that $s^{\beta} = \gamma L$, $\gamma = \mathrm{ord}(1)$, so that $\log(\gamma(1-\gamma)) \sim \mathrm{ord}(1)$) we have

$$u_i^{\beta} \simeq A_{ij} f_j^{ext}\frac{L}{N} + \frac{1}{4\pi\mu}\left[\delta_{ij} + \delta_{i1}\delta_{j1}\right] f_j^{ext}\left(\log(L/a) + \mathrm{ord}(1)\right)$$

$\therefore$ To within errors of $\dfrac{1}{\log(L/a)}$

$$u_i^{\beta} = \frac{1}{4\pi\mu}\left[\delta_{ij} + \delta_{i1}\delta_{j1}\right] f_j^{ext}\log(L/a),$$

with increasing accuracy as $a/L \to 0$.

Improvements can be made, eg. include potential dipole terms, to develop more accurate but more complex theories.

Thus in the x-direction, the tangential direction:

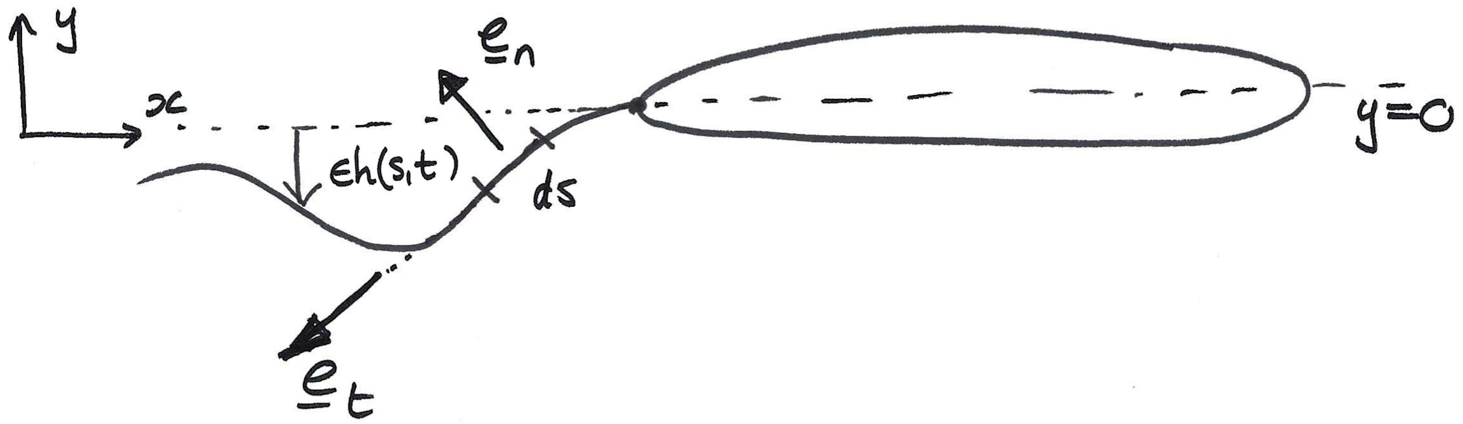
$$-f_T = U_T C_T \qquad C_T = \frac{2\pi\mu}{\log(L/a)}$$

Drag = - Force exerted on filament by N3

Similarly in the normal direction

$$-f_N = U_N C_N , \qquad C_N = \frac{4\pi\mu}{\log(L/a)} = 2C_T.$$

The speed of a swimming cell



* Consider a swimming cell with a slender body, eg. Leishmania major.

* Let $y=0$ correspond to the midplane of the flagellar beat. Small amplitude beating entails the flagellum location is given by

$$y = \epsilon h(s,t) \quad \text{with } \epsilon \ll 1.$$

↳ s is arclength

* Slender body approximated by a cylindrical filament with

$$h = 0 \text{ (no beating)}, \text{ radius } a_b, \text{ length } L_b,$$

so that

$$C^b_N = 2C^b_T = \frac{4\pi\mu}{\ln(L_b/a_b)}$$

← viscosity of medium

* Assume y-velocity and angular velocity are negligible ... not clear a priori when this is true, except possibly for sufficiently small $\epsilon h(s,t)$

See the example sheet for further details concerning this approximation

* Hence overall speed of cell is $U\underline{e}_x$, U unknown.

Find U

$$\underline{0} = (\text{Drag Force on flagellum}) + (\text{Drag Force on cell body})$$

Flagellum

dropping $O(\epsilon^2)$ terms

$$\underline{e}_t = (-1, \epsilon h_s), \quad \underline{e}_n = (\epsilon h_s, 1), \quad \underline{u} = (U, \epsilon h_t).$$

Drag force per unit length

$$\underline{f} = -\left[C_N(\underline{e}_n \cdot \underline{u})\underline{e}_n + C_T(\underline{e}_t \cdot \underline{u})\underline{e}_t\right]$$

$$= -\left[(C_N - C_T)(\underline{e}_n \cdot \underline{u})\underline{e}_n + C_T\underline{u}\right]$$

Hence

$$\underline{f} \cdot \underline{e}_x = -\left[(C_N - C_T)(\underline{e}_n \cdot \underline{u})(\underline{e}_n \cdot \underline{e}_x) + C_T U\right]$$

$$= -\left[(C_N - C_T)\left[(\epsilon h_s)U(\epsilon h_s) + (\epsilon h_t)(\epsilon h_s)\right] + C_T U\right]$$

$$= -\left[(C_N - C_T)\epsilon^2 h_s^2 U + (C_N - C_T)\epsilon^2 h_t h_s + C_T U\right].$$

Cell body

$h = 0$ for cell body

Drag per unit length in x-direction, $\underline{e}_x \cdot \underline{f}_b = -C_T^b U$

$$\therefore \quad 0 = \text{Total Drag in x-direction}$$

$$= -\left(C_T^b U L_b\right) - \int_0^L ds \left[(C_N - C_T)\,\epsilon^2 h_s^2 U + (C_N - C_T)\,\epsilon^2 h_s h_t + C_T U\right]$$

$(1 + O(\epsilon^2))$ correction $\therefore$ drop

$$\therefore \left[C_T^b L_b + C_T L + (C_N - C_T)\,\epsilon^2 \int_0^L ds\, h_s^2\right] U$$

$$= -(C_N - C_T)\,\epsilon^2 \int_0^L h_s h_t\, ds$$

$$\therefore \quad U = \frac{(C_T - C_N)}{C_b^T L_b + C_T L}\,\epsilon^2 \int_0^L h_s h_t\, ds$$

at leading order.